On the Radiation of Sound from an Unflanged Circular Pipe

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A rigorous and explicit solution is obtained for the problem of sound radiation from an unflanged circular pipe, assuming axially symmetric excitation. The solution is valid throughout the wave-length range of dominant mode (plane wave) propagation in the pipe. The reflection coefficient for the velocity potential within the pipe and the power-gain function, embodying the characteristics of the radiation pattern, are evaluated numerically. The absorption cross section of the pipe for a plane wave incident from external space, and the gain function for this direction, are found to satisfy a reciprocity relation. In particular, the absorption cross section for normal incidence is just the area of the mouth. At low frequencies of vibration, the velocity potential within the pipe is the same as if the pipe were lengthened by a certain fraction of the radius and the open end behaved as a loop. The exact value of the end correction turns out to be 0.6133.

I. INTRODUCTION

IT is known that nearly complete reflection of a dominant mode sound wave occurs at the open end of a pipe of circular cross section, if the diameter is small compared to the wave-length. Within the pipe, the velocity potential is the same as if the pipe were lengthened by a certain fraction of its radius and the open end behaved as a loop.

An approximate calculation of the end correction was performed by Lord Rayleigh,[1] who assumed the open end of the pipe fitted with an infinite flange. In the absence of theoretical information, the influence of the flange was investigated experimentally. From his own work, and that of Bosanquet,[2] Rayleigh obtained the probable value of the unflanged end correction as 0.6 the radius of the pipe. This value was substantially confirmed by the subsequent experiments of Blaikley[3] (0.576), Boehm[4] (0.656), and Bate[5] (0.66). Bosanquet, Blaikley, and Anderson and Ostensen[6] found small changes in the magnitude of the end correction, as the wave-length was varied.

[1] Lord Rayleigh, *Theory of Sound(Macmillan and Company, London, 1940)*, Vol. II, Chapter 16 and Appendix A; Phil. Mag. **3**, 456 (1877). L. V. King, Phil. Mag. **21**, 128 (1936).

[2] R. H. M. Bosanquet, Phil. Mag. **4**, 216 (1877).

[3] D. J. Blaikley, Phil. Mag. **7**, 339 (1879).

[4] W. M. Boehm, Phys. Rev. **31**, 341 (1910).

[5] A. E. Bate, Phil. Mag. **10**, 617 (1930); **24**, 453 (1937).

[6] S. H. Anderson and F. C. Ostensen, Phys. Rev. **31**, 267 (1928).

II. STATEMENT OF THE PROBLEM AND RESULTS

We consider a steady-state situation, in which air vibrations are communicated to free space from the interior of an open-ended, rigid circular pipe of negligible wall thickness. If the incident wave-length lies in the proper range, only dominant mode (plane) waves can propagate in the pipe. When the dominant mode waves fall on the open end, part of the incident energy is returned in reflected waves of the same type and the remainder is carried away by waves propagating into the external space.

Our purpose is to determine rigorously the amplitude and phase of the reflected propagating wave in the pipe and the amplitude of the diverging spherical wave at large distance from the mouth. The principal results are as follows.

With the end of the pipe chosen as a reference plane, the reflection coefficient for the dominant mode component of the velocity potential is given by

$$R=-|R|e^{2ikl},$$

where

$$|R|=\exp\left\{-\frac{2ka}{\pi}\int_0^{ka}\frac{\tan^{-1}(-J_1(x)/N_1(x))}{x[(ka)^2-x^2]^{\frac{1}{2}}}dx\right\},$$

and

$$\frac{l}{a}=\frac{1}{\pi}\int_0^{ka}\frac{\log\{\pi J_1(x)[(J_1(x))^2+(N_1(x))^2]^{\frac{1}{2}}\}}{x[(ka)^2-x^2]^{\frac{1}{2}}}dx+\frac{1}{\pi}\int_0^{\infty}\frac{\log[1/(2I_1(x)K_1(x))]}{x[x^2+(ka)^2]^{\frac{1}{2}}}dx.$$

Here and in the following a denotes the radius of the pipe, $k=2\pi/\lambda$ the propagation constant of sound waves in free space, and λ the associated wave-length. J_1, N_1 and I_1, K_1 designate the first-order cylinder functions of real and imaginary argument (Appendix A).

In the long wave-length (or low frequency) limit, $ka \ll 1$, the reflection coefficient assumes the form

$$R=-e^{2ikl},$$

where

$$\frac{l}{a}=\frac{1}{\pi}\int_0^\infty \frac{1}{x^2}\log\frac{1}{2I_1(x)K_1(x)}dx=0.6133$$

is the exact value of the end correction, to be compared with experimental determinations ranging from 0.58 to 0.66.

The angular distribution of the emitted radiation, which is symmetrical about the axis of the pipe, is described by the power-gain function

$$\mathcal{G}(\vartheta)=\frac{4}{\pi \sin^2\vartheta}\frac{J_1(ka\sin\vartheta)}{[(J_1(ka\sin\vartheta))^2+(N_1(ka\sin\vartheta))^2]^{\frac{1}{2}}}\times\frac{|R|}{1-|R|^2}\exp\left[\frac{2ka\cos\vartheta}{\pi}P\times\int_0^{ka}\frac{x\tan^{-1}(-J_1(x)/N_1(x))dx}{[x^2-(ka\sin\vartheta)^2][x^2+(ka)^2]^{\frac{1}{2}}}\right],$$

defined relative to an isotropically radiating point source. The angle ϑ is measured from the axis of the pipe, and P signifies that the integral is to be understood as a principal value. The gain does not vanish for any direction in space; null directions and secondary maxima appear in the radiation pattern when the pipe sustains more than one propagating mode.

The maximum value of the gain occurs in the forward direction, and is simply

$$\mathcal{G}(0)=(ka)^2/(1-|R|^2),$$

a monotonically increasing function of frequency. Another indication of the greater directivity in the radiation pattern at high frequencies is found in the ratio

$$\mathcal{G}(\pi)/\mathcal{G}(0)=|R|^2.$$

The gain in the direction at right angles to the axis of the pipe also assumes a relatively simple form,

$$\mathcal{G}(\pi/2)=\frac{4}{\pi}\frac{J_1(ka)}{[(J_1(ka))^2+(N_1(ka))^2]^{\frac{1}{2}}}\frac{|R|}{1-|R|^2}.$$

These analytical expressions have been obtained from the solution of an integral equation by a Fourier transform method, and are rigorously correct provided that only dominant

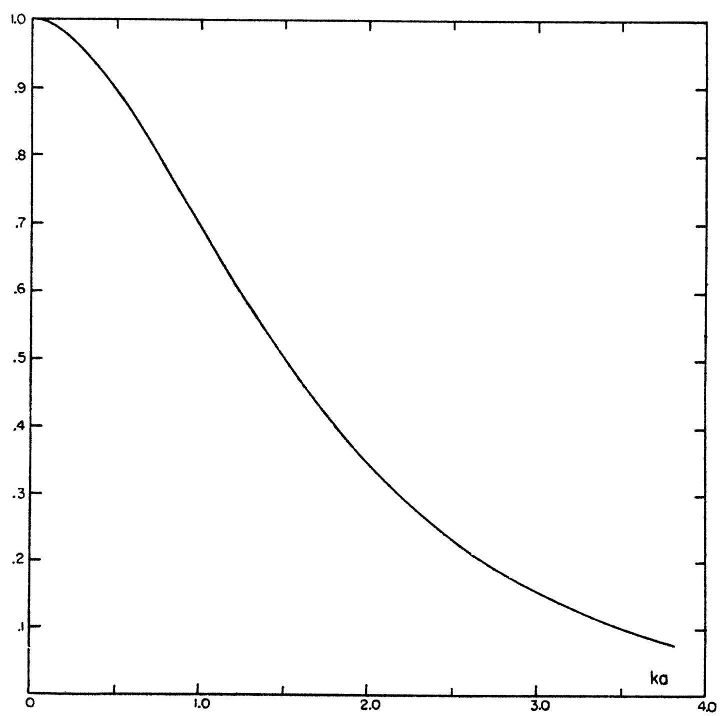

FIG. 1. Magnitude of the velocity potential reflection coefficient as a function of $ka=2\pi a/\lambda$.

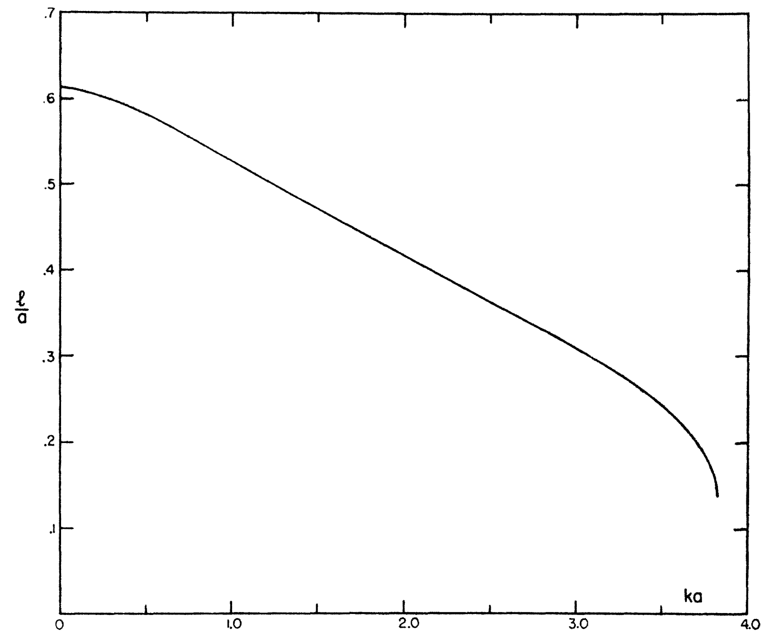

FIG. 2. End correction, in units of the pipe radius, as a function of ka.

mode propagation occurs in the pipe. This condition is realized by restricting values of the characteristic parameter, ka, in accordance with the inequality $0 < ka < 3.832$.[7] There is no essential difficulty in extending the results to a larger frequency range where reflected waves of several propagating modes are generated by the incident dominant mode waves. On the other hand, this method is not simply adapted to the problem of a flanged pipe.

The results of numerical calculation are displayed in the curves of Figs. 1–4.

The known radiation characteristics of the pipe determine the absorption of energy from an externally incident plane wave. This type of reciprocity is familiar in electromagnetic theory, for the intensity of radiation emitted by an antenna in a given direction is proportional to the absorption of radiation arriving from the same direction.

The absorption cross section, obtained on dividing the power transmitted into the pipe by the power incident per unit area, is

$$\sigma(\vartheta) = \pi a^2(\mathcal{G}(\vartheta)/\mathcal{G}(0)),$$

where, ϑ denotes the angle between the direction of incidence of the plane wave and the axis of the pipe. This result is also rigorous in the wave-length range of the dominant mode. The range is reduced, however, to $0 < ka < 1.841$, in the case of oblique incidence ($\vartheta \neq 0$), owing to the excitation of all modes, including non-symmetric, at the mouth of the pipe.

The absorption cross section for normal incidence ($\vartheta = 0$) is just the area of the mouth, independent of the wave-length, in the range $0 < ka < 3.832$.

III. DESCRIPTION OF PHYSICAL QUANTITIES

A well-known existence theorem states that the electromagnetic fields within a region are uniquely determined by the values of the tangential components of the electric or magnetic fields on the bounding surface of the region. Similarly, in the problem under consideration, the entire acoustic field can be derived from its boundary values on the surface of the pipe, regarded as an obstacle imbedded in free space. In this section we shall obtain the dependence of the physical quantities on these (as yet unknown) boundary values.

The fundamental field variable is the scalar velocity potential $\psi(\mathbf{r})$, which satisfies the wave equation

$$(\nabla^2 + k^2)\psi(\mathbf{r}) = 0 \qquad \text{(III. 1)}$$

[7] The cut-off frequencies of the various modes are determined by the zeros of the derivative of integral order Bessel functions; for the symmetric modes these are identical with the zeros of the first-order Bessel function.

for harmonic time variation e^{-ikct} (c = velocity of sound propagation in free space), and has vanishing normal derivative on the surface of the pipe. If the pipe is excited internally, and sustains only the dominant mode, the asymptotic forms of the velocity potential are (Fig. 5)

$$\psi(\mathbf{r}) \sim Ae^{ikz}+Be^{-ikz}, \quad z\rightarrow -\infty \tag{III. 2a}$$

within the pipe, and

$$\psi(\mathbf{r}) \sim f(\vartheta)e^{ikr}/r, \quad r\rightarrow \infty \tag{III. 2b}$$

outside the pipe. With this type of excitation, the entire field is axially symmetric.

