Theory of High Frequency Gas Discharges. III. High Frequency Breakdown[1]

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(Received August 18, 1947)

In search of a definition of "breakdown" of a gas under microwave fields, the density of electrons is investigated as a function of the field strength under the following simplifying assumptions. Ionization occurs as result of single impacts between gas atoms and sufficiently fast electrons. The gas has an infinite volume, and negative ions are not formed, so that the only mechanism for electron removal is recombination with positive ions. The calculations show that, at a certain field strength, the density of electrons rises sharply. The simple model, therefore, leads to a breakdown phenomenon, at field strengths not far from observed values. Gases treated are helium and neon under frequencies of 3000 megacycles.

I

AS was observed in article I[3] of this series, a simple theory of breakdown, concerned with the motion of a free electron in an alternating field such that the electron may reach ionizing velocities after acceleration through a half-period of the field, fails since it predicts values of the field strength which are many times too high. The alternative to this procedure is to construct a theoretical discharge characteristic, that is, a plot of current against field strength for given values of the gas pressure and field frequency. If such a curve exhibits breakdown it will appear as a sharp rise in the current for field strengths above a more or less definite value.

The current density may be written

$$j=ne\langle v_x\rangle_{Av},$$

where n is the electron density and the field is taken to have the direction of the x axis. Since no sudden change in the value of $\langle v_x\rangle_{Av}$ is to be expected as breakdown is reached, it will be assumed that the phenomenon of breakdown may be investigated adequately by studying the dependence of n upon the field strength. In particular n/N, where N is the gas particle density, will be plotted against the field strength E.

We shall make several assumptions which serve to restrict the physical situation being discussed. First, it will be considered that the field is uniform and that the discharge vessel is effectively infinite in extent. Hence the effects of electron diffusion to the boundaries of the discharge vessel and secondary processes occurring at these boundaries will be neglected. Second, it will be assumed that the gas molecules do not form negative ions. Specifically, pure helium and neon will be considered. Hence electrons can be produced by ionizing electron collisions with gas atoms as well as by cosmic radiation and radioactive background and can be removed by volume recombination of electrons and positive ions. Third, we shall assume that the electron distribution function may be written in the form

$$f=f_0{}^0+\frac{v_x}{v}[f_1{}^1\cos\omega t+g_1{}^1\sin\omega t]. \qquad (1)$$

This implies that the gas is in a steady state (constant electron density) and, in view of the conclusions of article II[4], that the field frequency lies in the microwave region.

Actually, the discharge vessel is finite and diffusion plays an important role. Information on the capture cross sections of positive ions seems scant, but these cross sections are often believed to be small. Indeed most investigators in the field of gas discharges would be disposed to ignore recombination in the presence of diffusion. We have deliberately considered here the case of an infinite gas, thus ignoring diffusion,

[1] Part of a dissertation presented to the Faculty of the Graduate School of Yale University in candidacy for the degree of Doctor of Philosophy. This work has been assisted by the Office of Naval Research, United States Navy, under Contract N6ori-44.

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[3] H. Margenau, Phys. Rev. **73**, 297 (1948).

[4] H. Margenau and L. M. Hartman, Phys. Rev. **73**, 309 (1948).

mainly because this model is easiest to deal with. Initial expectations are that, if anything like breakdown were to emerge at all, it would occur at field strengths quite different from those observed experimentally. Instead of that, however, it was found that the model chosen agreed much better with what is known than could be anticipated. It does produce a sharp rise in electron concentration at a certain field strength, and this field strength, for a great variety of recombination cross sections, is only a few volts per cm below that observed. It seems, therefore, that the present approach does not miss the central issue, although it cannot prove the absence of diffusion. We hope to include this mechanism in subsequent calculations.

Let θ denote the rate at which electrons are being produced per unit volume by external agencies such as cosmic radiation. This is of the order of 10 electrons per cubic centimeter per second at atmospheric pressure. Furthermore, let the electronic mean free paths for ionization and recombination be denoted, respectively, by λ_i and λ_r. Then, normalizing the distribution function to unity, we have the following equation to describe the steady state:

$$\frac{\partial n}{\partial t}=\theta+n\int_0^\infty \frac{v}{\lambda_i} f_0{}^0\cdot 4\pi v^2 dv - n\int_0^\infty \frac{v}{\lambda_r} f_0{}^0\cdot 4\pi v^2 dv=0. \quad (2)$$

