Theory of High Frequency Gas Discharges. II. Harmonic Components of the Distribution Function[1]

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The type of distribution function employed in the foregoing paper is not sufficiently general to be applicable to all conditions of frequency and field strength. To examine its limitations, and the limitations of corresponding d.c. treatments, the function is developed as a series in Legendre polynomials of v_x/v, x being the field direction, and Fourier functions of ωt, ω being the field frequency. Attention is limited to steady states and to elastic collisions between electrons and gas molecules. By solving the resulting recurrence equations a number of successive approximations has been obtained, and from each approximation the range of validity of the preceding one is determined. Questions of mathematical convergence are not dealt with, since the physical meaning of the results is usually clear and reasonable. The current through the gas is shown to take on d.c. character when $\omega\lambda \ll (m/M)v$, λ being the mean free path and v the (mean) velocity of the electrons producing the current.

I

IN the first article of this series[2] the distribution function of electrons in a gas under a uniform, sinusoidal electric field was assumed to be approximately isotropic, the current being accounted for by a small additive term. Thus

$$f(\mathbf{v}, t) = f^0(v) + v_x\varphi(v, t),$$

provided the field is taken in the direction of the x-axis with a magnitude given by

$$E \cos\omega t. \qquad (1)$$

An expansion of the distribution function in this form, however, cannot be expected to hold for all frequencies of the impressed field. For very low frequencies even the isotropic part must be a function of the time and at high field strengths one cannot expect that non-linear terms in v_x are absent. A more general expansion will be investigated in the present paper, and conditions will be found under which the simple expansion above may be used.

When diffusion is negligible, the distribution function under a field in the direction of the x-axis may be taken to be a function of three variables: time (t), electronic speed (v), and α ($=v_x/v$). The number of electrons contained within an annular element of velocity space is

$$dn = f(\alpha, v, t)2\pi v^2 dv d\alpha. \qquad (2)$$

The most general expansion of the distribution function under these conditions and for a field with magnitude given by (1) may be written

$$f(\alpha, v, t) = \sum_{l=0}^{\infty} P_l(\alpha)\Big\{f_0{}^l(v) + \sum_{m=1}^{\infty}[f_m{}^l(v)\cos m\omega t + g_m{}^l(v)\sin m\omega t]\Big\}, \qquad (3)$$

where $P_l(\alpha)$ is the Legendre polynomial of degree l. The electron density becomes:

$$n = 4\pi\int_0^\infty \{f_0{}^0 + \sum_m (f_m{}^0 \cos m\omega t + g_m{}^0 \sin m\omega t\}v^2 dv \qquad (4)$$

while the current density is

$$I_x = \frac{4}{3}\pi e\int_0^\infty \{f_0{}' + \sum_m (f_m{}' \cos m\omega t + g_m{}' \sin m\omega t)\}v^3 dv, \qquad (5)$$

$$I_y = I_z = 0.$$

To determine the functions $f_m{}^l$ and $g_m{}^l$ the Boltzmann equation is employed. In terms of the variables used here this equation may be

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[2] H. Margenau, Phys. Rev. 73, 297 (1948).

written:

$$\frac{\partial f}{\partial t}+\gamma\cos\omega t\left[\alpha\frac{\partial f}{\partial v}+\frac{1}{v}(1-\alpha^2)\frac{\partial f}{\partial\alpha}\right]=\frac{\delta f}{\delta t},\tag{6}$$

where $\gamma=eE/m$ and the operator $\delta/\delta t$ expresses the time rate of increase as the result of collisions of all types between electrons and molecules of the gas. If the distribution function in the form of (3) is substituted in (6) and use is made of the orthogonal properties of both Legendre polynomials and the Fourier terms, a set of recurrence relations is obtained, the general equation for $m>1$ being:

$$\frac{\delta}{\delta t}\begin{pmatrix}f_m{}^l\\ g_m{}^l\end{pmatrix}=\begin{pmatrix}+g_m{}^l\\ -f_m{}^l\end{pmatrix}m\omega+\frac{\gamma}{2}\left\{\frac{l+1}{2l+3}\frac{1}{v^{l+2}}\frac{d}{dv}\left[v^{l+2}\begin{pmatrix}f_{m-1}^{l+1}+f_{m+1}^{l+1}\\ g_{m-1}^{l+1}+g_{m+1}^{l+1}\end{pmatrix}\right]+\frac{l}{2l-1}v^{l-1}\frac{d}{dv}\left[v^{1-l}\begin{pmatrix}f_{m-1}^{l-1}+f_{m+1}^{l-1}\\ g_{m-1}^{l-1}+f_{m+1}^{l-1}\end{pmatrix}\right]\right\},\tag{7}$$

where either the upper or the lower expressions are chosen in all parentheses.

For $m=0$:

$$\frac{\delta f_0{}^l}{\delta t}=\frac{\gamma}{2}\left\{\frac{l+1}{2l+3}v^{-l-2}\frac{d}{dv}[v^{l+2}f_1{}^{l+1}]+\frac{l}{2l-1}v^{l-1}\frac{d}{dv}[v^{1-l}f_1{}^{l-1}]\right\}.\tag{8}$$

For $m=1$:

$$\frac{\delta}{\delta t}\begin{pmatrix}f_1{}^l\\ g_1{}^l\end{pmatrix}=\begin{pmatrix}+g_1{}^l\\ -f_1{}^l\end{pmatrix}\omega+\frac{\gamma}{2}\left\{\frac{l+1}{2l+3}v^{-l-2}\frac{d}{dv}\left[v^{l+2}\begin{pmatrix}2f_0{}^{l+1}+f_2{}^{l+1}\\ g_2{}^{l+1}\end{pmatrix}\right]+\frac{l}{2l-1}v^{l-1}\frac{d}{dv}\left[v^{1-l}\begin{pmatrix}2f_0{}^{l-1}+f_2{}^{l-1}\\ g_2{}^{l-1}\end{pmatrix}\right]\right\}.\tag{9}$$

A similar treatment can be carried through for the d.c. distribution. The expansion of the distribution function is then given by:

$$f(\alpha,v)=\sum_l P_l(\alpha)f^l(v),\tag{10}$$

and the recurrence relations become:

$$\frac{\delta f^l}{\delta t}=\gamma_{\text{d.c.}}\left\{\frac{l+1}{2l+3}v^{-l-2}\frac{d}{dv}[v^{l+2}f^{l+1}]+\frac{l}{2l-1}v^{l-1}\frac{d}{dv}[v^{1-l}f^{l-1}]\right\}.\tag{11}$$

The notation $\gamma_{\text{d.c.}}$ serves as a reminder that the d.c. field strength corresponds to the a.c. r.m.s. value. As the frequency of the alternating field goes to zero the distribution function in expansion (3) reduces to (10) by means of the identification:

$$f^l=\lim_{\omega\to 0}\sum_m f_m{}^l.\tag{12}$$

If the electron density is constant in the time, as is assumed throughout this paper, an important consequence may be inferred from the recurrence relations (7), for then

$$\int f_0{}^0 4\pi v^2dv=n;\quad \int f_m{}^0v^2dv=\int g_m{}^0v^2dv=0,\ m>0.$$

But the recurrence relations connect in the same equation only those functions, $f_m{}^l$ and $g_m{}^l$, for which the sum of the upper and lower indices is either odd or even. Now the electron density is either greater than or equal to zero. If it is equal to zero, all terms of the distribution function vanish; if it is not equal to zero, it is given as above by the integral of $f_0{}^0$, which combines only with functions of even $l+m$. Hence, for a constant electron density one must set equal to zero all terms for which $m+l$ is odd. The distribution function then assumes the form:

$$f(\alpha,v,t)=(f_0{}^0+f_2{}^0\cos 2\omega t+g_2{}^0\sin 2\omega t+\cdots)+\alpha(f_1{}'\cos\omega t+g_1{}'\sin\omega t+f_3{}'\cos 3\omega t+g_3{}'\sin 3\omega t+\cdots)+\cdots.\tag{13}$$

This form of the distribution function will here be assumed to hold. Physically it means that a sinusoidal electric field can generate only even harmonics in the isotropic part of the distribution function and odd harmonics in the current density, e.g., no d.c. current component.