Let us apply Green's theorem in the form

$$\int [\psi(\mathbf{r})(\nabla^2+k^2)\phi(\mathbf{r})-\phi(\mathbf{r})(\nabla^2+k^2)\psi(\mathbf{r})]d\tau = \int [\psi(\mathbf{r})\mathbf{n}\cdot\nabla\phi(\mathbf{r})-\phi(\mathbf{r})\mathbf{n}\cdot\nabla\psi(\mathbf{r})]dS \tag{III. 3}$$

to the velocity potential $\psi(\mathbf{r})$ and the plane wave

$$\phi(\mathbf{r})=e^{-i\mathbf{k}\cdot\mathbf{r}}=e^{-ikr\cos\Theta}, \quad \cos\Theta=\cos\vartheta\cos\vartheta'+\sin\vartheta\sin\vartheta'\cos(\varphi-\varphi'), \tag{III. 4}$$

whose direction of propagation is specified by the angles (ϑ', φ'). The domain of integration, indicated in Fig. 5, is bounded by surfaces inside and outside the pipe; $\mathbf{n}$ is a unit vector which is normal to the bounding surface at each point and is directed outwards from the region. The length L is chosen sufficiently large so that the asymptotic forms (2) may be employed on the surfaces S_1, S_2.

The volume integral in (3) vanishes since $\psi(\mathbf{r})$ and $\phi(\mathbf{r})$ are solutions of the homogeneous wave Eq. (1). Separating the various parts of the surface integral, we find, with the use of (2) and the boundary condition for the radial derivative of the velocity potential at the surface of the pipe,

$$\begin{aligned}\int_{S_1}\Big[f(\vartheta)\frac{e^{ikr}}{r}\frac{\partial}{\partial r}e^{-ikr\cos\Theta}-e^{-ikr\cos\Theta}\frac{\partial}{\partial r}f(\vartheta)\frac{e^{ikr}}{r}\Big]dS \\ -\int_{S_2}\Big\{(Ae^{-ikL}+Be^{ikL})\frac{\partial}{\partial z}[\exp(-ikz\cos\vartheta'-ik\rho\sin\vartheta'\cos(\varphi-\varphi')]_{z=-L} \\ -\exp(ikL\cos\vartheta'-ik\rho\sin\vartheta'\cos(\varphi-\varphi'))\frac{\partial}{\partial z}(Ae^{ikz}+Be^{-ikz})_{z=-L}\Big\}dS \\ -\int_{S_3}[\psi(\mathbf{r})_{\rho=a+0}-\psi(\mathbf{r})_{\rho=a-0}]\frac{\partial}{\partial\rho}[\exp(-ikz\cos\vartheta'-ik\rho\sin\vartheta'\cos(\varphi-\varphi'))]_{\rho=a}dS=0.\end{aligned} \tag{III. 5}$$

The difference of the velocity potential on inner and outer surfaces of the pipe, which appears in the integral S_3, is a consequence of the oppositely directed normals at these surfaces.

The integration on the surface S_1 is effected by choosing the direction (ϑ', φ') as the polar axis; thus

$$\begin{aligned}&\int_{S_1}\Big[f(\vartheta)\frac{e^{ikr}}{r}\frac{\partial}{\partial r}e^{-ikr\cos\Theta}-e^{-ikr\cos\Theta}\frac{\partial}{\partial r}f(\vartheta)\frac{e^{ikr}}{r}\Big]dS \\ &=-ik\int_0^{2\pi}\int_0^{\pi}(1+\cos\Theta)f(\vartheta)e^{ikr(1-\cos\Theta)}r\sin\Theta d\Theta d\Phi+\int_0^{2\pi}\int_0^{\pi}f(\vartheta)e^{ikr(1-\cos\Theta)}\sin\Theta d\Theta d\Phi \\ &=-\int_0^{2\pi}\int_0^{\pi}(1+\cos\Theta)f(\vartheta)\Big[\frac{d}{d\Theta}e^{ikr(1-\cos\Theta)}\Big]d\Theta d\Phi+\int_0^{2\pi}\int_0^{\pi}f(\vartheta)e^{ikr(1-\cos\Theta)}\sin\Theta d\Theta d\Phi \\ &=-4\pi f(\vartheta')+\int_0^{2\pi}\int_0^{\pi}e^{ikr(1-\cos\Theta)}\frac{d}{d\Theta}f(\vartheta)d\Theta d\Phi.\end{aligned}$$

If the integration by parts is continued, the integral is developed in a series of inverse powers of r. Consequently,

$$\lim_{r\to\infty}\int_{S_1}\left[f(\vartheta)\frac{e^{ikr}}{r}\frac{\partial}{\partial r}e^{-ikr\cos\Theta}-e^{-ikr\cos\Theta}\frac{\partial}{\partial r}f(\vartheta)\frac{e^{ikr}}{r}\right]dS=-4\pi f(\vartheta'). \quad \text{(III. 6)}$$

If we return to (5) and perform some of the remaining integrations it follows that (dropping the primes on the angle ϑ)

$$f(\vartheta)=\lim_{L\to\infty}\left[\frac{ka}{2}\sin\vartheta J_1(ka\sin\vartheta)\int_{-L}^{0}H(z)e^{-ikz\cos\vartheta}dz + i\frac{a}{2}\frac{J_1(ka\sin\vartheta)}{\sin\vartheta}\{A(1+\cos\vartheta)e^{-ikL(1-\cos\vartheta)}-B(1-\cos\vartheta)e^{ikL(1+\cos\vartheta)}\}\right], \quad \text{(III. 7)}$$

where $H(z)$ is the discontinuity of the velocity potential on crossing the surface of the pipe

$$H(z)=\psi(\mathbf{r})_{\rho=a+0}-\psi(\mathbf{r})_{\rho=a-0}.$$

If we define a function of the complex variable ζ by

$$F(\zeta)=\lim_{L\to\infty}\left[\frac{a}{2}(k^2-\zeta^2)^{\frac{1}{2}}J_1((k^2-\zeta^2)^{\frac{1}{2}}a)\int_{-L}^{0}H(z)e^{-i\zeta z}dz + i\frac{a}{2}\frac{J_1((k^2-\zeta^2)^{\frac{1}{2}}a)}{(k^2-\zeta^2)^{\frac{1}{2}}}\{A(k+\zeta)e^{-iL(k-\zeta)}-B(k-\zeta)e^{iL(k+\zeta)}\}\right], \quad \text{(III. 8)}$$

it follows that when $Im\,\zeta>0$ (or $>|Im\,k|$) the terms involving A and B vanish in the limit $L\to\infty$, and

$$F(\zeta)=\frac{a}{2}(k^2-\zeta^2)^{\frac{1}{2}}J_1((k^2-\zeta^2)^{\frac{1}{2}}a)\times\int_{-\infty}^{0}H(z)e^{-i\zeta z}dz. \quad \text{(III. 9)}$$

Equation (7) may now be regarded as the analytic continuation of this function on the line $Im\,\zeta=0$ (or in the strip $-|Im\,k|\leqslant Im\,\zeta\leqslant|Im\,k|$), at the point $\zeta=k\cos\vartheta$:

$$F(k\cos\vartheta)=f(\vartheta)=\frac{ka}{2}\sin\vartheta J_1(ka\sin\vartheta)\times\int_{-\infty}^{0}H(z)e^{-ikz\cos\vartheta}dz. \quad \text{(III. 10)}$$

The integral

$$H(\zeta)=\int_{-\infty}^{0}H(z)e^{-i\zeta z}dz \quad \text{(III. 11)}$$

defines the Fourier transform of a function $H(z)$ which vanishes for positive z. $H(z)$ has the same asymptotic form as the velocity potential on the inner surface of the pipe (since the field outside decreases in magnitude with the nature of a spherical wave), and we infer from (2a) and (11) that the transform $H(\zeta)$ has simple poles at $\zeta=\pm k$.

Using (7) and (9), we find

$$f(0)=i\tfrac{1}{2}ka^2A=-\tfrac{1}{2}ka^2\lim_{\zeta\to k}(\zeta-k)H(\zeta)=-\tfrac{1}{2}ka^2\,\mathrm{Res}_{\zeta=k}H(\zeta), \quad \text{(III. 12)}$$

$$f(\pi)=-i\tfrac{1}{2}ka^2B=\tfrac{1}{2}ka^2\lim_{\zeta\to-k}(\zeta+k)H(\zeta)=\tfrac{1}{2}ka^2\,\mathrm{Res}_{\zeta=-k}H(\zeta), \quad \text{(III. 13)}$$

where $\mathrm{Res}_{\zeta=\pm k}H(\zeta)$ denotes the residue of the function $H(\zeta)$ at the poles $\zeta=\pm k$, respectively. Thus

$$A=i\,\mathrm{Res}_{\zeta=k}H(\zeta),\quad B=i\,\mathrm{Res}_{\zeta=-k}H(\zeta), \quad \text{(III. 14)}$$

and the reflection coefficient for the dominant mode component of the velocity potential takes the form

$$R=\frac{B}{A}=\frac{\mathrm{Res}_{\zeta=-k}H(\zeta)}{\mathrm{Res}_{\zeta=k}H(\zeta)}. \quad \text{(III. 15)}$$

To describe the radiation characteristics of the pipe, we calculate the power-gain function, which compares the intensity of radiation in a given direction with that of an isotropically radiating point source of equal power output.

We assume as the expression for the time average energy flow per unit area

$$\mathbf{S} = Re\frac{1}{ik}\psi^*(\mathbf{r})\nabla\psi(\mathbf{r}), \qquad \text{(III. 16)}$$

and verify that this corresponds to dissipationless transport of energy by the sound waves in free space, for

$$\nabla\cdot\mathbf{S} = Re\nabla\cdot\left(\frac{1}{ik}\psi^*(\mathbf{r})\nabla\psi(\mathbf{r})\right) = Re\frac{1}{ik}(|\nabla\psi(\mathbf{r})|^2 - k^2|\psi(\mathbf{r})|^2) = 0.$$

Through (16), the average power incident on the mouth of the pipe is found to be

$$P_{\text{inc}} = \pi a^2|A|^2, \qquad \text{(III. 17)}$$

and therefore the power leaving the end of the pipe for external space is

$$P_{\text{rad}} = \pi a^2(|A|^2 - |B|^2) = \pi a^2|A|^2(1-|R|^2). \qquad \text{(III. 18)}$$

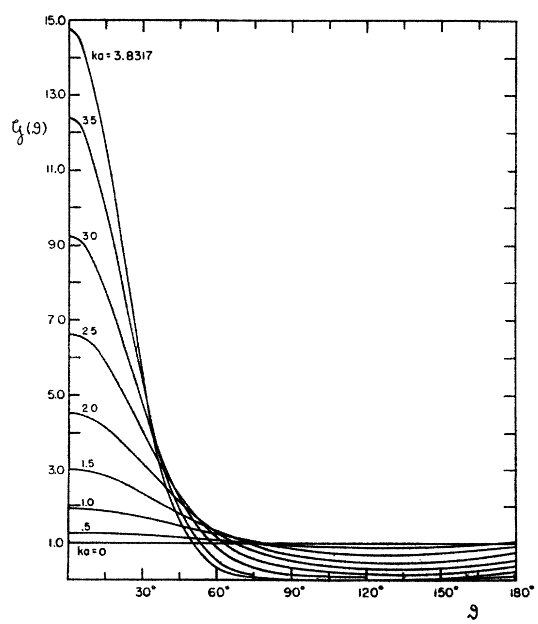

FIG. 3. The dependence of the power-gain function on angle for various values of ka.

Since $\nabla_r\psi(\mathbf{r}) \sim ik\psi(\mathbf{r})$, $r\to\infty$, the average power radiated into unit solid angle about the direction (ϑ, φ) is given by

$$P(\vartheta) = \text{Lim}_{r\to\infty} r^2|\psi(\mathbf{r})|^2 = |f(\vartheta)|^2; \qquad \text{(III.19)}$$

thus the power-gain function becomes

$$\mathcal{G}(\vartheta) = \frac{P(\vartheta)}{P_{\text{rad}}/4\pi} = \frac{4}{a^2}\frac{|f(\vartheta)|^2}{|A|^2(1-|R|^2)}. \qquad \text{(III. 20)}$$

Using the connections between the incident and reflected amplitudes A, B and the radiated amplitudes $f(0)$, $f(\pi)$ provided in (12) and (13), we find

$$\mathcal{G}(\vartheta) = \frac{(ka)^2}{1-|R|^2}\frac{|f(\vartheta)|^2}{|f(0)|^2}. \qquad \text{(III. 21)}$$

It follows readily that

$$\mathcal{G}(0) = \frac{(ka)^2}{1-|R|^2}, \qquad \text{(III. 22)}$$

and

$$\frac{\mathcal{G}(\pi)}{\mathcal{G}(0)} = \frac{|f(\pi)|^2}{|f(0)|^2} = |R|^2. \qquad \text{(III. 23)}$$

Using (10), (12), and (15), the gain function (21) takes the form

$$\mathcal{G}(\vartheta) = \frac{(k\sin\vartheta J_1(ka\sin\vartheta))^2|H(k\cos\vartheta)|^2}{|\text{Res}_{\zeta=k}H(\zeta)|^2 - |\text{Res}_{\zeta=-k}H(\zeta)|^2}. \qquad \text{(III. 24)}$$

Equations (15) and (24) provide the dependence of the important physical quantities on the transform $H(\zeta)$. We note that these expressions are independent of any constant multiplying $H(\zeta)$.

