system. A conclusive test of the general applicability of the present functions would be available through an experimental study of the concentric spherical system where the field is an inverse square function of the radius.

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Theory of High Frequency Gas Discharges. I. Methods for Calculating Electron Distribution Functions[1,2]

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After a brief summary of the "free-electron" theory of a.c. currents in gases, which is shown to be inadequate for microwaves, methods for calculating distribution functions for electrons with consideration of all types of collisions are developed and discussed. Conditions are specified under which non-elastic collisions may be introduced as integral and as differential expressions in the Boltzmann transfer equation. In Section VI this equation is solved approximately and the differences between direct current and h.f. current discharges are pointed out. Finally, Section VIII is concerned with an aspect of what is usually called Paschen's law; it indicates that a minimum should exist in the graph of voltage *vs.* pressure, and of voltage *vs.* frequency, both at constant current. This minimum comes when the frequency of the field is of the order of magnitude of the collision frequency of the electrons.

THIS series of papers presents a study of what happens when a gas at low pressure becomes conducting under high frequency electromagnetic waves. While the precise meaning of the term "high frequency" is defined in Article II, it will usually refer to situations in which the frequency of the electromagnetic waves is comparable with the frequency of collisions between free electrons and gas atoms.

Although one's first impression is to the contrary, such microwave discharges are probably less complicated with regard to the physical processes which they involve, and therefore possibly more instructive, than direct current discharges. This is true primarily because conditions can be arranged under which ionization occurs predominantly in the body of the gas, and secondary processes at the electrodes and at the walls of the discharge chamber are of lesser importance. In addition to this simplifying aspect, it is proper to assume that the gas acquires no net volume charge, its condition being similar to a d.c. plasma. Diffusion of electrons, being ambipolar, will be slow; it can, in a certain range of gas pressures, be made the dominant mechanism for the disposal of electrons. In another pressure range it can be made subordinate to volume recombination, permitting this mechanism to be studied.

Unfortunately, theory enjoys in this field at present the large amount of liberty that always goes with a scarcity of experimental data. The subject is new and measurements are difficult. Checks on assumptions are imposed only by comparison with known analogous situations in d.c. discharges, and with the few facts now available. But to show what theory can do is perhaps not wholly useless in advance of experimentation, since the experimenter may be aided

[1] The contents of this paper form part of Radiation Laboratory Report No. 967; they are based on work done for the Office of Scientific Research and Development under Contract OEMsr-262 with the Massachusetts Institute of Technology.

[2] Some of the developments in this paper are parallel to the work of T. Holstein, Phys. Rev. 70, 367 (1946). It is hoped that the difference in approach justifies the inclusion of those portions of the present paper which are already dealt with in Holstein's publication.

by knowing what kind of measurements are, and what are not, theoretically interesting.

The first article contains a development and summary of methods for calculating the energy distribution of electrons in a discharge, as a function of field strength, frequency of the field, and gas pressure. The effect of different physical processes on the shape of the distribution function will be investigated, but the work is based on the assumption that this function departs very little from being isotropic in the velocity space of the electrons and has the periodicity of the impressed electric field.

In Article II these assumptions are scrutinized and the physical conditions are determined under which various possible forms of the distribution function are usable approximations. This is done with neglect of inelastic collisions. The results enable one to set limits depending upon field strength, frequency, and gas pressure, beyond which customary formulas fail.

One of the methods discussed in the present paper is developed in Article III for the purpose of calculating the breakdown potential in a gas filling an infinite region, so that direct recombination is the only process which disposes of electrons, and volume ionization is assumed to be the only mechanism producing them. While it is difficult to define the meaning of a sharp breakdown potential, it does turn out that the number of electrons per unit volume rises quite suddenly at a certain field strength; the value of this more or less critical field strength is smaller than the breakdown fields which have been measured (in finite enclosures). The difference can be ascribed to other disposal mechanisms, such as diffusion of electrons to the walls of the vessel.

In Article IV some general consequences respecting current voltage characteristics of a.c. discharges are discussed, and attention is drawn to the form which the similarity principle, known from d.c. work, takes under conditions of microwave excitation.

I. FREE ELECTRONS IN ALTERNATING FIELDS

For the sake of orientation we first sketch the results of the simple theory[3] of electrons in an alternating field, a theory which neglects collisions. An electron, born with zero velocity at time t' and under the action of the field $Ee^{i\omega t}$ along the x-axis, has a velocity

$$\dot{x}=\frac{ieE}{m\omega}(\exp(i\omega t')-\exp(i\omega t)),$$

and hence a maximum velocity

$$v=\frac{eE}{m\omega}|1\mp\sin\omega t'|.$$

This is reached at $t=\pm\pi/\omega$, when the field reverses its direction. Electrons born at these instants will reach greater speeds than the others since the field has an entire half-period to accelerate them. The maximum speeds of these electrons are

$$v_{\max}=2eE/m\omega=0.94\times10^7E/\nu\ e\text{-volts},\quad(1)$$

if E is in volts/cm and ν is the frequency of the field. According to this formula, E must be of the order of 10,000 volts/cm in the microwave region ($\nu\approx10^{10}$/sec.) if an electron is to attain ionizing velocities.