II

Expressions involving the operator $\delta/\delta t$ may be easily evaluated. Again we shall study here only the case of elastic collisions. Consider that during an elastic collision at time t the electron leaves the volume element about α' and v' and enters that about α, v; or that it leaves α, v and proceeds to α'', v''. The positions of these three volume elements are mutually related by the nature and the geometry of the collision; the loss in speed was discussed in article I.[2] Consider further that there exists a well defined mean free path $\lambda_e(v)$ and let $A(\alpha, \alpha')d\alpha d\alpha'$ be the probability that during the collision the electron will proceed from $(\alpha', \alpha'+d\alpha')$ to $(\alpha, \alpha+d\alpha)$. Then:

$$\left.\frac{\delta f}{\delta t}\right|_e v^2 dv d\alpha = \int_{\alpha'} \frac{v'}{\lambda_e(v')} f(\alpha', v', t)\cdot v'^2 dv' A d\alpha d\alpha' - \frac{v}{\lambda_e(v)} f(\alpha, v, t)\cdot v^2 dv$$

$$= \sum_l \int P_l(\alpha')A(\alpha, \alpha')d\alpha' \left\{ \frac{v'^3}{\lambda_e(v')} dv' d\alpha \left[f_0{}^l(v') + \sum_m \left(f_m{}^l(v')\cos m\omega t + g_m{}^l(v')\sin m\omega t\right)\right]\right\}$$

$$- \frac{v}{\lambda_e(v)} \sum_l P_l(\alpha)d\alpha\left[f_0{}^l(v) + \sum_m \left(f_m{}^l(v)\cos m\omega t + g_m{}^l(v)\sin m\omega t\right)\right] v^2 dv. \quad (14)$$

By Eq. (3), this quantity also equals

$$\sum_l P_l(\alpha)d\alpha \left\{ \left.\frac{\delta f_0{}^l}{\delta t}\right|_e + \sum_m \left(\left.\frac{\delta f_m{}^l}{\delta t}\right|_e \cos m\omega t + \left.\frac{\delta g_m{}^l}{\delta t}\right|_e \sin m\omega t \right)\right\} v^2 dv. \quad (15)$$

We shall assume that scattering is isotropic, so that A becomes a constant and is equal to $\frac{1}{2}$. Then

$$\int P_l(\alpha')d\alpha' A d\alpha' = \delta_{0,l}.$$

Comparison of (14) and (15) yields

$$\text{(a)} \quad \left.\frac{\delta f_m{}^l}{\delta t}\right|_e = -\frac{v}{\lambda_e(v)} f_m{}^l, \quad l > 0$$

$$\text{(b)} \quad \left.\frac{\delta f_m{}^0}{\delta t}\right|_e v^2 dv = f_m{}^0(v')\frac{v'^3}{\lambda_e(v')}dv' - f_m{}^0(v)\frac{v^3}{\lambda_e(v)}dv, \quad (16)$$

and expressions of the same form hold for $g_m{}^l$ and $g_m{}^0$. Equation (16a) can be generalized and shown to be true for all, not only elastic collisions, if the meaning of λ is suitably changed. Thus:

$$\frac{\delta f_m{}^l}{\delta t} = -\frac{v}{\lambda(v)} f_m{}^l, \quad l > 0 \quad (17)$$

and similarly for $g_m{}^l$, where $\lambda(v)$ is the electron mean free path with respect to all collisions. For the case of elastic collisions (16b) may be written, in the form:

$$\left.\frac{\delta f_m{}^0}{\delta t}\right|_e = \frac{m}{Mv^2}\frac{d}{dv}\left(\frac{v^4}{\lambda_e}f_m{}^0\right) + \frac{kT}{Mv^2}\frac{d}{dv}\left(\frac{v^3}{\lambda_e}\frac{df_m{}^0}{dv}\right) \quad (18)$$

as discussed in article I[2].