As the final task of this section we shall derive the reciprocity relation between emission and absorption of energy by the pipe. For this purpose, we consider the independent situations in which plane waves are incident on the mouth from within and outside the pipe. The velocity potentials for the two fields $\psi_a(\mathbf{r})$, $\psi_b(\mathbf{r})$ have the asymptotic forms

$$\left.\begin{aligned}\psi_a(\mathbf{r}) &\sim Ae^{ikz} + Be^{-ikz}\\ \psi_b(\mathbf{r}) &\sim Ce^{-ikz}\end{aligned}\right\} z\to-\infty, \text{ inside the pipe}$$

and (III. 25)

$$\left.\begin{aligned}\psi_a(\mathbf{r}) &\sim f(\vartheta)\frac{e^{ikr}}{r}\\ \psi_b(\mathbf{r}) &\sim e^{-ikr\cos\Theta} + g(\vartheta,\varphi)\frac{e^{ikr}}{r}\end{aligned}\right\} r\to\infty, \text{ outside the pipe.}$$

Applying Green's theorem (3) to the functions $\psi_a(\mathbf{r})$, $\psi_b(\mathbf{r})$ in the closed region indicated in Fig. 5, we find that the integral over the bounding surface vanishes. There is no contribution to this integral from the inner and outer surfaces of the pipe, in consequence of the boundary condition for the velocity potentials, and thus

$$\int_{S_1}\left[\psi_a(\mathbf{r})\frac{\partial}{\partial r}\psi_b(\mathbf{r})-\psi_b(\mathbf{r})\frac{\partial}{\partial r}\psi_a(\mathbf{r})\right]dS$$
$$=\int_{S_2}\left[\psi_a(\mathbf{r})\frac{\partial}{\partial z}\psi_b(\mathbf{r})-\psi_b(\mathbf{r})\frac{\partial}{\partial z}\psi_a(\mathbf{r})\right]dS.$$

Inserting the forms (25), and following the procedure used in the derivation of (6), we find that

$$C=2f(\vartheta')/ika^2A.$$

Using (20) and (22), it follows that

$$|C|^2=\mathcal{G}(\vartheta')/\mathcal{G}(0),$$

and thus the absorption cross section of the pipe for a plane wave incident in the direction specified by the angles (ϑ, φ) is

$$\begin{aligned}\sigma(\vartheta)&=\pi a^2|C|^2\\&=\pi a^2\mathcal{G}(\vartheta)/\mathcal{G}(0).\end{aligned} \qquad \text{(III. 26)}$$

IV. INTEGRAL EQUATION FORMULATION

To proceed with the evaluation of the reflection coefficient and power-gain function, it is necessary to calculate the Fourier transform of the discontinuity of the velocity potential at the surface of the pipe. In this section we shall obtain an integral equation for the determination of the transform.

We begin by deriving a general expression for the velocity potential at an arbitrary point in space, in terms of its discontinuity at the surface of the pipe. The mathematical medium for this purpose is provided by the free space scalar Green's function

$$G(\mathbf{r}, \mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \qquad \text{(IV. 1)}$$

which satisfies the inhomogeneous wave equation

$$(\nabla^2+k^2)G(\mathbf{r}, \mathbf{r}')=-\delta(\mathbf{r}-\mathbf{r}'). \qquad \text{(IV. 2)}$$

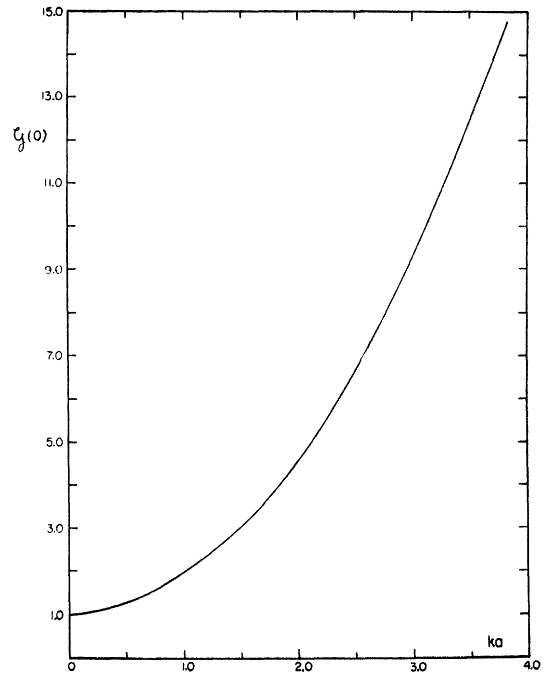

FIG. 4. The gain in the forward direction as a function of *ka*.

Upon applying Green's theorem in the form

$$\int[G(\mathbf{r}', \mathbf{r})(\nabla'^2+k^2)\psi(\mathbf{r}')-\psi(\mathbf{r}')(\nabla'^2+k^2)G(\mathbf{r}', \mathbf{r})]d\tau'$$
$$=\int[G(\mathbf{r}', \mathbf{r})\mathbf{n}\cdot\nabla'\psi(\mathbf{r}')-\psi(\mathbf{r}')\mathbf{n}\cdot\nabla'G(\mathbf{r}', \mathbf{r})]dS', \qquad \text{(IV. 3)}$$

where the integration is extended over all space and the boundary includes the inner and outer surfaces of the pipe, we find, using the boundary condition for the velocity potential at the surface of the pipe,

$$\begin{aligned}\psi(\mathbf{r})&=a\int_0^{2\pi}d\varphi'\int_{-\infty}^{0}\Psi(\varphi', z')\frac{\partial}{\partial\rho'}G(\mathbf{r}', \mathbf{r})_{\rho'=a}dz',\\ \Psi(\varphi, z)&=\psi(\mathbf{r})_{\rho=a+0}-\psi(\mathbf{r})_{\rho=a-0}.\end{aligned} \qquad \text{(IV. 4)}$$

As the notation indicates, the formulation is not restricted to the case of an axially symmetric field.

It should be noted that there is no contribution to (3) from that part of the surface which spans the interior of the pipe, since the magnitude of the Green's function decreases inversely as the

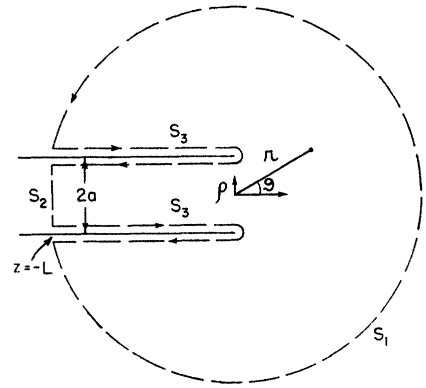

FIG. 5. Domain of integration for Eq. (III. 3).

distance between the points $\mathbf{r}$, $\mathbf{r}'$, and the velocity potential is bounded. The surface integral exterior to the pipe also vanishes, for, in consequence of the asymptotic forms

$$\psi(\mathbf{r}) \sim f(\mathbf{n})\frac{e^{ikr}}{r}, \qquad \mathbf{n}=\mathbf{r}/r,\ r\to\infty$$
$$G(\mathbf{r},\mathbf{r}') \sim \frac{\exp[ik(r-\mathbf{n}\cdot\mathbf{r}')]}{4\pi r},$$

it follows that

$$\lim_{r\to\infty} r^2\left[G(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r}\psi(\mathbf{r})-\psi(\mathbf{r})\frac{\partial}{\partial r}G(\mathbf{r},\mathbf{r}')\right]=0.$$

Equation (4) provides confirmation of the assertion that the entire field is determined by its boundary values at the surface of the pipe.

Upon requiring that the velocity potential (4) satisfy the boundary condition at the surface of the pipe, we are provided with the integral equation

$$\int_0^{2\pi} d\varphi' \int_{-\infty}^{0} \Psi(\varphi', z')\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'} \times G(\mathbf{r},\mathbf{r}')_{\rho,\,\rho'=a}dz'=0, \quad z<0 \qquad \text{(IV. 5)}$$

for the determination of $\Psi(\varphi, z)$.

To effect the angular integration in (5), it is convenient to express the Green's function as a Fourier series in the angular variable $(\varphi-\varphi')$. For this purpose, we rewrite (2) in the form[8]

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}+k^2\right) \times G(\rho,\rho',\varphi-\varphi',z-z') = -\frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi')\delta(z-z'); \qquad \text{(IV. 6)}$$

that the Green's function depends on the difference of the coordinates φ, φ' and z, z' is evident from (1).

Multiplying (6) by $e^{-i\zeta z}$ and integrating over all values of z, we find

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+k^2-\zeta^2\right)G(\rho,\rho',\varphi-\varphi',\zeta) = -\frac{\delta(\rho-\rho')}{\rho}\delta(\varphi-\varphi'). \qquad \text{(IV. 7)}$$

Here

$$G(\rho,\rho',\varphi-\varphi',\zeta)=\int_{-\infty}^{\infty} G(\rho,\rho',\varphi-\varphi',z)e^{-i\zeta z}dz$$
$$=\int_{-\infty}^{\infty}\frac{\exp[ik(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{\frac{1}{2}}]}{4\pi(\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')+z^2)^{\frac{1}{2}}}e^{-i\zeta z}dz \qquad \text{(IV. 8)}$$

is the Fourier transform of the Green's function with respect to z.

Introducing coordinates in the ζ-plane,

$$\zeta=\xi+i\eta,$$

it follows that

$$|G(\rho,\rho',\varphi-\varphi',\zeta)| \leqslant \int_{-\infty}^{\infty}|G(\rho,\rho',\varphi-\varphi',z)|e^{\eta z}dz.$$

Assuming the propagation constant k to have an arbitrarily small positive imaginary part[9] (which is eventually set equal to zero), we find,

[8] The three-dimensional representation of the delta-function satisfies the conditions $\delta(\mathbf{r}-\mathbf{r}')=0$, $\mathbf{r}\neq\mathbf{r}'$ and $\int\delta(\mathbf{r}-\mathbf{r}')d\tau=1$, where $d\tau=\rho d\rho d\varphi dz$ is the volume element in cylindrical coordinates.

[9] This corresponds to a small attenuation of sound waves traveling in free space.

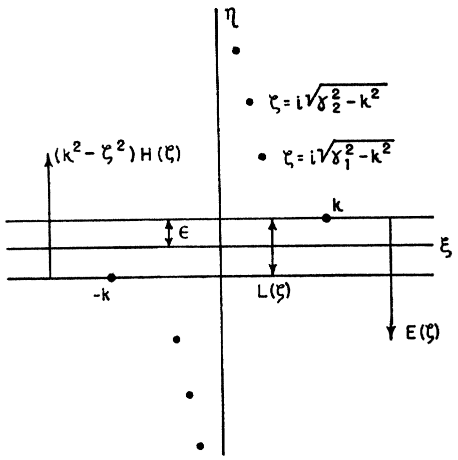

FIG. 6. Regions of regularity of transforms.

in consequence of the asymptotic form

$$G(\rho,\ \rho',\ \varphi-\varphi',\ z)\sim e^{ik|z|}/z, \quad |z|\rightarrow\infty,$$

that the transform $G(\rho,\ \rho',\ \varphi-\varphi',\ \zeta)$ is regular in the strip $|\eta|<\epsilon(=Im\ k)$. The operation of integration by parts twice, with discard of the integrated terms, in the derivation of the equation

$$\int_{-\infty}^{\infty}\frac{\partial^2}{\partial z^2}G(\rho,\ \rho',\ \varphi-\varphi',\ z)e^{-i\zeta z}dz$$
$$=\int_{-\infty}^{\infty}G(\rho,\ \rho',\ \varphi-\varphi',\ z)\frac{\partial^2}{\partial z^2}e^{-i\zeta z}dz$$
$$=-\zeta^2 G(\rho,\ \rho',\ \varphi-\varphi',\ \zeta)$$

is justified if ζ lies in the strip.

Introducing the expansion

$$G(\rho,\ \rho',\ \varphi-\varphi',\ \zeta)=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\varphi-\varphi')}G_m(\rho,\ \rho',\ \zeta) \qquad \text{(IV. 9)}$$

in (7), and multiplying through by $e^{-im\varphi}$, we find, after integrating over the range of the angle φ,

$$\left(\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right)+k^2-\zeta^2-\frac{m^2}{\rho^2}\right)G_m(\rho,\ \rho',\ \zeta)=-\frac{\delta(\rho-\rho')}{\rho}. \qquad \text{(IV. 10)}$$

When $\rho\neq\rho'$, the right-hand member of (10) vanishes, and we arrive at Bessel's differential equation.

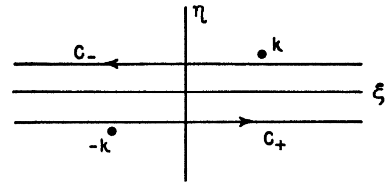

FIG. 7. Integration contours for $L_+(\zeta)$, $L_-(\zeta)$.