The ionization cross section will be assumed to be a linear function of the excess of the electronic energy over that necessary for ionization:

$$q_i(u)=\begin{cases} K(u-u_i) & u\geqslant u_i \\ 0 & u\leqslant u_i \end{cases} \quad (3)$$

where K is a constant and

$$u\equiv v^2, \quad eV_i\equiv \tfrac{1}{2}mu_i,$$

V_i being the ionization potential of the gas. Multiple ionization will be neglected. Hence, since the electron density and the positive ion density are equal, λ_i is defined by:

$$(N-n)q_i\lambda_i=1. \quad (4)$$

The recombination cross section has been investigated by Wessel and others[5] and has been shown to have the form:

$$q_r(u)=\frac{R}{u}, \quad (5)$$

where R is a constant. The mean free path is defined by:

$$nq_r\lambda_r=1. \quad (6)$$

Finally it is convenient to introduce a new variable δ defined by:

$$u\equiv \delta u_i.$$

Let

$$I_1\equiv\int_1^\infty f_0{}^0\cdot\delta(\delta-1)d\delta, \quad I_2\equiv\int_0^\infty f_0{}^0 d\delta.$$

Equation (2) may now be written:

$$\theta+2\pi n(N-n)Ku_i{}^3I_1-2\pi n^2Ru_iI_2=0. \quad (7)$$

This equation may be solved for n/N. We obtain:

$$n/N=\frac{I_1\left\{1+\left[1+\frac{2\theta}{\pi N^2}\left(\frac{Ku_i{}^3I_1+Ru_iI_2}{(Ku_i{}^3I_1)^2}\right)\right]^{\frac{1}{2}}\right\}}{2\left\{I_1+\frac{R}{Ku_i{}^2}I_2\right\}}. \quad (8)$$

For fields that are sufficiently strong θ may be neglected and (8) reduces to:

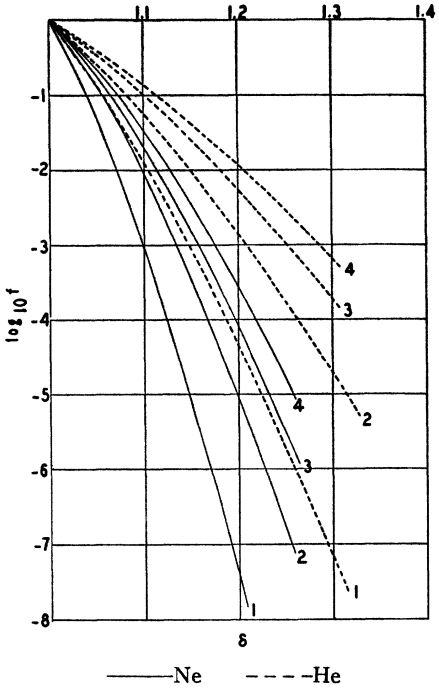
$$n/N=\frac{1}{1+\left(\frac{R}{Ku_i{}^2}\right)\left(\frac{I_2}{I_1}\right)}. \quad (8a)$$

In this form the electron density does not depend upon the normalization but only upon the shape of the distribution function. For small values of the field strength the integral I_1 may be neglected and (8) reduces to:

$$n/N=\left(\frac{\theta}{2\pi N^2Ru_iI_2}\right)^{\frac{1}{2}}. \quad (8b)$$

By means of these equations the entire discharge characteristic may be plotted. The problem, therefore, is one of determining the function $f_0{}^0$ and the integrals I_1 and I_2 for different values of the discharge parameters.

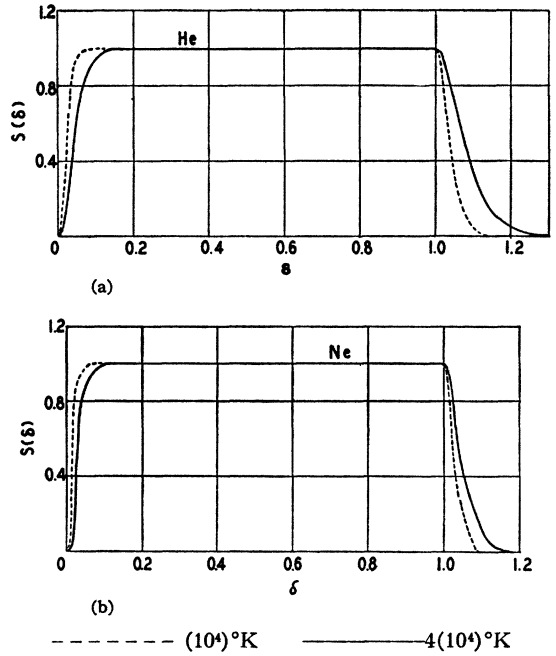
[5] W. Wessel, Ann. d. Physik 5, 611 (1930); E. C. G. Stueckelberg and P. M. Morse, Phys. Rev. 36, 16 (1930).



———Ne – – – –He

FIG. 1. Logarithm of distribution function *vs.* energy of electrons.