In the remainder of this paper, no attention will be given to inelastic collisions.

III

In order to make use of these results a consistent approximation procedure must be devised, since the recurrence relations, a quadruply infinite set of difference-differential equations, cannot in general be solved. Approximate solutions can be obtained if the distribution function (3), or (10) in the d.c. case, is terminated after several terms. In certain simple cases a solution may be found in closed form. It is then necessary to examine the conditions which must be imposed on the discharge parameters, consistent with the termination adopted. Two requirements must be satisfied if consistency is to be maintained:

(a) For a given pair of indices, m and l, both the sine and cosine terms must be included in the terminated distribution function.

(b) Only these recurrence equations are used for which the term in $\delta/\delta t$ does not vanish. That is, if the left-hand side of any member of (7) is assumed to vanish, the right-hand side must be wholly ignored.

Since a non-vanishing electron density requires the inclusion of $f_0{}^0$, or f^0 in the d.c. case, in the distribution function, and since the recurrence relation in $\delta f_0{}^0/\delta t$, or $\delta f^0/\delta t$, connects $f_0{}^0$, or f^0, only with f_1', or f', it follows that successive approximations are:

$$\text{d.c.}\quad \begin{aligned}&(a)\quad f=f^0,\\ &(b)\quad f=f^0+\alpha f',\\ &(c)\quad f=f^0+\alpha f'+\tfrac{1}{2}(3\alpha^2-1)f^2;\end{aligned} \tag{19}$$

$$\text{a.c.}\quad \begin{aligned}&(a)\quad f=f_0{}^0,\\ &(b)\quad f=f_0{}^0+\alpha(f_1'\cos\omega t+g_1'\sin\omega t),\\ &(c)\quad f=(f_0{}^0+f_2{}^0\cos 2\omega t+g_2{}^0\sin 2\omega t)\\ &\qquad +\alpha(f_1'\cos\omega t+g_1'\sin\omega t)\\ &\qquad +\tfrac{1}{2}(3\alpha^2-1)(f_0{}^2+f_2{}^2\cos 2\omega t\\ &\qquad +g_2{}^2\sin 2\omega t).\end{aligned} \tag{20}$$

The equations resulting from the use of (19a) and (20a) are

$$\delta f^0/\delta t=0;\quad \delta f_0{}^0/\delta t=0. \tag{21}$$

In both the d.c. and a.c. cases the operator $\delta/\delta t$ for $l=0$ has the form given by (18). Hence, on integrating Eqs. (21), the distribution becomes Maxwellian in first approximation, as expected. Similarly, the recurrence equations which correspond to (19b) are obtained immediately from Eqs. (11). They are:

$$\frac{\delta f^0}{\delta t}=\frac{\gamma_{\text{d.c.}}}{3v^2}\frac{d}{dv}(v^2f');\quad -\frac{v}{\lambda}f'=\gamma_{\text{d.c.}}\frac{df^0}{dv}. \tag{22}$$

These equations may be integrated directly. The solution is the Druyvesteyn distribution[3] ($T=0$) or the Davydov distribution[4] ($T\neq 0$). In the same way, the equations determining (20b) are obtained directly from (7):

$$\begin{aligned}&\frac{\delta f_0{}^0}{\delta t}=\frac{\gamma}{6v^2}\frac{d}{dv}(v^2f_1');\\ &-\frac{v}{\lambda}f_1'=\omega g_1'+\gamma\frac{df_0{}^0}{dv};\quad -\frac{v}{\lambda}g_1'=-\omega f_1'.\end{aligned} \tag{23}$$

The solution of these equations has been treated by Margenau.[5]