Since the Green's function (1) describes the propagation of a spherical wave from the source point $\mathbf{r}=\mathbf{r}'$, we look for a solution of (10) which involves cylindrical wave propagation outwards along the radius ρ. In view of the regularity of $G_m(\rho,\ \rho',\ \zeta)$ when either ρ or ρ' vanishes, and its symmetry in these coordinates, we write

$$\begin{aligned}G_m(\rho,\ \rho',\ \zeta)&=A_mH_m^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}\rho)\\&\quad\times J_m((k^2-\zeta^2)^{\frac{1}{2}}\rho'),\quad \rho>\rho',\\G_m(\rho,\ \rho',\ \zeta)&=A_mH_m^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}\rho')\\&\quad\times J_m((k^2-\zeta^2)^{\frac{1}{2}}\rho),\quad \rho<\rho'.\end{aligned} \qquad \text{(IV. 11)}$$

J_m is the mth order Bessel function, and $H_m^{(1)}=J_m+iN_m$ is the mth order Hankel function of the first kind.

To insure the proper sense of cylindrical wave propagation, it is necessary to specify the phases of the radical in (11): this we do on the real axis of the ζ-plane, adopting the values 0, $\pi/2$ for the phase in the sections $|\zeta|<k$, $>k$, respectively.

The same coefficient A_m occurs in the expressions (11), owing to the continuity of the function $G_m(\rho,\ \rho',\ \zeta)$ at $\rho=\rho'$. The value of A_m is obtained from the inhomogeneous term of the differential Eq. (10) on multiplying both sides by ρ and integrating in a small interval about ρ':

$$\rho\frac{d}{d\rho}G_m(\rho,\ \rho',\ \zeta)\Bigg|_{\rho=\rho'-\Delta}^{\rho=\rho'+\Delta}+\int_{\rho'-\Delta}^{\rho'+\Delta}\left(k^2-\zeta^2-\frac{m^2}{\rho^2}\right)\rho\times G_m(\rho,\ \rho',\ \zeta)d\rho=-1.$$

Recalling the continuity of the terms in the integrand, and passing to the limit $\Delta\rightarrow 0$, we find

$$\rho\frac{d}{d\rho}G_m(\rho,\ \rho',\ \zeta)\Bigg|_{\rho=\rho'-0}^{\rho=\rho'+0}=-1.$$

Introducing the expressions (11), and employing the relation

$$H_m^{(1)\prime}(z)J_m(z)-J_m'(z)H_m^{(1)}(z)=2i/\pi z,$$

it follows that

$$A_m=\tfrac{1}{2}\pi i.$$

Thus, using $\rho_>$, $\rho_<$ to denote the larger and smaller of the coordinates ρ, ρ' respectively,

$$G_m(\rho,\rho',\zeta)=\frac{\pi i}{2}H_m^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}\rho_>)\times J_m((k^2-\zeta^2)^{\frac{1}{2}}\rho_<), \quad \text{(IV. 12)}$$

and by the Fourier inversion formula,

$$G(\rho,\rho',\varphi-\varphi',z-z')=\frac{i}{8\pi}\int_{-\infty+i\eta}^{\infty+i\eta}\sum_{m=-\infty}^{\infty}e^{im(\varphi-\varphi')}H_m^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}\rho_>)J_m((k^2-\zeta^2)^{\frac{1}{2}}\rho_<)e^{i\zeta(z-z')}d\zeta$$

$$=\frac{i}{8\pi}\int_{-\infty+i\eta}^{\infty+i\eta}H_0^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}(\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi'))^{\frac{1}{2}}]e^{i\zeta(z-z')}d\zeta. \quad \text{(IV. 13)}$$

The integration contour in (13) is a straight line in the region of regularity of the Green's function transform.

Returning to the integral Eq. (5), and assuming

$$\Psi(\varphi,z)=e^{im\varphi}H_m(z), \quad \text{(IV. 14)}$$

we find

$$\int_{-\infty}^{0}H_m(z')K_m(z-z')dz'=0,\quad z<0, \quad \text{(IV. 15)}$$

where

$$K_m(z)=\int_0^{2\pi}e^{im\varphi'}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}\times G(\rho,\rho',\varphi-\varphi',z)_{\rho,\,\rho'=a}d\varphi'. \quad \text{(IV. 16)}$$

Thus, using the Green's function representation (13), it follows that

$$K_m(z)=\frac{1}{2\pi}\int_{-\infty+i\eta}^{\infty+i\eta}K_m(\zeta)e^{i\zeta z}d\zeta, \quad \text{(IV. 17)}$$

with

$$K_m(\zeta)=\tfrac{1}{2}\pi i(k^2-\zeta^2)H_m^{(1)\prime}((k^2-\zeta^2)^{\frac{1}{2}}a)\times J_m'((k^2-\zeta^2)^{\frac{1}{2}}a). \quad \text{(IV. 18)}$$

The primes in (18) denote differentiations with respect to the argument of the cylinder functions.

The integral Eq. (15) resembles the Wiener-Hopf type,[10] and may be solved by application of a Fourier transform method. In the course of solution, we use the fact that the asymptotic form of $H_m(z)$, for large negative values of z, is the same as that of the velocity potential on the inner surface of the pipe (see remarks following (III. 11)).

The formulation thus far is completely general and applies directly if the pipe is excited in any set of modes with a common angular dependence of the type (14). This generality is not required for present purposes since our interest relates only to the case of symmetric excitation ($m=0$) in accordance with the program outlined in the preceding sections.

With wave-lengths permitting only dominant mode propagation, our problem therefore is to

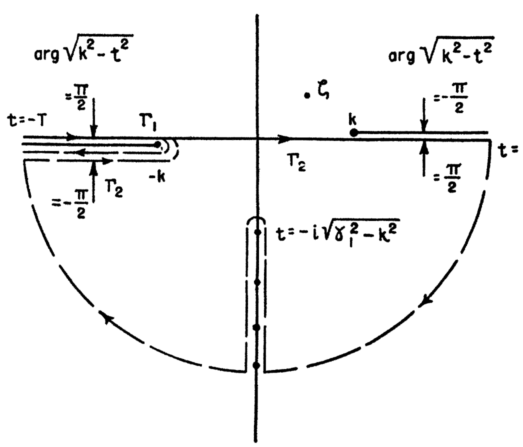

FIG. 8. Integration contours for Eq. (V. 17).

[10] R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (Am. Math. Soc. Colloquium Publications, New York, 1934), Vol. XIX; E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, London, 1937), Chapter IV, p. 339; E. Reissner, J. Math. and Phys. M. I. T. (20) 5, 219 (1941).

obtain a solution $H_0(z)$ of the integral equation

$$\int_{-\infty}^{0} H_0(z')K_0(z-z')dz'=0, \quad z<0, \quad \text{(IV. 19)}$$

with the asymptotic form (omitting an unessential constant factor)

$$H_0(z)\sim Ae^{ikz}+Be^{-ikz}, \quad z\to-\infty. \quad \text{(IV. 20)}$$

For an extended frequency interval, we must include in (20) additional reflected waves of the higher symmetric modes, with appropriate propagation constants.

In concluding this section we note an alternative integral equation formulation of the problem, originating with the division of space into the two regions $\rho \gtreqless a$. Using (3) in conjunction with appropriate Green's functions, the velocity potential in each region is expressed in terms of its radial derivative on the surface $\rho=a$, $z>0$. The requirement of continuity for the velocity potential on crossing this surface provides an inhomogeneous integral equation of the Wiener-Hopf type for the determination of the common radial derivative.

The transform of the radial derivative allows a simple calculation of the physical quantities; the results are identical with those obtained from the formulation in terms of the discontinuity of the velocity potential at the surface of the pipe.

If an infinite flange is fitted to the open end of the pipe, the new formulation requires modification only in the construction of the Green's function for the region $\rho>a$. Using the method of images, the new Green's function is obtained by adding to (1) a similar expression in which the sign of z' is reversed. The kernel of the resulting inhomogeneous integral equation thus includes a term depending on $z+z'$, which considerably complicates the Fourier transform solution.

V. FOURIER TRANSFORM SOLUTION OF THE INTEGRAL EQUATION

In preparation for the Fourier transform solution of (IV. 19), we consider an extended integral equation (in which subscripts are omitted):

$$\int_{-\infty}^{\infty} H(z')K(z-z')dz'=\begin{cases}E(z), & z\geqslant 0\\ 0, & z<0,\end{cases}$$

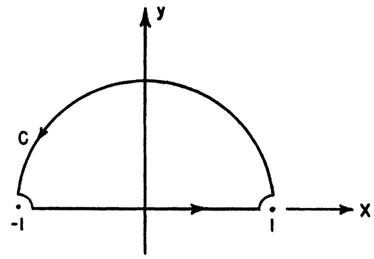

FIG. 9. Integration contour for Eq. (VI. 11).

where

$$H(z)=\begin{cases}0, & z>0\\ H_0(z), & z\leqslant 0,\end{cases} \quad \text{(V. 1)}$$

and

$$K(z)=\int_0^{2\pi}\frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'}G(\rho,\rho',\varphi-\varphi',z)_{\rho,\,\rho'=a}d\varphi'.$$

By construction, the integral Eq. (1) is equivalent to (IV. 19), for $z<0$; the function $E(z)$ is defined by (1) for $z\geqslant 0$.

Multiplying (1) by $e^{-i\zeta z}$ and integrating over all values of z,

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}H(z')K(z-z')dz'\right)e^{i\zeta z} =\int_0^{\infty}E(z)e^{-i\zeta z}dz;$$

thus, with a change of variable, we have

$$\int_{-\infty}^{0}H(z')e^{-i\zeta z'}dz'\int_{-\infty}^{\infty}K(t)e^{-i\zeta t}dt =\int_0^{\infty}E(z)e^{-i\zeta z}dz,$$

and

$$H(\zeta)K(\zeta)=E(\zeta), \quad \text{(V. 2)}$$

where $H(\zeta)$, $K(\zeta)$, and $E(\zeta)$ denote the Fourier transforms of the respective functions.

The transform Eq. (2) has significance if both members are regular in a common domain of the ζ-plane. To verify the existence of such a domain we examine the nature of the individual transforms.

From the discussion following (III. 11), we recall that the transform

$$H(\zeta)=\int_{-\infty}^{0}H(z)e^{-i\zeta z}dz=\int_{-\infty}^{0}H(z)e^{-i\xi z+\eta z}dz \quad \text{(V. 3)}$$

has simple poles at $\zeta=\pm k$. In consequence of the asymptotic form (IV. 20), it follows that

$$|H(\zeta)|\leqslant\int_{-\infty}^{0}|H(z)|e^{\eta z}dz$$

is bounded in the region $\eta>\epsilon$ $(=Im\ k)$.

The transform $K(\zeta)$ is obtained from (IV. 18) by setting $m=0$ and using the relations (A. 4), (A. 5) of the Appendix:

$$K(\zeta)=\tfrac{1}{2}\pi i(k^2-\zeta^2)\times H_1^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}a)J_1((k^2-\zeta^2)^{\frac{1}{2}}a). \quad \text{(V. 4)}$$

Inserting the explicit form (IV. 1) of the Green's function in (1), and performing the indicated differentiations, we find

$$K(z)\sim e^{ik|z|}/z^3,\quad |z|\rightarrow\infty;$$

thus the transform $K(\zeta)$ is regular in the strip $|\eta|<\epsilon$.

Although $K(\zeta)$ vanishes at $\zeta=\pm k$, these are branch points of the function. If we write

$$K(\zeta)=\tfrac{1}{2}(k^2-\zeta^2)L(\zeta),$$
$$L(\zeta)=\pi iH_1^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}a)J_1((k^2-\zeta^2)^{\frac{1}{2}}a), \quad \text{(V. 5)}$$

the function $L(\zeta)$ is regular in the strip $|\eta|<\epsilon$, and has the value unity at the branch points.

We next establish the important result that $L(\zeta)$ does not vanish in this strip. To do so, we note first that $H_1^{(1)}(z)$ has no zeros for which $-\pi/2\leqslant\arg z\leqslant 3\pi/2$.[11] Since this phase interval contains all the values appropriate to the function $(k^2-\zeta^2)^{\frac{1}{2}}$ in the ζ-plane, we conclude that $H_1^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}a)$ has no zeros. The zeros of $J_1((k^2-\zeta^2)^{\frac{1}{2}}a)$ occur at $\zeta=\pm k$,[12] and at $\zeta=\pm(\gamma_n^2-k^2)^{\frac{1}{2}}$, where $J_1(\gamma_n a)=0$. However, the latter set, comprising the attenuation constants for the non-propagating symmetric modes of the pipe, have imaginary parts with magnitudes in excess of ϵ, and lie outside the strip.