1. $(10^4)°K$
2. $2(10^4)$
3. $3(10^4)$
4. $4(10^4)$



– – – – – – – $(10^4)°K$ ———————$4(10^4)°K$

FIG. 2. Rate of increase of electrons having velocities below δ *vs.* δ.
Normalized to unity at $\delta = 1.0$

II

It has been shown in the first two articles of this series[3,4] that, if the distribution function is written in the form of Eq. (1), f_0^0 must satisfy a differential equation having the form

$$\frac{\gamma}{6v^2}\frac{d}{dv}\left[\frac{(\gamma\lambda)v^3}{v^2+(\omega\lambda)^2}\frac{df}{dv}\right]+\frac{\delta f}{\delta t}=0, \tag{9}$$

where

$$\gamma\equiv\frac{eE}{m};$$

$$\frac{\delta f}{\delta t}=\frac{m}{Mv^2}\frac{d}{dv}\left(\frac{v^4}{\lambda_e}f\right)+\frac{kT}{Mv^2}\frac{d}{dv}\left(\frac{v^3}{\lambda_e}\frac{df}{dv}\right)+\frac{1}{4\pi v^2}\frac{dS}{dv}. \tag{10}$$

(Here and in what follows, we write f in place of f_0^0.) All non-elastic collisions are accounted for by the proper choice of the function S. The large majority of collisions occurring in a discharge at a given electronic energy will be elastic, since the magnitude of the distribution function will fall off rapidly in the energy range immediately above the energy necessary for ionization of the molecules and even in this region the cross sections for ionization and excitation are small as compared with the total collision cross section. We shall assume, therefore, as an approximation in the following that

$$\lambda \doteqdot \lambda_e.$$

Introducing two parameters defined by

$$u_1\equiv(\omega\lambda);\quad u_2\equiv(2\gamma\lambda),$$

we may integrate (9) and obtain

$$f=\left(\frac{3M}{\pi}\right)F\int_u^\infty\frac{(u+u_1)\lambda S du}{u^2F[Mu_2^2+24(u+u_1)kT]}, \tag{11}$$

where

$$F=\exp\left\{-\int\frac{\frac{m}{2}du}{kT+\frac{Mu_2^2}{24(u+u_1)}}\right\}. \tag{12}$$

$F(u)$, therefore, is the form that the isotropic part of the distribution function assumes when only elastic collisions are taken into account.

A number of special cases are of interest.

Since a differential equation of the same form as (9) is obtained for a d.c. field, both the microwave and d.c. distributions may be handled by the present treatment.

(a) $u_1=0$

$$f=\left(\frac{3M}{2\pi}\right)F\int_u^\infty \frac{\lambda S du}{uF[Mu_2^2+12ukT]}, \quad (13)$$

where

$$F=\exp\left\{-\int\frac{\frac{m}{2}du}{kT+\frac{Mu_2^2}{12u}}\right\}.$$

$F(u)$ is the Davydov distribution.[6] Note that u_2 here refers to the r.m.s. field.

(b) $u_1=0$, $T=0$

$$f=\left(\frac{3}{2\pi}\right)F\int_u^\infty \frac{\lambda S du}{uu_2^2F}, \quad (14)$$

where

$$F=\exp\left\{-\frac{3mu^2}{Mu_2^2}\right\}.$$

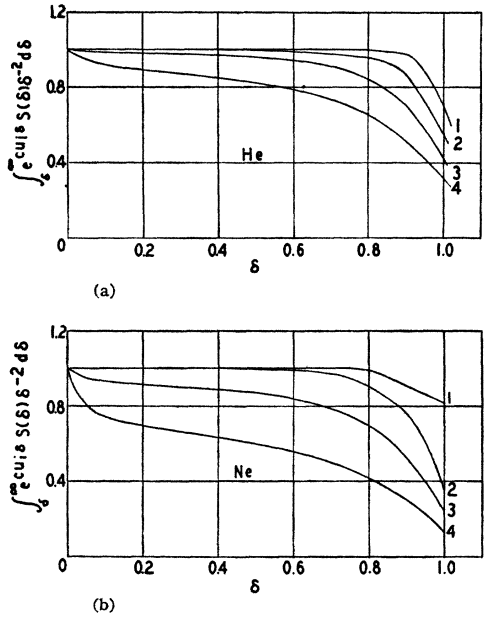


FIG. 3. $\int_u^\infty e^{cu}u^{-2}S(u)du$ vs. $\frac{u}{u_i}$.

Normalized to unity at $\delta=0$

1. $(10^4)°K$
2. $2(10^4)$
3. $3(10^4)$
4. $4(10^4)$

[6] B. Davydov, Physik. Zeits. Sowjetunion **8**, 59 (1935).

$F(u)$ is the Druyvesteyn distribution.[7] Again u_2 refers to the r.m.s. field. This result is given by Chapman and Cowling.[8]

(c) $T=0$

$$f=\left(\frac{3}{\pi}\right)F\int_u^\infty \frac{(u+u_1)\lambda S du}{u^2u_2^2F}, \quad (15)$$

where

$$F=\exp\left\{-\frac{6mu(u+2u_1)}{Mu_2^2}\right\}.$$

This result has already been presented in article I[3] of the present series.