IV

Under a d.c. field the distribution function in the form of (19c) is given by the solution of the following set of recurrence equations:

$$\begin{aligned}&\frac{\delta f^0}{\delta t}=\frac{\gamma_{\text{d.c.}}}{3v^2}\frac{d}{dv}(v^2f');\\ &-\frac{v}{\lambda}f'=\gamma_{\text{d.c.}}\left\{\frac{2}{5v^2}\frac{d}{dv}v^3f^2+\frac{df^0}{dv}\right\};\\ &-\frac{v}{\lambda}f^2=\frac{2}{3}\gamma_{\text{d.c.}}v\frac{d}{dv}(v^{-1}f').\end{aligned} \tag{24}$$

Rather than attempting to handle Eqs. (24), it is of interest to learn when their solution will reduce approximately to that of Eqs. (22). This will be the case when:

$$|f^2|\ll|f'|. \tag{25}$$

Considering here only the case of low temperatures, let us assume that f^0 and f' satisfy Eqs. (22). Then for $T=0$:

$$f^0=A\exp\left\{-\frac{3mv^4}{4M(\gamma_{\text{d.c.}}\lambda)^2}\right\}. \tag{26}$$

We evaluate f^1 by the second of Eqs. (22) and f^2 by the third of Eqs. (24). On forming the ratio of f^2 to f' it is seen that condition (25) is satisfied provided

$$\left|\frac{\eta v^2}{\gamma_{\text{d.c.}}\lambda}-\frac{2\gamma_{\text{d.c.}}\lambda}{3v^2}\right|\ll 1,\quad \text{where } \eta=2m/M.$$

Physically, η is the mean energy loss of an electron per collision. It is convenient to introduce a variable χ defined by

$$v^2=\chi\langle v^2\rangle$$

and to express this inequality in terms of χ. The mean value of v^2 is given by

$$\langle v^2\rangle=(8/3)^{\frac{1}{2}}\frac{\Gamma(5/4)}{\Gamma(3/4)}\eta^{-\frac{1}{2}}\gamma_{\text{d.c.}}\lambda.$$

We then have, finally,

$$\eta^{\frac{1}{2}}\left|1.21\chi-\frac{0.55}{\chi}\right|\ll 1. \tag{27}$$

Since η is of the order of 10^{-4} this is seen to be satisfied over a fairly wide range of energies about the mean energy, e.g., $10^{-1}\leqslant\chi\leqslant\chi$. This

[3] M. J. Druyvesteyn, Physica 10, 61 (1930).
[4] B. Davydov, Physik. Zeits. Sovjetunion, 8, 59 (1935).
[5] H. Margenau, Phys. Rev. 69, 508 (1946).

result is independent of the field strength. Only the very fast and the very slow electrons have a distribution in energy which differs from the Druyvesteyn distribution. These exceptions are not unexpected. The very fast electrons are those whose motion in the field has been largely unimpeded by collisions with molecules, hence those with abnormally long free times. The very slow electrons, on the other hand, are those whose recent history has been relatively unaffected by the field, that is, those which have made a great many collisions with the molecules of the gas and whose distribution, therefore, is not independent of the gas temperature and must approach a Maxwellian form.

V

Under an a.c. field the situation is considerably more complex. The equations which determine the distribution function written in the form (20c) are:

$$\text{(a)}\quad \frac{\chi}{6v^2}\frac{d}{dv}(v^2 f_1') = \frac{\delta f_0^0}{\delta t},$$

$$\text{(b)}\quad 2\omega g_2^0 + \frac{\gamma}{6v^2}\frac{d}{dv}(v^2 f_1') = \frac{\delta f_2^0}{\delta t},$$

$$\text{(c)}\quad -2\omega f_2^0 + \frac{\gamma}{6v^2}\frac{d}{dv}(v^2 g_1') = \frac{\delta g_2^0}{\delta t},$$

$$\text{(d)}\quad \omega g_1' + \frac{\gamma}{5v^3}\frac{d}{dv}[v^3(2f_0^2+f_2^2)] + \frac{\gamma}{2}\frac{d}{dv}(2f_0^0+f_2^0) = -\frac{v}{\lambda}f_1',$$