The results show that $H(\zeta)$ and $K(\zeta)$ are regular in separate domains of the ζ-plane. However, the product $H(\zeta)K(\zeta)$ may be separated into a pair of factors which are regular in a common domain. This is accomplished by writing $K(\zeta)$ in the form (5), and associating the factor $(k^2-\zeta^2)/2$ with $H(\zeta)$. The function $(k^2-\zeta^2)H(\zeta)$ is regular for $\eta\geqslant-\epsilon$, and thus has a common domain of regularity with $L(\zeta)$.

A discussion of the transform $E(\zeta)$ is based on (1); inserting the asymptotic form of the kernel, we find

$$E(z)\sim e^{ikz}\int_{-\infty}^{0}H(z')\frac{e^{-ikz'}}{(z-z')^3}dz',\quad z\rightarrow\infty.$$

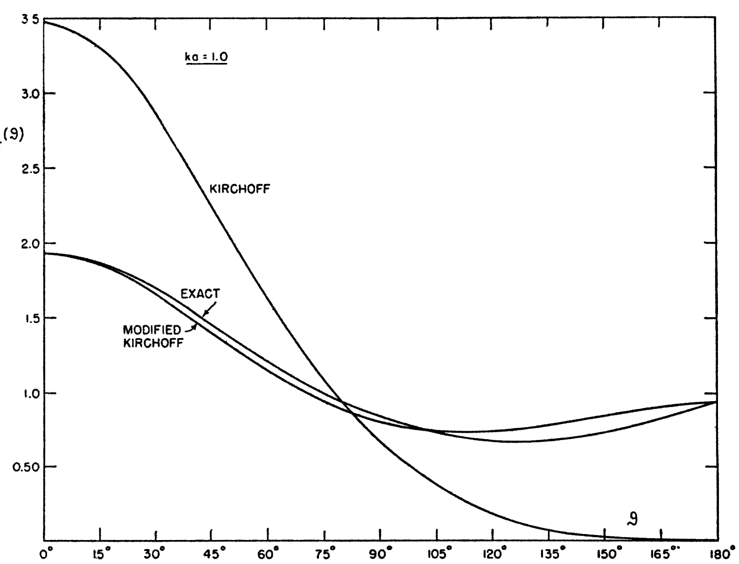

FIG. 10. Comparison of exact and approximate gain functions.

[11] This follows from an investigation of the zeros of $K_1(z)=-(\pi/2)H_1^{(1)}(iz)$; G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, Teddington, England, 1945), p. 511.

[12] The zeros of $J_1((k^2-\zeta^2)^{\frac{1}{2}}a)$ at $\zeta=\pm k$ are compensated by the singulartities of $H_1^{(1)}((k^2-\zeta^2)^{\frac{1}{2}}a)$, resulting in a finite value for $L(\zeta)$.

In estimating the z dependence of the integral, it is not permissible to neglect z' in the denominator; if this is done, the integral has the value of the transform $H(\zeta)$ at the pole $\zeta=k$ (corresponding to the incident wave Ae^{ikz}). However, if we replace $H(z)$ by e^{ikz}, it follows that

$$E(z)\sim e^{ikz}/z^2, \quad z\rightarrow\infty.$$

This result may be verified by noting that $E(z)$ is proportional to the ρ-derivative of the velocity potential on the surface $\rho=a$ and utilizing the asymptotic form (III. 2b).

The transform

$$E(\zeta)=\int_0^\infty E(z)e^{-i\zeta z}dz=\int_0^\infty E(z)e^{-i\xi z+\eta z}dz \quad \text{(V. 6)}$$

is thus regular in the region $\eta\leqslant\epsilon$.

Collecting results, we find that the strip $|\eta|<\epsilon$ is a common region of regularity for the functions $(k^2-\zeta^2)H(\zeta)$, $L(\zeta)$, and $E(\zeta)$, appearing in the modified form of the transform Eq. (2):

$$(k^2-\zeta^2)H(\zeta)L(\zeta)=2E(\zeta). \quad \text{(V. 7)}$$

It is convenient to designate the regions $\eta>-\epsilon$, $\eta<\epsilon$ as the upper and lower half-planes, respectively (Fig. 6).

In order to solve the transform Eq. (7), we represent $L(\zeta)$ as a quotient of functions $L_+(\zeta)$, $L_-(\zeta)$, which are regular (and not zero) in the upper and lower half-planes, respectively. If the expression

$$L(\zeta)=L_+(\zeta)/L_-(\zeta) \quad \text{(V. 8)}$$

is inserted in (7), it follows that

$$(k^2-\zeta^2)H(\zeta)L_+(\zeta)=2E(\zeta)L_-(\zeta). \quad \text{(V. 9)}$$

The left-hand side of (9) is regular in the upper half-plane and the right-hand side in the lower half-plane. Both are regular in the strip $|\eta|<\epsilon$ and may be considered as analytic continuations of each other; together they define an integral function throughout the finite ζ-plane.

It turns out that the integral function has algebraic behavior at infinity in the respective half-planes and is in fact, a constant. Thus, the transforms $H(\zeta)$, $E(\zeta)$ bear a simple relation to $L_+(\zeta)$, $L_-(\zeta)$, respectively.

We next consider the explicit determination of $L_+(\zeta)$, $L_-(\zeta)$. The function $\log L(\zeta)$ is regular in the strip $|\eta|<\epsilon$, on the branch for which $\log 1=0$. Applying Cauchy's integral formula to a rectangular domain (with sides parallel to the coordinate axes) within the strip, we have

$$\log L(\zeta)=\frac{1}{2\pi i}\oint\frac{\log L(t)}{t-\zeta}dt$$
$$=\frac{1}{2\pi i}\oint\frac{\log\{\pi i H_1^{(1)}[(k^2-t^2)^{\frac{1}{2}}a]J_1[(k^2-t^2)^{\frac{1}{2}}a]\}}{t-\zeta}dt, \quad \text{(V. 10)}$$

where ζ is an internal point.

As $|t|\rightarrow\infty$, $\arg(k^2-t^2)^{\frac{1}{2}}=\pi/2$ within the strip, and we find with the help of (A. 11),

$$\log L(t)\sim\log|t|, \quad |t|\rightarrow\infty, \quad |Im\, t|<\epsilon.$$

Thus, there are vanishing contributions to the integral (10) from the vertical sections of the rectangular contour as these are displaced to infinity.

Consequently,

$$\log L(\zeta)=\log L_+(\zeta)-\log L_-(\zeta)$$
$$=\frac{1}{2\pi i}\int_{C_+}\frac{\log L(t)}{t-\zeta}dt+\frac{1}{2\pi i}\int_{C_-}\frac{\log L(t)}{t-\zeta}dt, \quad \text{(V. 11)}$$

where C_+, C_- designate infinite straight line contours in the strip $|Im\, t|<\epsilon$ (Fig. 7).

The function

$$L_+(\zeta)=\exp\left[\frac{1}{2\pi i}\int_{C_+}\frac{\log\{\pi i H_1^{(1)}[(k^2-t^2)^{\frac{1}{2}}a]J_1[(k^2-t^2)^{\frac{1}{2}}a]\}}{t-\zeta}dt\right] \quad \text{(V. 12)}$$

is regular and different from zero in the upper half-plane provided ζ does not lie on the contour C_+.

Similarly,

$$L_-(\zeta)=\exp\left[-\frac{1}{2\pi i}\int_{C_-}\frac{\log\{\pi i H_1^{(1)}[(k^2-t^2)^{\frac{1}{2}}a]J_1[(k^2-t^2)^{\frac{1}{2}}a]\}}{t-\zeta}dt\right] \tag{V. 13}$$

defines a non-vanishing regular function in the lower half-plane if ζ does lie on the contour C_-.

First Evaluation of $L_+(\zeta)$, $L_-(\zeta)$

The conversion of (12) and (13) into forms suitable for analytical and numerical manipulation is based on certain deformations of the contours C_+, C_-.

First, we note that in the limit $\epsilon\rightarrow 0$ the contours C_+, C_- coincide with the real axis of the t-plane, except for indentations at the points $t=\pm k$ and $t=\zeta$, if ζ is real. The contours are indented above $t=-k$, below $t=k$, and oppositely with respect to $t=\zeta$ (C_+ below, C_- above), thus preserving the correspondence of points in the ζ-plane with the regions of regularity of the functions $L_+(\zeta)$, $L_-(\zeta)$.

Actually, the logarithmic functions in (12) and (13) vanish at $t=\pm k$, so that the integrations are conducted entirely along the real axis with an indentation only if ζ is real.

Inserting the phase of the radical appropriate to the sections of the real axis $|t|<k$, $|t|>k$, and using (A. 6), (A. 7), we find

$$\begin{Bmatrix} L_+(\zeta) \\ L_-(\zeta)\end{Bmatrix}=\exp\left[\frac{1}{2\pi i}\int_{-k}^{k}\frac{\log\{\pi i H_1^{(1)}[(k^2-t^2)^{\frac{1}{2}}a]J_1[(k^2-t^2)^{\frac{1}{2}}a]\}}{t-\zeta}dt\right.$$
$$\left.+\frac{\zeta}{\pi i}\int_{k}^{\infty}\frac{\log\{2K_1[(t^2-k^2)^{\frac{1}{2}}a]I_1[(t^2-k^2)^{\frac{1}{2}}a]\}}{t^2-\zeta^2}dt\right], \tag{V. 14}$$
$$L_+(-\zeta)=1/L_-(\zeta).$$

We note, on referring to (A. 12), that the integrals along the sections $t>k$, $t<-k$ are individually non-convergent.

When ζ is real, the singular integral of (14) is calculated as the principal value $\pm\pi i$ (residue of the integrand at the pole $t=\zeta$). The choice of sign corresponds to the sense of deformation of the contour, being positive for $L_+(\zeta)$ and negative for $L_-(\zeta)$.

Accordingly, we find

$$L_+(k\cos\vartheta)=[\pi i H_1^{(1)}(ka\sin\vartheta)J_1(ka\sin\vartheta)]^{\frac{1}{2}}\cdot\exp\left[i\frac{ka\cos\vartheta}{\pi}P\int_0^{ka}\frac{x\log[\pi i H_1^{(1)}(x)J_1(x)]}{[x^2-(ka\sin\vartheta)^2][(ka)^2-x^2]^{\frac{1}{2}}}dx\right.$$
$$\left.+i\frac{ka\cos\vartheta}{\pi}\int_0^{\infty}\frac{x\log[1/2I_1(x)K_1(x)]}{[x^2+(ka\sin\vartheta)^2][x^2+(ka)^2]^{\frac{1}{2}}}dx\right], \tag{V. 15}$$

where P designates the principal value.

The principal value is not required if ζ coincides with either of the branch points $\pm k$, and it turns out that

$$L_+(k)=\frac{1}{L_+(-k)}=\exp\left[i\frac{ka}{\pi}\int_0^{ka}\frac{\log[\pi i H_1^{(1)}(x)J_1(x)]}{x[(ka)^2-x^2]^{\frac{1}{2}}}dx+i\frac{ka}{\pi}\int_0^{\infty}\frac{\log[1/(2I_1(x)K_1(x))]}{x[x^2+(ka)^2]^{\frac{1}{2}}}dx\right]. \tag{V. 16}$$

Second Evaluation of $L_+(\zeta)$, $L_-(\zeta)$

To avoid questions of convergence with those parts of the integrals (12) and (13) for which $|Re\, t|>Re\, k$, we begin with sections of the contours C_+, C_- extending from $t=-T$ to $t=T$. Later we proceed to the limit $T\rightarrow\infty$.

The abbreviated contour C_+' may be replaced by the contours Γ_1, Γ_2 shown in Fig. 8. Γ_1 is extended along the sides of a horizontal branch cut which is drawn from the point $t=-k$, and Γ_2 proceeds along the under side of the branch cut on to the point $t=T$ of the original contour C_+.

In this discussion, we employ the logarithmic derivative

$$\frac{d}{d\zeta}\log L_+(\zeta)=\frac{1}{2\pi i}\int_{C_+'}\frac{\log\{\pi i H_1^{(1)}[(k^2-t^2)^{\frac{1}{2}}a]J_1[(k^2-t^2)^{\frac{1}{2}}a]\}}{(t-\zeta)^2}dt, \tag{V. 17}$$

from which $L_+(\zeta)$ is determined, exclusive of a multiplicative constant, by a single integration. If the point $t=\zeta$ is located in the upper half-plane, C_+' or any equivalent contours may be freely deformed in the lower half-plane provided we do not pass through singularities of the logarithm in the integrand of (17).