(d) $u_1=0$, $u_2=0$

$$f=\left(\frac{M}{8\pi KT}\right)F\int_u^\infty \frac{\lambda S du}{u^2F}, \quad (16)$$

where

$$F=\exp\left\{-\frac{mu}{2kT}\right\}.$$

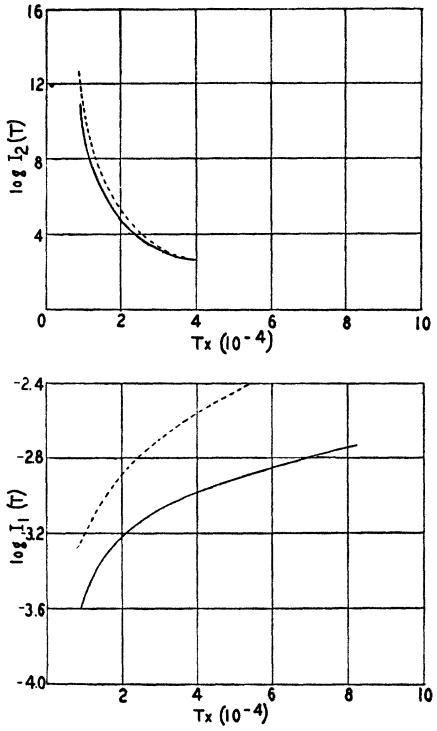


FIG. 4. Plot of log$I$ vs. T.

[7] M. J. Druyvesteyn, Physica **10**, 61 (1930).

[8] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1939), p. 354.

$F(u)$ is here a Maxwellian distribution corresponding to a temperature T. This, therefore, represents thermal excitation of the gas.

(e) $u_1 \gg u$. If this inequality holds over the entire range of u within which the magnitude of the distribution function is significant, it has been shown in article II[4,9] that $F(u)$ is Maxwellian at an effective "temperature" T' given by

$$T' = T\left[1+\frac{Mu_2^2}{24u_1kT}\right]. \tag{17}$$

If (17) is used to eliminate T from Eq. (11), the distribution function becomes:

$$f=\left(\frac{M}{8\pi kT'}\right)F\int_u^\infty \frac{\lambda S du}{u^2 F}. \tag{18}$$

This is of the same form as (16). Hence, it follows that insofar as the electron distribution function is concerned the effect of an alternating field of sufficiently high frequency is equivalent to that of raising the temperature of the gas and conversely. Thus a thermal "discharge" and a high frequency discharge can be studied simultaneously so long as T' is defined.

III

The results outlined in the preceding section can be given practical meaning only if the form of the function $S(u)$ is specified. From the discussion of this function in article I[3] it is clear that this specification will involve a choice as to the nature and number of inelastic processes which are deemed important for the problem in question. We shall consider two cases. The first, the excitation of a single, discrete state of the molecule, will permit us to develop a method of using the results of the preceding section to compute f. The second case will include ionization of the molecule and volume recombination of electrons and positive ions but will neglect the excitation of quantized states. We shall use the latter to compute the electron density.

Consider then that only a single type of inelastic collision occurs. Denote the cross section for this process by $q_s(u)$ and let the value of u corresponding to the excitation energy be denoted by u_s. Then from article I[3] we have:

$$S(u)=2\pi N\int_u^{u+u_s} f\cdot q_s\cdot u du. \tag{19}$$

If this is used in conjunction with Eq. (16) for thermal excitation or with any of the other integral representations of f, the finite upper limit of the integral in (19) leads to an integro-differential equation in f or a differentio-difference equation and no useful result is obtained. We note, however, that we may rewrite Eq. (19) as follows:

$$\begin{aligned} S(u)&=2\pi N\int_u^\infty f\cdot q_s\cdot u du \\ &\quad -2\pi N\sum_{j=1}^{\infty}\int_{u+ju_s}^{u+(j+1)u_s} f\cdot q_s\cdot u du \qquad (20)\\ &=2\pi N\int_u^\infty f\cdot q_s\cdot u du-\sum_{j=1}^{\infty}S(u+ju_s). \end{aligned}$$

The entire range of u has thus been subdivided into the finite ranges:

$$(0, u_s),\ (u_s, 2u_s),\ (2u_s, 3u_s),\ \cdots.$$

Since f goes to zero strongly as u increases, there will be some range of u within which f is so small that it may be assumed to vanish in the next higher range without causing appreciable error Hence

$$\begin{aligned} &(1)\ u\geqslant nu_s \quad f\doteqdot S\doteqdot 0\\ &(2)\ (n-1)u_s\leqslant u\leqslant nu_s\\ &\qquad\qquad S=2\pi N\int_u^\infty f\cdot q_s\cdot u du\\ &(3)\ (n-2)u_s\leqslant u\leqslant (n-1)u_s\\ &\qquad\qquad S=2\pi N\int_u^\infty f\cdot q_s\cdot u du-S(u+u_s) \qquad (21)\\ &\cdots\\ &(n+1)\ 0\leqslant u\leqslant u_s\\ &\qquad\qquad S=2\pi N\int_{u_s}^\infty f\cdot q_s\cdot u du-\sum_{j=1}^{n-1}S(u+ju_s). \end{aligned}$$