$$\text{(e)}\quad -\omega f_1' + \frac{\gamma}{5v^3}\frac{d}{dv}(v^3 g_2^2) + \frac{\gamma}{2}\frac{dg_2^0}{dv} = -\frac{v}{\lambda}g_1', \qquad (28)$$

$$\text{(f)}\quad \frac{\gamma v}{3}\frac{d}{dv}(v^{-1}f_1') = -\frac{v}{\lambda}f_0^2,$$

$$\text{(g)}\quad 2\omega g_2^2 + \frac{\gamma v}{3}\frac{d}{dv}(v^{-1}f_1') = -\frac{v}{\lambda}f_2^2,$$

$$\text{(h)}\quad -2\omega f_2^2 + \frac{\gamma v}{3}\frac{d}{dv}(v^{-1}g_1') = -\frac{v}{\lambda}g_2^2,$$

where the form of $\delta/\delta t$ for elastic collisions is again given by (18). These equations do not submit to an exact solution. We shall assume first, as in the d.c. case, that the distribution function may be written approximately in the form (20b) and therefore start with the solution of Eqs. (23). Equation (20c) differs from (20b) by inclusion of (a) isotropic terms f_2^0, and g_2^0, (b) non-isotropic terms f_0^2, f_2^2, and g_2^2. The conditions under which set (a) is negligible are different from those under which set (b) is negligible. Inspection of Eqs. (28) shows that these conditions can be studied separately. In the remainder of this section, we give attention to set b.

According to (28f, g, h), the functions f_0^2, f_2^2, and g_2^2 are expressed in terms of f_1' and g_1'. Now these latter develop peculiarities when v is either very large or very small in comparison with $\omega\lambda$, and are quite regular when $v \sim \omega\lambda$. If, therefore, the roles played by f_0^2, f_2^2, g_2^2 are investigated for these two limiting cases (for which the analysis is easy), predictions for the intermediate range can safely be made.

There are two cases of interest.

(A) $(\omega\lambda) \ll v$ over the important range of the distribution function. Then the first terms on the left of (28g) and (28h) may be neglected. g_2^2 is smaller in absolute value than f_0^2, and f_2^2 is equal to f_0^2. Furthermore

$$f_0^0 \doteq A \exp\left\{-\frac{3\eta v^4}{4(\gamma\lambda)^2}\right\}, \qquad (29)$$

if we restrict ourselves to low temperatures. If f_1' is evaluated by means of (23) and the variable χ is introduced as before, the mean "energy" being given by:

$$\bar{v}^2 = \frac{2}{\sqrt{3}}\frac{\Gamma(5/4)}{\Gamma(3/4)}\eta^{-\frac{1}{2}}\gamma\lambda,$$

we obtain finally the result that the terms in $P_2(\alpha)$ may be neglected provided that

$$\eta^{\frac{1}{2}}\left|0.86\chi - \frac{0.39}{\chi}\right| \ll 1. \qquad (30)$$

The situation is essentially the same as that under a d.c. field and the same remarks apply here.

(B) $(\omega\lambda) \gg v$ over the range of v within which the distribution function is significant. Then f_0^0

is Maxwellian[5] at an effective "temperature" T' given by:

$$T'=T\left[1+\frac{M}{6kT}\left(\frac{\gamma\lambda}{\omega\lambda}\right)^2\right] \quad (31)$$

and

$$f_1'=\frac{mv^2\gamma}{kT'\omega^2\lambda}f_0^0; \quad g_1'=\frac{\omega\lambda}{v}f_1'. \quad (32)$$