Combining the integrands of (17) on the upper and lower sections of Γ_1 we find, using (A. 6)–(A. 9),

$$\frac{d}{d\zeta}\log L_+(\zeta)\bigg|_{\Gamma_1}=-\frac{1}{2\pi i}\int_k^T\log\left\{\frac{K_1([t^2-k^2]^{\frac{1}{2}}a)+\pi e^{-i\pi/2}I_1([t^2-k^2]^{\frac{1}{2}}a)}{K_1[(t^2-k^2)^{\frac{1}{2}}a]}\right\}\frac{dt}{(t+\zeta)^2}. \tag{V. 18}$$

It follows from (A. 13) that (18) diverges logarithmically in the limit $T\rightarrow\infty$; if we supply the factor $\exp[i(\pi/2)-2a(t^2-k^2)^{\frac{1}{2}}]$ to the argument of the logarithm, the singular dependence may be isolated, with the result

$$\frac{d}{d\zeta}\log L_+(\zeta)\bigg|_{\Gamma_1}=\frac{ia}{\pi}\lim_{T\rightarrow\infty}\log\frac{2T}{k}+\frac{1}{4}\frac{1}{k+\zeta}-\frac{ia}{\pi}\left[1+\frac{2\zeta}{(k^2-\zeta^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{k-\zeta}{k+\zeta}\right)^{\frac{1}{2}}\right]$$
$$-\frac{1}{2\pi i}\int_k^\infty\log\left\{\frac{e^{i\pi/2}K_1[(t^2-k^2)^{\frac{1}{2}}a]+\pi I_1[(t^2-k^2)^{\frac{1}{2}}a]}{K_1[(t^2-k^2)^{\frac{1}{2}}a]}\exp[-2a(t^2-k^2)^{\frac{1}{2}}]\right\}\frac{dt}{(t+\zeta)^2}. \tag{V. 19}$$

Next, we close the contour Γ_2 as shown in Fig. 8, proceeding along circular arcs in the lower half-plane and on opposite sides of a branch cut along the negative imaginary axis (which is drawn through the zeros $t=-i(\gamma_n^2-k^2)^{\frac{1}{2}}$ of the Bessel function $J_1[(k^2-t^2)^{\frac{1}{2}}a]$). On this complete contour (17) vanishes since the integrand is analytic at all interior points. Thus the integral along Γ_2 may be expressed in terms of integrals along the remainder of the closed contour; if these be traversed in the same sense, we have

$$\int_{\Gamma_2}=-\int_{\text{arcs}}-\int_{\text{branch cut}} \tag{V. 20}$$

Referring to (A. 11), it follows that the integral (17) along a circular arc in the third quadrant, on which $-\pi/2<\arg(k^2-t^2)^{\frac{1}{2}}<0$, takes the constant value

$$\frac{1}{2\pi i}\int_{\text{arc}}\frac{dt}{t^2}(2i)(ita)=\frac{a}{2} \tag{V. 21}$$

in the limit $T\rightarrow\infty$. The integral along a circular arc in the fourth quadrant, on which $0<\arg(k^2-t^2)^{\frac{1}{2}}<\pi/2$, vanishes in this limit.

The value of the remaining integral in (20) depends upon the difference in phase of the integrand on opposite sides of the branch cut along the negative imaginary axis.

Using the asymptotic expansion

$$\gamma_n a\sim(n+\tfrac{1}{4})\pi+0\left(\frac{1}{n}\right),\quad n\gg1, \tag{V. 22}$$

we find that the number of zeros of $J_1[(k^2-t^2)^{\frac{1}{2}}a]$ contained on the section of the negative imaginary axis $|Im\, t| < T$, is

$$n_T \sim [\gamma_n a/\pi] = [Ta/\pi], \quad Ta/\pi \gg 1;$$

here $[Ta/\pi]$ is the largest integer not exceeding Ta/π.

Thus, since the functions $H_1^{(1)}([k^2-t^2]^{\frac{1}{2}}a)$ and

$$J_1([k^2-t^2]^{\frac{1}{2}}a) \Big/ \prod_{n=1}^{[Ta/\pi]} [t+i(\gamma_n^2-k^2)^{\frac{1}{2}}]$$

have the same phase on opposite sides of the branch cut, the integral becomes

$$\frac{1}{2\pi i}\int_{\text{branch cut}} \log \prod_{n=1}^{[Ta/\pi]} [t+i(\gamma_n^2-k^2)^{\frac{1}{2}}]\frac{dt}{(t-\zeta)^2} = \sum_{n=1}^{[Ta/\pi]} \int_{t=-i(\gamma_n^2-k^2)^{\frac{1}{2}}}^{-iT} \frac{dt}{(t-\zeta)^2}$$

$$= \sum_{n=1}^{[Ta/\pi]} \left(\frac{1}{\zeta+iT} - \frac{1}{\zeta+i[\gamma_n^2-k^2]^{\frac{1}{2}}}\right). \quad \text{(V. 23)}$$

Further, since

$$\lim_{m\to\infty}\left[\sum_{n=1}^{m}\frac{1}{n} - \log m\right] = \log \gamma = 0.5772,$$

we have

$$\lim_{T\to\infty}\sum_{n=1}^{[Ta/\pi]}\left[\frac{1}{\zeta+iT} - \frac{1}{\zeta+(\gamma_n^2-k^2)^{\frac{1}{2}}}\right] = \frac{ia}{\pi}\lim_{T\to\infty}\log\frac{Ta\gamma}{\pi} - \frac{ia}{\pi} - \sum_{n=1}^{\infty}\left[\frac{1}{\zeta+i(\gamma_n^2-k^2)^{\frac{1}{2}}} + \frac{ia}{n\pi}\right]. \quad \text{(V. 24)}$$

Collecting the results (21) and (24) for use in (20), we find

$$\left.\frac{d}{d\zeta}\log L_+(\zeta)\right|_{\Gamma_2} = -\frac{ia}{\pi}\lim_{T\to\infty}\log\frac{Ta\gamma}{\pi} + \frac{ia}{\pi} - \frac{a}{2} + \sum_{n=1}^{\infty}\left[\frac{1}{\zeta+i(\gamma_n^2-k^2)^{\frac{1}{2}}} + \frac{ia}{n\pi}\right]. \quad \text{(V. 25)}$$

When (19) and (25) are combined, the arbitrary parameter T conveniently disappears and we have

$$\frac{d}{d\zeta}\log L_+(\zeta) = \frac{1}{4(k+\zeta)} + \frac{ia}{\pi}\log\frac{2\pi i}{\gamma ka} - \frac{2i\zeta a}{\pi(k^2-\zeta^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{k-\zeta}{k+\zeta}\right)^{\frac{1}{2}} + \sum_{n=1}^{\infty}\left[\frac{1}{\zeta+i(\gamma_n^2-k^2)^{\frac{1}{2}}} + \frac{ia}{n\pi}\right]$$

$$-\frac{1}{2\pi i}\int_k^\infty \log\left\{\frac{e^{i\pi/2}K_1([t^2-k^2]^{\frac{1}{2}}a) + \pi I_1([t^2-k^2]^{\frac{1}{2}}a)}{K_1[(t^2-k^2)^{\frac{1}{2}}a]} \times \exp[-2a(t^2-k^2)^{\frac{1}{2}}]\right\}\frac{dt}{(t+\zeta)^2}. \quad \text{(V. 26)}$$

Integrating with respect to ζ and supplying an integration constant, it follows that

$$L_+(\zeta) = C((k+\zeta)a)^{\frac{1}{4}}F_+(\zeta)\prod_{n=1}^{\infty}\left\{\left[1-\left(\frac{k}{\gamma_n}\right)^2\right]^{\frac{1}{2}} - \frac{i\zeta}{\gamma_n}\right\}e^{i\zeta a/n\pi}$$

$$\cdot\exp\left[\frac{i\zeta a}{\pi}\left(\log\frac{2\pi i}{\gamma ka}+1\right) + \frac{2ia}{\pi}(k^2-\zeta^2)^{\frac{1}{2}}\tan^{-1}\left(\frac{k-\zeta}{k+\zeta}\right)^{\frac{1}{2}}\right], \quad \text{(V. 27)}$$

where

$$F_+(\zeta) = \exp\left[\frac{1}{2\pi i}\int_k^\infty \log\left\{\frac{e^{i\pi/2}K_1[(t^2-k^2)^{\frac{1}{2}}a] + \pi I_1[(t^2-k^2)^{\frac{1}{2}}a]}{K_1[(t^2-k^2)^{\frac{1}{2}}a]}\exp[-2a(t^2-k^2)^{\frac{1}{2}}]\right\}\frac{dt}{t+\zeta}\right]. \quad \text{(V. 28)}$$

The infinite product in (27) converges uniformly in the upper half-plane. This result follows from (22) since for any value of ζ

$$\{[1-(k/\gamma_n)^2]^{\frac{1}{2}}-i\zeta/\gamma_n\}e^{i\zeta a/n\pi}\sim 1+a_n(\zeta),\quad a_n(\zeta)=O(1/n^2),\quad Im\,\zeta>0,\quad n\gg\max([ka],[|\zeta a|]).$$ [13]

On deforming the contour C_- in the upper half-plane, and employing similar procedures, we find (in view of the symmetry of $L(\zeta)$ with regard to ζ)

$$\frac{1}{L_-(\zeta)}=L_+(-\zeta)=C((k-\zeta)a)^{\frac{1}{2}}\frac{1}{F_-(\zeta)}\prod_{n=1}^{\infty}\left\{\left[1-\left(\frac{k}{\gamma_n}\right)^2\right]^{\frac{1}{2}}+\frac{i\zeta}{\gamma_n}\right\}e^{-i\zeta a/n\pi}.$$

$$\cdot\exp\left[-\frac{i\zeta a}{\pi}\log\left(\frac{2\pi i}{\gamma ka}+1\right)+\frac{2ia}{\pi}(k^2-\zeta^2)^{\frac{1}{2}}\tan^{-1}\left(\frac{k+\zeta}{k-\zeta}\right)^{\frac{1}{2}}\right],\quad \text{(V. 29)}$$

where

$$F_-(\zeta)=\frac{1}{F_+(-\zeta)}=\exp\left[-\frac{1}{2\pi i}\int_k^{\infty}\log\left\{\frac{e^{i\pi/2}K_1[(t^2-k^2)^{\frac{1}{2}}a]+\pi I_1[(t^2-k^2)^{\frac{1}{2}}a]}{K_1[(t^2-k^2)^{\frac{1}{2}}a]}\right.\right.$$

$$\left.\left.\times\exp[-2a(t^2-k^2)^{\frac{1}{2}}]\right\}\frac{dt}{t-\zeta}\right].\quad \text{(V. 30)}$$

To determine the constant C we multiply (27) and (29) and obtain, on referring to (8),

$$\pi i H_1^{(1)}[(k^2-\zeta^2)^{\frac{1}{2}}a]J_1[(k^2-\zeta^2)^{\frac{1}{2}}a]$$

$$=C^2((k^2-\zeta^2)a^2)^{\frac{1}{2}}F_+(\zeta)F_+(-\zeta)\exp[ia(k^2-\zeta^2)^{\frac{1}{2}}]\prod_{n=1}^{\infty}\left(1-\frac{k^2-\zeta^2}{\gamma_n^2}\right).\quad \text{(V. 31)}$$

Introducing the product representation

$$J_1(za)=\frac{za}{2}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\gamma_n^2}\right)$$

in (31), it follows that

$$C^2F_+(\zeta)F_+(-\zeta)=\frac{\pi i}{2}((k^2-\zeta^2)a^2)^{\frac{1}{2}}H_1^{(1)}([k^2-\zeta^2]^{\frac{1}{2}}a)\exp[-ia(k^2-\zeta^2)^{\frac{1}{2}}].\quad \text{(V. 32)}$$

We next examine the form of this equation as $|\zeta|\to\infty$ within the strip $|\eta|<\epsilon$. Employing the asymptotic expressions (A. 10) and

$$F_+(\pm\zeta)\sim\exp\left[\pm\frac{1}{2\pi i}\frac{1}{\zeta a}\int_0^{\infty}\log\left\{\frac{e^{i\pi/2}K_1(x)+\pi I_1(x)}{k_1(x)}e^{-2x}\right\}\frac{xdx}{(x^2+(ka)^2)^{\frac{1}{2}}}\right]\to 1,\quad |\zeta|\to\infty$$

we find

$$C^2=(\tfrac{1}{2}\pi)^{\frac{1}{2}}e^{-i\pi/4}.\quad \text{(V. 33)}$$

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[13] The condition for uniform convergence is stated in E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Teddington, 1944), p. 49.