Expression (21.2) may now be combined with (11) or one of its reduced forms; $S(u)$ may be eliminated and the result is a second-order

[9] See also H. Margenau, Phys. Rev. 69, 508 (1946).

differential equation in f. The solution of this equation is the distribution function in the corresponding range of energy. Similarly (21.3) may then be combined with (11) to yield a differential equation in f in the next lower energy range since now $S(u+u_s)$ is known as a function of u. The solution is then fitted to the preceding solution, and so on. In this way the distribution function may be found over the entire range of u. A cumulative error can be avoided by using the solution corresponding to (21.2) to compute f in the energy range corresponding to (21.1) and then repeating the process until consistency has been achieved. This method of procedure would appear to be impractical were it not for the fortunate circumstance that under certain conditions only two steps are required. Thus, for example, in the case of the rare gases the distribution function may be considered to be negligibly small for energies greater than that corresponding to twice the excitation potential. When this is the case (21) may be reduced to:

$$\begin{aligned} &\text{(a) } u \geqslant u_s \qquad S = 2\pi N \int_u^\infty f \cdot q_s \cdot u\,du \\ &\text{(b) } 0 \leqslant u \leqslant u_s \quad S = S(u_s) - S(u+u_s). \end{aligned} \tag{22}$$

A similar procedure is possible when only ionization and volume recombination are assumed to occur. Again following article I[3] and making use of the neutral molecule and positive ion particle densities in (4) and (6) we have:

$$\begin{aligned} S(u) = 2 \cdot 2\pi(N-n) &\int_u^{2u+u_i} f \cdot q_i \cdot u\,du \\ &+ 2\pi(N-n) \int_0^u f \cdot q_i \cdot u\,du \\ &- 2\pi n \int_0^u f \cdot q_r \cdot u\,du. \end{aligned} \tag{23}$$

Furthermore, from the definition of a steady state we must also have:

$$\begin{aligned} S(\infty) = 2\pi(N-n) &\int_0^\infty f \cdot q_i \cdot u\,du \\ &- 2\pi n \int_0^\infty f \cdot q_r \cdot u\,du = 0. \end{aligned} \tag{24}$$

The cross section for recombination becomes negligibly small for energies of the order of that corresponding to the ionization potential or larger; the ionization cross section is zero for energies less than this value. In addition and in line with the foregoing procedure we shall assume that the distribution function may be neglected for energies greater than that corresponding to *three* times the ionization potential. Hence, making use of these approximations and the condition (24), we may easily show that:

(a) $u \geqslant u_i$

$$\begin{aligned} S(u) \doteqdot 2 \cdot 2\pi(N-n) &\int_u^\infty f \cdot q_i \cdot u\,du \\ &+ 2\pi(N-n) \int_{u_i}^u f \cdot q_i \cdot u\,du \\ &- 2\pi n \int_0^\infty f \cdot q_r \cdot u\,du \\ &= 2\pi(N-n) \int_u^\infty f \cdot q_i \cdot u\,du. \end{aligned} \tag{25}$$

(b) $0 \leqslant u \leqslant u_i$

$$\begin{aligned} S(u) = 2 \cdot 2\pi(N-n) &\int_{u_i}^{2u+u_i} f \cdot q_i \cdot u\,du \\ &- 2\pi n \int_0^u f \cdot q_r \cdot u\,du \\ = \{S(u_i) &- S(2u+u_i)\} \\ &+ \left\{ 2\pi n \int_u^\infty f \cdot q_r \cdot u\,du - S(2u+u_i) \right\}. \end{aligned}$$

The second expression in braces in (25b) vanishes at both ends of the interval: accurately at zero, approximately at u_i since both terms become negligibly small. Within the interval, as u increases, both terms decrease monotonically in magnitude. In a first approximation, therefore, this term will be neglected. To anticipate the results of the calculation, it will be found that when $n \ll N$, which is still true in the region of breakdown, this approximation is a good one.