Substituting these expressions in (28f, g, h) and solving for f_0^2, f_2^2, and g_2^2, we obtain the following expressions:

$$\frac{f_0^2}{f_1'}=\frac{m\gamma\lambda}{kT'}\left(1-\frac{kT'}{mv^2}\right),$$
$$\frac{f_2^2}{f_1'}=\frac{1}{6}\frac{m\gamma\lambda}{kT'}, \quad (33)$$
$$\frac{g_2^2}{f_1'}=\frac{1}{4}\frac{v}{\omega\lambda}\frac{m\gamma\lambda}{kT'}\left(1-\frac{2kT'}{3mv^2}\right).$$

Since now $\frac{1}{2}m\langle v^2\rangle=\frac{3}{2}kT'$ it follows that these ratios will all be small if

$$\frac{m\gamma\lambda}{kT'}\ll 1, \quad (34)$$

except for very small values of v. The expression on the left of (34) is usually negligible except in the neighborhood of its maximum with respect to γ which occurs at

$$\gamma=\omega\left(\frac{6kT}{M}\right)^{\frac{1}{2}}. \quad (35)$$

Now γ/ω is roughly the speed acquired by a free electron in one cycle, $(6kT'/M)^{\frac{1}{2}}$ roughly the speed of the molecules. When these are equal, therefore, the necessity for including higher Legendre terms may arise.

As an example we note that for helium at 300°K and at a frequency of 3000 megacycles, Eq. (35) defines a field strength of about 2 volts/cm, and the value of $m\gamma\lambda/kT'$ corresponding to this field strength reaches 1 for pressures of a few millimeters. The present analysis indicates that for pressures as low as this and for field strengths of this critical value, higher Legendre terms may have to be included

Similar conclusions hold when $\omega\lambda\approx v$. Summarizing the results, one may say that the functions of set (b) can usually be ignored. They are of importance only for electron energies differing widely from the mean, and possibly, at small pressures, when condition (35) is true. For these cases they are given by Eqs. (30) and (33).

VI

The situation is somewhat different with respect to the time-dependent isotropic terms in (20c). To neglect them,

$$\left|\frac{f_2^0}{f_0^0}\right|\ll 1; \quad \left|\frac{g_2^0}{f_0^0}\right|\ll 1 \quad (36)$$

must be true. We estimate these expressions from the recurrence Eqs. (28b) and (28c) directly. Assume that the right-hand numbers of these equations are small. We then have at once approximate expressions for f_2^0 and q_2^0 which satisfy automatically the condition of normalization to a constant electron density. These expressions may be used on the right of Eqs. (28b) and (28c) to obtain a second approximation to f_2^0 and g_2^0, etc. This scheme converges well and indicates that the first approximation to g_2^0 is satisfactory. The functions f_0^0 and f_1' are calculated from Eqs. (23). Then, with restriction to zero temperature, one finds

$$g_2^0 \doteq -\frac{\eta v}{\omega\lambda}\left\{1-\frac{3\eta v^2(v^2+\omega^2\lambda^2)}{4\gamma^2\lambda^2}\right\}f_0^0.$$

Provided v is not too far from the mean velocity, g_2^0 is appreciable only when $\omega\lambda$ is of the order of ηv or less. Similarly, we obtain from the second approximation for f_2^0 (again neglecting T)

$$f_2^0 \doteq \frac{\eta}{4v^2}\frac{d}{dv}\left\{v^3\left(1+\frac{\eta v^2}{(\omega\lambda)^2}\right)f_0^0\right\},$$

in the neighborhood of the mean velocity. If $\eta^{\frac{1}{2}}v<\omega\lambda$ this reduces further and one finds

$$\left|\frac{f_2^0}{f_0^0}\right|=0(\eta)\ll 1.$$

Hence it follows that for $(\omega\lambda)>\eta^{\frac{1}{2}}v$ both f_2^0 and g_2^0 may be neglected. Numerically this means that under discharge conditions $(\omega\lambda)$ must exceed about 10^7 cm sec.$^{-1}$.[6] The distribution function in the form of (20b) therefore is valid only for frequencies higher than $\eta^{\frac{1}{2}}v/\lambda$.

A similar investigation shows that for very low frequencies, $g_2{}^0$ approaches zero while $f_2{}^0$ remains finite.