Consequently,

$$L_+(k\cos\vartheta)=\left\{\frac{\pi ka}{2}(1+\cos\vartheta)\right\}^{\frac{1}{2}}\left\{\frac{2J_1(ka\sin\vartheta)}{ka\sin\vartheta}\right\}^{\frac{1}{2}}\exp\left[-\frac{i\pi}{8}+\frac{ika\cos\vartheta}{\pi}\left(\log\frac{2\pi i}{\gamma ka}+1\right)+i\vartheta\frac{ka\sin\vartheta}{\pi}\right.$$
$$-i\sum_{n=1}^{\infty}\left(\sin^{-1}\frac{k\cos\vartheta}{(\gamma_n^2-k^2\sin^2\vartheta)^{\frac{1}{2}}}-\frac{ka\cos\vartheta}{n\pi}\right)$$
$$\left.+\frac{1}{2\pi i}\int_0^{\infty}\frac{\log\left\{\dfrac{e^{i\pi/2}K_1(x)+\pi I_1(x)}{K_1(x)}e^{-2x}\right\}}{(x^2+(ka)^2)^{\frac{1}{2}}+ka\cos\vartheta}\frac{xdx}{(x^2+(ka)^2)^{\frac{1}{2}}}\right]. \quad \text{(V. 34)}$$

Equation (32) allows a simple determination of the behavior of $F_+(\zeta)$ near the point $\zeta=-k$. On approximating the Hankel function of small argument by means of (A. 1)–(A. 3) and using (33), it follows that

$$F_+(\zeta)\sim\left(\frac{2}{\pi}\right)^{\frac{1}{2}}e^{i\pi/4}\frac{(2ka)^{-\frac{1}{2}}((k+\zeta)a)^{-\frac{1}{2}}}{F_+(k)},\quad \zeta\to-k. \quad \text{(V. 35)}$$

Combining (35) with (27) and (28), we find

$$L_+(k)=\frac{1}{L_+(-k)}=(\pi ka)^{\frac{1}{2}}\exp\left[-\frac{i\pi}{8}+\frac{ika}{\pi}\left(\log\frac{2\pi i}{\gamma ka}+1\right)-i\sum_{n=1}^{\infty}\left(\sin^{-1}\frac{k}{\gamma_n}-\frac{ka}{n\pi}\right)\right.$$
$$\left.+\frac{1}{2\pi i}\int_0^{\infty}\frac{\log\left\{\dfrac{e^{i\pi/2}K_1(x)+\pi I_1(x)}{K_1(x)}e^{-2x}\right\}}{(x^2+(ka)^2)^{\frac{1}{2}}+ka}\frac{xdx}{(x^2+(ka)^2)^{\frac{1}{2}}}\right] \quad \text{(V. 36)}$$

Further, on inserting the value of C in (32), and setting $\zeta=0$, it follows that

$$\int_0^{\infty}\log\left\{\frac{e^{i\pi/2}K_1(x)+\pi I_1(x)}{K_1(x)}e^{-2x}\right\}\frac{xdx}{x^2+(ka)^2}=\pi i\log\left\{\left(\frac{\pi ka}{2}\right)^{\frac{1}{2}}\exp[i(\tfrac{3}{4}\pi-ka)]H_1^{(1)}(ka)\right\}; \quad \text{(V. 37)}$$

finally, on extracting the real part of this integral, and proceeding to the limit $ka\to0$, we find

$$\int_0^{\infty}\log\left\{\left[1+\left(\frac{\pi I_1(x)}{K_1(x)}\right)^2\right]^{\frac{1}{2}}e^{-2x}\right\}\frac{dx}{x}$$
$$=\lim_{ka\to0}Re\left[\pi i\log\left\{\left(\frac{\pi ka}{2}\right)^{\frac{1}{2}}\exp[i(\tfrac{3}{4}\pi-ka)]H_1^{(1)}(ka)\right\}\right]=-\pi^2/4. \quad \text{(V. 38)}$$

Having completed the decomposition of $L(\zeta)$, we return to the transform Eq. (9). In order to determine the nature of the integral function defined by this equation, we examine the form of both members as $|\zeta|\to\infty$ in their respective half-planes of regularity.

According to (3), the order of magnitude of $H(\zeta)$ in the upper half-plane is determined by the behavior of $H(z)$ for small negative values of z. The velocity potentials on inner and outer surfaces of the pipe are continuous functions of z which assume equal values on the surface $\rho=a$, $z>0$. Therefore, $H(z)$ is a continuous function of z which vanishes for positive z, and we may write

$$H(z)\sim z^{\alpha},\quad \alpha>0,\quad z\to0^-. \quad \text{(V. 39)}$$

We thus deduce from (3)

$$H(\zeta)\sim(-i\zeta)^{-\alpha-1}, \quad |\zeta|\to\infty, \quad Im\ \zeta>0. \tag{V. 40}$$

Similarly, the order of magnitude of $E(\zeta)$ in the lower half-plane is determined by the behavior of $E(z)$ for small positive values of z. $E(z)$ has an integrable singularity at $z=0$ (being related to the component of the particle velocity along the cylindrical radius, at the periphery of the pipe), and we write

$$E(z)\sim z^{-\beta}, \quad \beta<1, \quad z\to 0^+. \tag{V. 41}$$

We thus deduce from (6),

$$E(\zeta)\sim(i\zeta)^{\beta-1}, \quad |\zeta|\to\infty, \quad Im\ \zeta<0. \tag{V. 42}$$

The asymptotic forms of $L_+(\zeta)$, $L_-(\zeta)$ are given in Appendix B; combining these with (40) and (42), we find:

$$\begin{aligned}\text{integral function} &= (k^2-\zeta^2)H(\zeta)L_+(\zeta)\sim(-i\zeta)^{\frac{1}{2}-\alpha} && \text{upper half-plane}\\ &= 2E(\zeta)L_-(\zeta)\sim(i\zeta)^{\beta-\frac{1}{2}}, \quad |\zeta|\to\infty && \text{lower half-plane.}\end{aligned}$$

In view of the bounds on α, β it follows that the integral function cannot become infinite, in the upper and lower half-planes, as rapidly as the square root of ζ. Such an integral function is a polynomial of degree less than $\frac{1}{2}$, i.e., a constant. Consequently, the values of α, β are each $\frac{1}{2}$, and $H(\zeta)$, $E(\zeta)$ are specified by the known functions $L_+(\zeta)$, $L_-(\zeta)$ in accordance with the equations

$$H(\zeta)=C/[(k^2-\zeta^2)L_+(\zeta)], \quad E(\zeta)=C/[2L_-(\zeta)], \tag{V. 43}$$

in which C is a constant.

VI. EVALUATION OF PHYSICAL QUANTITIES

It is a simple matter now to evaluate the physical quantities by means of the formulation developed in Section III.

To obtain the velocity potential reflection coefficient, we insert the values (compare (V. 43))

$$\mathrm{Res}_{\zeta=\pm k}H(\zeta)=\mp C/2kL_+(\pm k)$$

in (III. 15), and find

$$R=-L_+(k)/L_+(-k). \tag{VI. 1}$$

Employing (V. 16), it follows that

$$R=-|R|e^{2ikl}=-(L_+(k))^2, \tag{VI. 2}$$

where

$$|R|=|L_+(k)|^2=\exp\left\{-\frac{2ka}{\pi}\int_0^{ka}\frac{\tan^{-1}(-J_1(x)/N_1(x))}{x[(ka)^2-x^2]^{\frac{1}{2}}}dx,\right\} \tag{VI. 3}$$

and

$$\frac{l}{a}=\frac{1}{\pi}\int_0^{ka}\frac{\log\{\pi J_1(x)[(J_1(x))^2+(N_1(x))^2]^{\frac{1}{2}}\}}{x[(ka)^2-x^2]^{\frac{1}{2}}}dx+\frac{1}{\pi}\int_0^{\infty}\frac{\log\{1/[2I_1(x)K_1(x)]\}}{x(x^2+(ka)^2)^{\frac{1}{2}}}dx. \tag{VI. 4}$$

Alternatively, using (V. 36), (V. 38), we find

$$|R|=(\pi ka)^{\frac{1}{2}}\exp\left[-ka+\frac{1}{\pi}\int_0^{\infty}\tan^{-1}\left(\frac{K_1(x)}{\pi I_1(x)}\right)\left(1-\frac{ka}{(x^2+(ka)^2)^{\frac{1}{2}}}\right)\frac{dx}{x}\right], \tag{VI. 5}$$

and

$$\frac{l}{a}=\frac{1}{\pi}\left(\log\frac{2\pi}{\gamma ka}+1\right)-\frac{1}{ka}\sum_{n=1}^{\infty}\left(\sin^{-1}\frac{k}{\gamma_n}-\frac{ka}{n\pi}\right)+\frac{1}{2\pi}\int_0^{\infty}\log\left\{\left[1+\left(\frac{\pi I_1(x)}{K_1(x)}\right)^2\right]^{\frac{1}{2}}e^{-2x}\right\}\frac{dx}{x(x^2+(ka)^2)^{\frac{1}{2}}}. \tag{VI. 6}$$

To obtain the power-gain function, we insert the expression

$$H(k\cos\vartheta)=C/[(k\sin\vartheta)^2L_+(k\cos\vartheta)]$$

in (III. 25), and find, with reference to (1), (3),

$$\mathcal{G}(\vartheta)=\left(\frac{2J_1(ka\sin\vartheta)}{\sin\vartheta}\right)^2\frac{|R|}{1-|R|^2}\frac{1}{|L_+(k\cos\vartheta)|^2}. \tag{VI. 7}$$

Employing (V. 15), it follows that

$$\mathcal{G}(\vartheta)=\frac{4}{\pi\sin^2\vartheta}\frac{J_1(ka\sin\vartheta)}{[(J_1(ka\sin\vartheta))^2+(N_1(ka\sin\vartheta))^2]^{\frac{1}{2}}}\frac{|R|}{1-|R|^2}$$

$$\times\exp\left[\frac{2ka\cos\vartheta}{\pi}P\int_0^{ka}\frac{x\tan^{-1}(-J_1(x)/N_1(x))dx}{[x^2-(ka\sin\vartheta)^2][x^2+(ka)^2]^{\frac{1}{2}}}\right]. \tag{VI. 8}$$

From (3) and (8) we may readily verify the simple forms of the gain function appropriate to the directions $\vartheta=0,\ \pi/2,\ \pi$, as given in Section I.

Alternatively, using (V. 34) in conjunction with (7), we find

$$\mathcal{G}(\vartheta)=2\left(\frac{2ka}{\pi}\right)^{\frac{1}{2}}\frac{J_1(ka\sin\vartheta)}{\sin\vartheta(1+\cos\vartheta)^{\frac{1}{2}}}\frac{|R|}{1-|R|^2}$$

$$\times\exp\left[ka\cos\vartheta-\frac{1}{\pi}\int_0^\infty\frac{\tan^{-1}(K_1(x)/\pi I_1(x))}{(x^2+(ka)^2)^{\frac{1}{2}}+ka\cos\vartheta}\frac{xdx}{(x^2+(ka)^2)^{\frac{1}{2}}}\right]. \tag{VI. 9}$$

The identical simplification of (9), for the special directions considered above, may be confirmed with the use of (V. 37) and (5).

It remains to be shown that the gain function is correctly normalized. According to the definition (III. 20), this implies that the result of integrating (8) or (9) over the complete solid angle subtended at the mouth of the pipe is 4π. Since the gain function is symmetric about the axis of the pipe, this condition may be stated as

$$\int_0^\pi\mathcal{G}(\vartheta)\sin\vartheta d\vartheta=2. \tag{VI. 10}$$

We now verify (10) for the first form of the gain function. Let us consider the integral

$$I=\int_C\exp\left[-\frac{1}{\pi}\int_{-1}^1\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\frac{1}{2}(z+(1/z))}\right]\frac{dz}{z^2-1}, \tag{VI. 11}$$

extended over the contour shown in Fig. 9. If z lies on the arc of the unit circle, the t integration contour is indented above the pole $t=\frac{1}{2}(z+1/z)=\cos\vartheta$. This pole disappears if z is located on the section of the real axis, $|z|<1$, since $|\frac{1}{2}(z+(1/z))|>1$ and thus lies outside the range of the t integral. Furthermore, the integrand of (11) is analytic on the indentations at $z=\pm1$ (of radius δ) and everywhere within the contour since $Im\ t=0$, and $Im\ \frac{1}{2}(z+1/z)=-\frac{1}{2}(1/|z|-|z|)\sin\vartheta<0$.

Thus the integral I vanishes, and we deduce that the result of integrating along the arc of the unit circle and the section of the real axis is $\pi i/2$ (sum of residues of the integrand at the poles

$z=\pm 1$). Using (3), we find

$$\int_{\delta}^{\pi-\delta} \exp\left[-\frac{1}{\pi}\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\cos\vartheta}\right]\frac{d\vartheta}{2\sin\vartheta}$$
$$+\int_{-1+\delta}^{1-\delta} \exp\left[-\frac{1}{\pi}\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\frac{1}{2}(x+1/x)}\right]\frac{dx}{x^2-1}$$
$$=\frac{\pi i}{2}\frac{1}{2}\left(\exp\left[-\frac{1}{\pi}\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-1}\right]\right.$$
$$\left.-\exp\left[-\frac{1}{\pi}\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t+1}\right]\right)=\frac{\pi i}{4}\left(\frac{1}{|R|}-|R|\right). \quad \text{(VI. 12)}$$

Since

$$Im \exp\left\{i\tan^{-1}\left(-\frac{J_1(ka\sin\vartheta)}{N_1(ka\sin\vartheta)}\right)\right\}=\frac{J_1(ka\sin\vartheta)}{[(J_1(ka\sin\vartheta))^2+(N_1(ka\sin\vartheta))^2]^{\frac{1}{2}}},$$

it follows that

$$Im \exp\left\{-\frac{1}{\pi}\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\cos\vartheta}\right\}$$
$$=\frac{J_1(ka\sin\vartheta)\exp\left\{-\frac{1}{\pi}P\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\cos\vartheta}\right\}}{[(J_1(ka\sin\vartheta))^2+(N_1(ka\sin\vartheta))^2]^{\frac{1}{2}}}. \quad \text{(VI. 13)}$$

The contour of integration in (13) is indented above the point $t=\cos\vartheta$, and P designates the principal value.