IV

We are now in a position to find an expression for the distribution function. In the remainder

of this paper only thermal or quasi-thermal (high frequency) excitation will be discussed. Consider first the case of excitation of a single discrete level. The function $S(u)$ is given by (22). We shall assume a linear cross section of the form:

$$q_s(u)=\begin{cases}K(u-u_s) & u\geqslant u_s\\ 0 & u\leqslant u_s\end{cases} \tag{26}$$

$S(u)$ may be eliminated between (22a) and (16):

$$(f)''+\left[\frac{2}{u}+c\right](f)'+\left[\frac{\alpha}{u}-\beta\right](f)=0, \tag{27}$$

where

$$c\equiv\frac{m}{2kT};\quad \alpha\equiv 2c+\beta u_s;\quad \beta\equiv\frac{cKM}{2mq};\quad Nq\lambda=1$$

and the mean free path, λ, has been assumed for convenience to be a constant in the energy. The second term of (27) is removed by the substitution:

$$f=u^{-1}e^{-\frac{1}{2}cu}g. \tag{28}$$

We have finally:

$$g''+\left\{-\frac{1}{4}+\frac{k}{z}\right\}g=0,$$

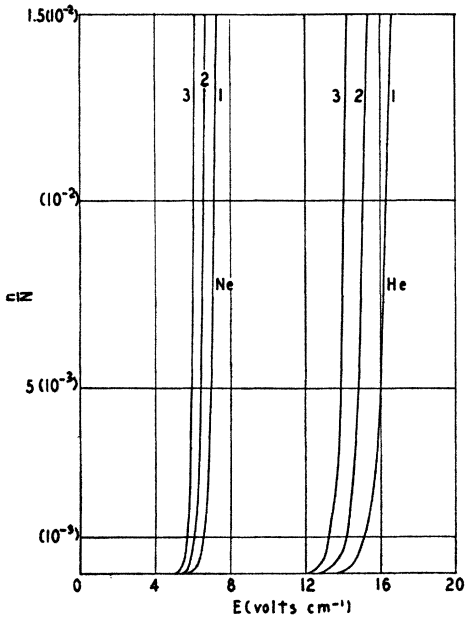


FIG. 5. Plot of n/N *vs.* field strength.

$p<1$-mm Hg
$\nu=3(10^9)$ sec.$^{-1}$

1. $R=10^{-6}$ cm^4 sec.$^{-2}$
2. $R=10^{-7}$
3. $R=10^{-8}$

where

$$z\equiv(c^2+4\beta)^{\frac{1}{2}}u;\quad k\equiv\frac{\alpha-c}{(c^2+4\beta)^{\frac{1}{2}}}. \tag{29}$$

This is a special form of the differential equation of the confluent hypergeometric function. Since $z>cu$ and $W_{-k,\frac{1}{2}}(-z)$ increases as $\exp(\frac{1}{2}z)$ for large values of z, we must choose as the solution consistent with the boundary condition that f must vanish at infinity:

$$g=AW_{k,\frac{1}{2}}(z). \tag{30}$$

Hence, by means of (16) and (22) the distribution function may be found over the entire range of u. There are four steps:

$$\begin{aligned}&\text{(a)}\quad u\geqslant u_s\quad f=Au^{-1}e^{-\frac{1}{2}cu}W_{k,\frac{1}{2}}(z),\\ &\text{(b)}\quad u\geqslant u_s\quad S=2\pi N\int_u^\infty f\cdot K_s(u-u_s)udu,\\ &\text{(c)}\quad u\leqslant u_s\quad S=S(u_s)-S(u+u_s),\\ &\text{(d)}\quad u\leqslant u_s\quad f=\left(\frac{M\lambda}{8\pi kT}\right)e^{-cu}\int_u^\infty\frac{e^{cu}Sdu}{u^2}.\end{aligned} \tag{31}$$

For the case of ionization and volume recombination (no excitation of discrete levels) we note that since the form of (25a) is identical with that of (22a) the same solution may be used as before, if a linear cross section in the form of (3) is adopted. Hence, the distribution function may again be computed over the entire range of u by means of four steps. We shall assume henceforth that $n\ll N$. Then Eqs. (31) remain valid for ionization provided we make the following simple changes:

u_s must be replaced by u_i; the constant K_s by the appropriate K_i, and Eq. (31c) reads

$$S=S(u_i)-S(2u+u_i). \tag{32}$$

The parameters of the Whittaker function are defined as in the first case. It is interesting to note that in first approximation the form of the distribution function does not depend upon any specific assumptions concerning the recombination cross section but only upon its existence in the equation of the steady state.

V

The distribution function has been calculated by means of (31) and (32) for helium and neon, since only for these two, among the rare gases, can the assumption of a constant mean free path

be justified even in approximation. The calculation was made for four temperatures: (10^4), $2(10^4)$, $3(10^4)$, $4(10^4)$°K. The results could then be interpreted in terms of the high frequency field by means of the relation, obtained from (17):

$$E=7.98(10^{-11})\nu\left[\frac{T'-T}{A}\right]^{\frac{1}{2}}. \tag{33}$$

Here E is measured in volts per centimeter, ν is the field frequency, A is the molecular weight of the gas molecule, and T is taken to be room temperature. The frequency of the field was assumed to be $3(10^9)$ sec.$^{-1}$ since measurements exist at this frequency. The treatment as one of "thermal" ionization is then valid for pressures below about 1-mm Hg, in accordance with the conditions attending the use of (17).