To summarize, therefore, we have three distinct frequency regions in each of which the distribution function assumes a characteristic form:

(a) $\omega\lambda \ll \eta v$

$$f=(f_0{}^0+f_2{}^0\cos 2\omega t)+\alpha f_1{}'\cos\omega t,$$

(b) $\omega\lambda \approx \eta v$

$$f=(f_0{}^0+f_2{}^0\cos 2\omega t+g_2{}^0\sin 2\omega t)+\alpha(f_1{}'\cos\omega t+g_1{}'\sin\omega t), \tag{37}$$

(c) $\omega\lambda \gg \eta^{\frac{1}{2}}v$

$$f=f_0{}^0+\alpha(f_1{}'\cos\omega t+g_1{}'\sin\omega t).$$

A d.c. situation is one for which condition (a) is true.

VII

The distribution function in the form of (37a) is amenable to exact solution. The equations to be satisfied are found from (28):

$$\text{(a)}\qquad \frac{\gamma}{6v^2}\frac{d}{dv}(v^2f_1{}')=\frac{\delta f_0{}^0}{\delta t}=\frac{\delta f_2{}^0}{\delta t},$$

$$\text{(b)}\qquad \frac{\gamma}{2}\frac{d}{dv}(2f_0{}^0+f_2{}^0)=-\frac{v}{\lambda}f_1{}'. \tag{38}$$

We define

$$F=2f_0{}^0+f_2{}^0;\quad G=f_0{}^0-f_2{}^0. \tag{39}$$

[6] It is interesting to note that in the measurements of L. Rohde (Ann. d. Physik. (5) 12 (1932), especially p. 585) the variation of the critical discharge potential with frequency seemed to cease in the region of $\omega\lambda$ below this value. This is probably an indication of the fact that $f_2{}^0$ becomes important, giving the discharge certain direct-current features which will be discussed in the next section.

Then from Eqs. (38):

$$\frac{\delta G}{\delta t}=0;\quad G=A\exp\left(-\frac{mv^2}{2kT}\right). \tag{40}$$

Also

$$\frac{\gamma}{2v^2}\frac{d}{dv}(v^2f_1{}')=\frac{\delta F}{\delta t},\quad \frac{\gamma}{2}\frac{dF}{dv}=-\frac{v}{\lambda}f_1{}'$$

or

$$F=B\exp\left\{-\int\frac{\frac{m}{2}dv^2}{kT+\frac{M(\gamma\lambda)^2}{4v^2}}\right\}. \tag{41}$$

We write

$$\alpha\equiv\frac{mM\gamma^2\lambda^2}{8k^2T^2},\quad x=\frac{mv^2}{2kT}. \tag{42}$$

Then, integrating (41):

$$2f_0{}^0+f_2{}^0=B\left(1+\frac{x}{\alpha}\right)^{\alpha}$$

or

$$\begin{aligned}&\text{(a)}\qquad f_0{}^0-f_2{}^0=Ae^{-x},\\ &\text{(b)}\qquad f_0{}^0+f_2{}^0=\frac{1}{3}\left\{2B\left(1+\frac{x}{\alpha}\right)^{\alpha}-A\right\}e^{-x}.\end{aligned} \tag{43}$$

But $f_0{}^0\mp f_2{}^0$ is the form taken on by the isotropic part of f when $\omega t=\pi/2$ or 0, respectively, i.e., when the field is zero or a maximum. Hence, once every half-cycle, when the magnitude of the field is zero, the isotropic part of the distribution function is Maxwellian; when the field is a maximum, the isotropic part of the distribution has the form given by (43b). This shows precisely how the theory of a.c. discharges goes over continuously into the d.c. theory. The constants A and B are determined by normalizing $f_0{}^0$ to the value n, $f_2{}^0$ to zero. On calculation they are found to differ somewhat from the corresponding constant in the d.c. distribution function—as they should because of the neglect of higher harmonics in (37a), but become identical with the latter at small field strengths.