On extracting the imaginary part of (12), we find by referring to (8) and (13) and passing to the limit $\delta\to 0$,

$$\int_0^{\pi}\frac{J_1(ka\sin\vartheta)}{[(J_1(ka\sin\vartheta))^2+(N_1(ka\sin\vartheta))^2]^{\frac{1}{2}}}\exp\left\{-\frac{1}{\pi}P\int_{-1}^{1}\tan^{-1}\left\{-\frac{J_1[(1-t^2)^{\frac{1}{2}}ka]}{N_1[(1-t^2)^{\frac{1}{2}}ka]}\right\}\frac{dt}{t-\cos\vartheta}\right\}$$
$$=\frac{\pi}{8}\frac{1-|R|^2}{|R|}\int_0^{\pi}\mathcal{G}(\vartheta)\sin\vartheta d\vartheta=\frac{\pi}{4}\frac{1-|R|^2}{|R|},$$

or,

$$\int_0^{\pi}\mathcal{G}(\vartheta)\sin\vartheta d\vartheta=2.$$

VII. APPROXIMATION FORMULAS AND METHODS

The exact formulas for the magnitude of the reflection coefficient admit simple approximations when suitable restrictions are imposed on the values of ka.

Equation (VI. 3) provides a convenient basis for approximation in the range $ka<1$; this range corresponds to low frequency sound vibrations or wave-lengths large compared to the diameter of the pipe. On expanding the arc tangent in ascending powers of the argument, and employing the corresponding series representations of the Bessel and Neumann functions, (A. 1), (A. 2), we find

$$|R|\doteq\exp[-(ka)^2/2]\left(1+\frac{1}{6}(ka)^4\left[\log\frac{1}{\gamma ka}+\frac{19}{12}\right]\right),\quad ka<1. \quad \text{(VII. 1)}$$

At $ka=1$, the value of $|R|$ computed from (1) is in excess of the correct value by less than 3 percent; the deviations decrease with smaller values of ka.

An approximation for l/a in this range is more difficult to obtain; Fig. 2 indicates, however, that here the deviations from the static value ($l/a=0.6133$, $ka=0$) are small.

For values of ka greater than unity, we use (VI. 5) as the basis for approximation. The result of expanding the integrand in inverse powers of ka is

$$\int_0^\infty \tan^{-1}\left(\frac{K_1(x)}{\pi I_1(x)}\right)\left(1-\frac{ka}{(x^2+(ka)^2)^{\frac{1}{2}}}\right)\frac{dx}{x}$$
$$=\frac{1}{2(ka)^2}\int_0^\infty x\tan^{-1}\left(\frac{K_1(x)}{\pi I_1(x)}\right)dx+O\left(\frac{1}{(ka)^4}\right);$$

thus, on retaining only the first term of the expansion, and noting from (V. 37) that

$$\int_0^\infty x\tan^{-1}\left(\frac{K_1(x)}{\pi I_1(x)}\right)dx=\pi(ka)^2$$
$$\times\lim_{ka\to\infty}\left\{\left[\frac{\pi ka}{2}\{(J_1(ka))^2+(N_1(ka))^2\}\right]^{\frac{1}{2}}\right\}=\frac{3\pi}{16},$$

we find

$$|R|\doteq(\pi ka)^{\frac{1}{2}}e^{-ka}\left(1+\frac{3}{32}\frac{1}{(ka)^2}\right),\quad ka>1.\quad\text{(VII. 2)}$$

At $ka=1$, Eq. (2) yields a value smaller than the correct one by about 3 percent; the deviations are less than 1 percent for $ka>2$.

In consequence of the logarithmic singularity of the first integrand in (VI. 4) at the zeros of the Bessel function, the values of l/a abruptly decrease near $ka=3.832$.

It is of interest to compare the results of a rigorous formulation of the problem with those obtained by approximation methods; the comparison serves to determine the applicability of these methods in problems of a related nature, whose exact solution is difficult to obtain (e.g., for cylindrical pipes with different cross sections).

Approximation methods for the calculation of the reflection coefficient are not easily devised; however, it is possible to account in a simple manner for the directional properties of the radiation field. In the latter connection, we proceed via an assumed expression for the discontinuity of the velocity potential at the surface of the pipe. Representing this quantity as a combination of incident and reflected dominant mode waves,

$$H(z)=e^{ikz}+Re^{-ikz},\quad z\leqslant 0,\qquad =0,\quad z>0,\quad\text{(VII. 3)}$$

where R is the reflection coefficient, it follows that

$$H(\zeta)=i\left(\frac{1}{\zeta-k}+\frac{R}{\zeta+k}\right).\quad\text{(VII. 4)}$$

Substituting from (4) in (III. 24), we find, after inserting a normalization factor,

$$\mathcal{G}(\vartheta)=\frac{1}{N}\left(\frac{J_1(ka\sin\vartheta)}{\sin\vartheta}\right)^2\frac{1}{1-|R|^2}[(1+\cos\vartheta)^2$$
$$-2\sin^2\vartheta Re\,R+(1-\cos^2\vartheta)|R|^2].\quad\text{(VII. 5)}$$

We note that (5) has the correct functional dependence on $|R|$, when evaluated in the directions $\vartheta=0,\pi$. On imposing the normalization condition (VI. 10), we find

$$N=\frac{1+|R|^2}{1-|R|^2}-\frac{1}{2ka}\frac{|1+R|^2}{1-|R|^2}\int_0^{2ka}J_0(t)dt$$
$$+\frac{2}{ka}\frac{Re\,R}{1-|R|^2}J_1(2ka).\quad\text{(VII. 6)}$$

If the reflected wave is omitted in (3), we obtain, instead of (5),

$$\mathcal{G}(\vartheta)=\frac{1}{N}\left(\frac{1+\cos\vartheta}{\sin\vartheta}J_1(ka\sin\vartheta)\right)^2,\quad\text{(VII. 7)}$$

where the normalization factor is given by

$$N=1-\frac{1}{2ka}\int_0^{2ka}J_0(t)dt,$$
$$\doteq\tfrac{1}{3}(ka)^2,\quad ka\ll 1,\qquad\text{(VII. 8)}$$
$$\doteq 1-1/2ka,\quad ka\gg 1.$$

Equation (7) corresponds to the result of the Kirchoff approximation, in which the radiation field is calculated from the incident field and its normal derivatives at the mouth of the pipe. The Kirchoff approximation is least accurate at low frequencies, predicting a non-isotropic gain function in the limit $ka \to 0$ (compare Fig. 3) which decreases from a value 3 in the forward direction to 0 in the backward direction.

A considerable improvement on the Kirchoff result is achieved by inserting the correct values of the reflection coefficient in (5) and (6). The accuracy of this modified Kirchoff formula is shown by the comparison curves of Fig. 10, calculated for the value $ka=1$. This feature is in accord with the fact that the radiation field is derived principally from the surface discontinuity of the velocity potential on a section of the pipe, terminating at the mouth, whose linear dimension is comparable to the wave-length. At low frequencies, the length of this section is large compared to the transverse dimension of the pipe, and the radiation field can be accurately derived from the asymptotic form of the velocity potential within the pipe.

Rigorous solutions have also been obtained for the problem of electromagnetic radiation from a semi-infinite circular wave guide. We expect to publish the details shortly.

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APPENDIX A

Summary of Formulas Involving First-Order Cylinder Functions

$$J_1(z)=\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+1}}{m!(m+1)!}, \qquad \text{(A. 1)}$$

$$N_1(z)=\frac{2}{\pi}\log(z/2)\cdot J_1(z)-\frac{2}{\pi z}-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m(z/2)^{2m+1}}{m!(m+1)!}\times(\psi(m+1)+\psi(m+2)),$$

$$\psi(m+1)=\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}-\log\gamma,$$

$$\log\gamma=0.5772, \qquad \text{(A. 2)}$$

$$H_1^{(1)}(z)=J_1(z)+iN_1(z), \qquad \text{(A. 3)}$$

$$J_1(z)=-dJ_0(z)/dz, \qquad \text{(A. 4)}$$

$$H_1^{(1)}(z)=-dH_0^{(1)}(z)/dz, \qquad \text{(A. 5)}$$

$$H_1^{(1)}(iz)=-2K_1(z)/\pi, \qquad \text{(A. 6)}$$

$$J_1(iz)=iI_1(z), \qquad \text{(A. 7)}$$

$$H_1^{(1)}(e^{-i\pi/2}z)=2K_1(z)/\pi+2e^{-i\pi/2}I_1(z), \qquad \text{(A. 8)}$$

$$J_1(e^{-i\pi/2}z)=-e^{i\pi/2}I_1(z). \qquad \text{(A. 9)}$$

The remaining formulas are asymptotic expansions valid when $|z|$ is large compared to unity.

$$H_1^{(1)}(z)\sim(2/\pi z)^{\frac{1}{2}}e^{i(z-3\pi/4)}\left(1-\frac{3}{8iz}+\cdots\right),$$

$$-\pi<\arg z<2\pi, \qquad \text{(A. 10)}$$

$$H_1^{(1)}(z)J_1(z)\sim\frac{1}{\pi z}[1+e^{2i(z-3\pi/4)}]\left(1-\frac{3}{8iz}+\cdots\right)-\frac{i}{\pi z}[1-e^{2i(z-3\pi/4)}]\left(\frac{3}{8z}-\cdots\right),$$

$$|\arg z|<\pi \qquad \text{(A. 11)}$$

$$I_1(z)K_1(z)\sim\frac{1}{2z}\left(1-\frac{15}{64z^2}+\cdots\right)+\frac{e^{-2z+3\pi i/2}}{2z}\left(1+\frac{3}{4z}-\cdots\right),$$

$$-\pi/2<\arg z<3\pi/2, \qquad \text{(A. 12)}$$

$$\frac{I_1(z)}{K_1(z)}\sim\frac{1}{\pi}e^{2z}\left(1-\frac{3}{4z}+\cdots\right)+\frac{1}{\pi}e^{-2z+3\pi i/2},$$

$$-\pi/2<\arg z<3\pi/2. \qquad \text{(A. 13)}$$

APPENDIX B

For the determination of the integral function defined by the transform Eq. (V. 9), we require the behavior of $L_+(\zeta)$, $L_-(\zeta)$ when $|\zeta|\to\infty$, in the upper and lower half-planes, respectively. Referring to (V. 14), we note first that

$$\int_{-k}^{k}\frac{\log[\pi i H_1^{(1)}((k^2-t^2)^{\frac{1}{2}}a)J_1((k^2-t^2)^{\frac{1}{2}}a)]}{t-\zeta}dt\sim-\frac{2}{\zeta a}\int_0^{ka}\frac{x\log[\pi i H_1^{(1)}(x)J_1(x)]}{((ka)^2-x^2)^{\frac{1}{2}}}dx=O(1/\zeta),$$

$$|\zeta|\to\infty,\quad Im\ \zeta>0.$$

Furthermore, using (A. 12),

$$\int_k^\infty\frac{\log[2K_1((t^2-k^2)^{\frac{1}{2}}a)I_1((t^2-k^2)^{\frac{1}{2}}a)]}{t^2-\zeta^2}dt=\int_k^\infty\frac{\log(1/ta)}{t^2-\zeta^2}dt+\int_k^\infty\frac{\log[2K_1((t^2-k^2)^{\frac{1}{2}}a)I_1((t^2-k^2)^{\frac{1}{2}}a)ta]}{t^2-\zeta^2}dt$$

$$\sim\frac{1}{i\zeta}\int_{k/-i\zeta}^\infty\frac{\log(1/-i\zeta av)}{v^2+1}dv-\frac{1}{\zeta^2 a}\int_0^\infty\frac{\log[2K_1(x)I_1(x)(x^2+(ka)^2)^{\frac{1}{2}}]}{(x^2+(ka)^2)^{\frac{1}{2}}}xdx$$

$$\sim(\pi/2i\zeta)\log(-i\zeta)+O(1/\zeta),\quad |\zeta|\to\infty,\quad Im\ \zeta>0.$$

Thus,

$$L_+(\zeta)\sim(-i\zeta)^{-\frac{1}{2}},\quad |\zeta|\to\infty,\quad Im\ \zeta>0,\tag{B. 1}$$

and by (V. 14),

$$L_-(\zeta)=(1/L_+(-\zeta))\sim(i\zeta)^{\frac{1}{2}},\quad |\zeta|\to\infty,\quad Im\ \zeta<0.\tag{B. 2}$$