The slopes, K_i, of the ionization cross sections for helium and neon were measured from the published curves of Druyvesteyn and Penning.[10] The total collision cross sections were obtained by averaging over the collision probabilities as published by Brode.[11]

Direct evaluation of the Whittaker function in (31a) was found to be possible only for helium and at the temperature $4(10^4)$°K. This was due, first, to the fact that the large values of the parameter k required by the problem necessitated the evaluation of a very large number of terms of either the power series or asymptotic series expansions of the Whittaker function, and, second, to the circumstance that over the range of z in question we are near the zero of the coefficient of g in Eq. (29) and hence were forced to use an impracticably large number of significant figures. The matter was resolved by obtaining approximate, numerical solutions of Eq. (29) directly. Let z_0 denote the value of z corresponding to u_i. Then for $z\geqslant z_1>z_0$ the J.W.K.B. solution was found to be satisfactory. A suitable value of z_1 was that corresponding to $\delta=1.05$, where the variable δ may be defined, as in article I of this paper, by either of the relations:

$$u=\delta u_i, \quad z=\delta z_0. \tag{34}$$

[10] M. J. Druyvesteyn and F. M. Penning, Rev. Mod. Phys. 12, 92 (1940).
[11] R. B. Brode, Rev. Mod. Phys. 5, 263 (1933).

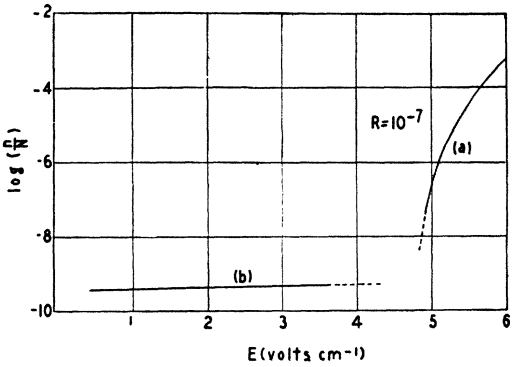


FIG. 6. Plot of $\log n/N$ *vs.* field strength.
Neon
$\nu=3(10^9)$ sec.$^{-1}$ $p=1$-mm Hg

Thus:

$$W_{k,\frac{1}{4}}(z)\doteq\left[\frac{\rho(z_1)}{\rho(z)}\right]^{\frac{1}{2}}\exp\left\{-\int_{z_1}^{z}\rho(z)dz\right\} \qquad z\geqslant z_1>z_0 \tag{35}$$

where

$$\rho^2(z)\equiv\left(\frac{1}{4}-\frac{k}{z}\right). \tag{36}$$

For $z_0\leqslant z\leqslant z_1$ we may rewrite Eq. (29):

$$g''-\frac{1}{16k}(z-4k)g=0.$$

This has the solution:

$$W_{k,\frac{1}{4}}(z)=(z-4k)^{\frac{1}{2}}\left\{AJ_{1/3}\left[\frac{i(z-4k)^{\frac{3}{2}}}{6k^{\frac{1}{2}}}\right]+BJ_{-1/3}\left[\frac{i(z-4k)^{\frac{3}{2}}}{6k^{\frac{1}{2}}}\right]\right\} \qquad z_0\leqslant z\leqslant z_1. \tag{37}$$

The solutions (35) and (37) may be equated at z_1. An additional relation in A and B is obtained by equating the derivatives at this point. Thus:

(a) $$W_{k,\frac{1}{4}}'(z)=-\left\{\rho+\frac{k}{4\rho^2z^2}\right\}W_{k,\frac{1}{4}}(z) \qquad z\geqslant z_1,$$

(b) $$W_{k,\frac{1}{4}}'(z)=\frac{i}{4k^{\frac{1}{2}}}(z-4k)\times\left\{AJ_{-2/3}\left[\frac{i(z-4k)^{\frac{3}{2}}}{6k^{\frac{1}{2}}}\right]-BJ_{2/3}\left[\frac{i(z-4k)^{\frac{3}{2}}}{6k^{\frac{1}{2}}}\right]\right\} \qquad z_1\geqslant z\geqslant z_0, \tag{38}$$

and

$$A=\frac{aJ_{2/3}+bJ_{-1/3}}{J_{1/3}J_{2/3}+J_{-1/3}J_{-2/3}}\qquad z=z_1, \tag{39}$$

$$B=\frac{aJ_{-2/3}-bJ_{1/3}}{J_{1/3}J_{2/3}+J_{-1/3}J_{-2/3}},$$

where

$$a\equiv\frac{W_{k,\frac{1}{2}}(z)}{(z-4k)^{\frac{1}{4}}},\qquad b\equiv-\frac{i4k^{\frac{1}{2}}W_{k,\frac{1}{2}}'(z)}{(z-4k)}.$$

The Bessel functions were evaluated at z_1 by interpolation from the tables of Jahnke-Emde,[12] at z_0 by using the series expansion for the Bessel functions,[13] the first three terms being sufficient.

It was found convenient to let $f(u_i)=1$ throughout the calculation. A check of the approximation outlined above was possible for helium at the temperature $4(10^4)$°K since at this temperature the asymptotic expansion of the Whittaker function could be used. Agreement to within five percent was found over the range of energies considered. In Fig. 1 the common logarithm of f is plotted as a function of δ for the four temperatures considered and for both helium and neon.

$S(\delta)$ is plotted in Fig. 2 for both helium and neon and for the two extreme temperatures considered. The function has a maximum at $\delta=1$. The curve becomes progressively flatter for $\delta<1$ as the temperature is reduced and may be approximated by a horizontal, straight line over most of this range for the temperatures in question.

In Fig. 3 is plotted the integral of (31d) for both gases. These curves illustrate the way in which the distribution function approaches a Maxwellian distribution for $\delta<1$ as the temperature is reduced. At a temperature of (10^4)°K the integral is effectively constant over the whole range of δ less than about 0.8 and, as seen by inspection of (31d), the distribution function is accordingly Maxwellian in this range. The integrand of (31d) actually possesses two maxima: one, quite sharp, near the origin, and a second, considerably broader, near 1.0. As the temperature decreases the latter maximum increases rapidly in height relative to the former.

In Fig. 4 the common logarithm of the integrals I_1 and I_2 as defined in article I is plotted as a function of the temperature. In Fig. 5 the quantity n/N is plotted as a function of the field strength; it was calculated by means of Eq. (8a), the relation between the field strength and the temperature being given by (33). The range of n/N is an arbitrary one satisfying the relation $n\ll N$. The proper values of R (cf. Eq. (5)) remain uncertain, and the values indicated on the graphs are taken as being probably representative, on the basis of Wessel's calculations for doubly charged helium ions, for the singly charged ions here involved. Several facts should be noted. First, the characteristics for neon are steeper than for helium. Second, the characteristic becomes steeper as the value chosen for R is reduced. Third, the characteristic becomes steeper as the field strength is reduced and hence as n/N becomes smaller. This last is better illustrated in Fig. 6 where the common logarithm of n/N is plotted as a function of the field strength for neon and where, in addition, the weak field characteristic is plotted with the use of (8b). The latter calculation was carried out by assuming that the distribution is Maxwellian, that ionization by electron-molecule collision could be neglected for the small field strengths involved, and that 10 ion pairs per cubic centimeter per second are produced by external agencies at atmospheric pressure. The calculation was made for a pressure of 1-mm Hg and for $R=(10^{-7})$. Similar results are obtained for other values of R and for other pressures. In order to connect the two curves it would be necessary to use (8) in its full form.

It is clear from Figs. 5 and 6 that the phenomenon of breakdown into a Townsend type discharge may be studied by this method and that the essential mechanisms cooperating to produce breakdown may well be ionizing electron-molecule encounters and volume recombination of electrons and positive ions. Rough values of the field strengths at breakdown may be estimated from Fig. 5, e.g., 5 volts/cm for neon and 13 volts/cm for helium. These may be

[12] E. Jahnke and F. Emde, *Funktionentafeln* (Dover Publishers, New York, 1943), p. 235.

[13] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, New York, 1935), p. 358.

compared with the observed values obtained at M.I.T.[14] The experiments indicate that at about 1-cm Hg the r.m.s. fields at breakdown are about 9 volts/cm for neon and 19 volts/cm for helium. The fact that the calculated values are too low is encouraging, for it leaves room for disposal of electrons by other mechanisms. The calculated results here are independent of the pressure, but the approximation upon which they are based breaks down for pressures above about 1-mm Hg. On the other hand, because of the small volume of the discharge vessel used for the measurements at M.I.T. the results below about 1-cm Hg are diffusion dependent. Additional calculations must be carried out at higher pressures, therefore, before a detailed comparison between experiment and theory will be in order. Taking these considerations into account and also the facts that inelastic collisions have been neglected, that additional mechanisms such as radiation trapping, diffusion, etc., will have a significant effect upon the shape of the characteristic, agreement between experimental results and the predictions of the present treatment appear to be satisfactory.

It should be noted finally that specific assumptions concerning the detailed shape of the cross sections involved have little effect upon the results of these calculations. Additional calculations made for helium with the slope of the ionization cross section multiplied by 2 shift the positions of the curves of n/N towards higher fields by less than 2 volts/cm. Similarly, an even smaller effect is introduced by varying the exponent of u in the recombination cross section (5) about the value unity.

[14] S. C. Brown, report at M.I.T. Conference on Physical Electronics, March, 1947; material unpublished.