The displacement is

$$x=\frac{eE}{m\omega^2}(1+i\omega t\cdot\exp(i\omega t')-\exp(i\omega t)).$$

It corresponds to an oscillation of amplitude

$$x_0=eE/m\omega^2=0.44\times10^{14}E/\nu^2\text{ cm},\quad(2)$$

if E is expressed in volts/cm; this oscillation is superposed on a drift with velocity $(eE/m\omega)\times\sin\omega t'$.

For microwaves at ionizing field strengths as given by Eq. (1)

$$x_0\approx10^{-2}\text{ cm}.\quad(3)$$

The mean current density $ne\langle\dot{x}\rangle_{\text{Av}}$, the average being taken over all t', is

$$J=-(ine^2E/m\omega)e^{i\omega t}.$$

It is in quadrature with the field and yields the well-known conductivity

$$J/Ee^{i\omega t}\equiv\sigma=-ine^2/m\omega.$$

The conditions under which this simple theory is applicable are these: (a) the frequency of the

[3] G. Mierdel, Ann. d. Physik **85**, 612 (1928); K. K. Darrow, Bell Sys. Tech. J., XI, 576 (1932); XII, 91 (1933); L. B. Loeb, *Fundamental Processes of Electrical Discharge in Gases* (J. Wiley and Sons, Inc., New York, 1939).

waves, ν, must be much greater than the collision frequency, ν_{coll}; (b) x_0 must be much smaller than the mean free path, λ, of the electrons. Using kinetic-theory expressions for λ and ν_{coll}, these conditions become:

$$x_0 \ll 1/Nq, \quad \text{(a)}$$

$$\nu \gg Nq\bar{v}, \quad \text{(b)}$$

provided N is the number density of gas molecules, q their collision cross section, and $\bar{v}$ their mean velocity. At a pressure of 1 cm Hg, $(Nq)^{-1}$ is about 10^{-2} cm; hence the value given by Eq. (3) violates condition (a). At electron speeds prevailing in a discharge $Nq\bar{v}$ is about 10^{10} sec.$^{-1}$; hence microwave frequencies violate condition (b).

The simple theory here sketched is therefore inadequate to provide information about microwave discharges.

II. COLLISIONS AND THEIR EFFECT ON THE DISTRIBUTION FUNCTION

To take account of collisions it is necessary to introduce a distribution function, f, whose value is the number of electrons having velocities about $\mathbf{v}$ and enclosed in a volume about $\mathbf{r}$ at time t. It is a function of $\mathbf{r}$, $\mathbf{v}$, t and will be normalized so that

$$\int\int f(\mathbf{v}, \mathbf{r}, t)d\mathbf{v}d\mathbf{r}$$

is the total number of electrons present, and

$$\int f(\mathbf{v}, \mathbf{r}, t)d\mathbf{v} = n(\mathbf{r}, t) \quad (4)$$

the number density of electrons. If n is not a function of t, the state represented by f is said to be steady; if f is not a function of $\mathbf{r}$, the distribution is uniform; if f is a function only of $|\mathbf{v}|$, it is isotropic.

The Boltzmann transfer equation,[4] together with certain physical conditions, determines the distribution function. It reads

$$\partial f/\partial t + \mathbf{v}\cdot\nabla f + \mathbf{a}\cdot\nabla f = \delta f/\delta t. \quad (5)$$

Here $\partial f/\partial t$ is the smooth local rate of change, while $\delta f/\delta t$ represents the rate of change occasioned only by collisions, but by all collisions, elastic and inelastic, including those which result in ionization or capture. The symbols ∇ and ∇ denote, respectively, the gradients in $\mathbf{r}$-space and in $\mathbf{v}$-space.

The presence of the term $\delta f/\delta t$, which makes the value of f at $\mathbf{v}$ depend on the number of electrons at very different velocities, marks Eq. (5) as a partial differentio-integral equation, for which no general method of solution is known. Even explicit evaluation of $\delta f/\delta t$ is possible only with the use of specific assumptions concerning the collisions as well as the form of the distribution function. As to the latter, we suppose for the present that f does not depart much from being isotropic, and that the small non-isotropic part has the spatial symmetry of the field. Let us also at first assume a uniform density of electrons (thus disregarding diffusion). Then

$$f(\mathbf{v}) = f^{(0)}(v) + v_x\phi(v) \quad (6)$$

and ϕ is presumably small. The term $\delta f/\delta t$ now consists of two parts, $\delta f^{(0)}/\delta t$ and $\delta(v_x\phi(v))/\delta t$, and these require separate treatment in detail.

The general formula for $\delta f/\delta t$, well known from kinetic theory, is obtained as follows. Different types of collision (such as elastic, or causing excitation of molecules between a given pair of levels, or ionizing, and so forth) of which an electron is capable will be labeled by an index μ.[5] Collisions of one electron with another are here disregarded as unlikely. If we denote by $\mathbf{v}_\mu'$ the velocity before encounter of an electron which is about to suffer a collision of type μ, and which after collision will have velocity $\mathbf{v}$, the rate of *increase* of electrons at $\mathbf{v}$ is given by

$$v_\mu' N q_\mu(\mathbf{v}_\mu') f(\mathbf{v}_\mu') d\mathbf{v}_\mu'. \quad (7)$$

To see this, we recall that $v_\mu'\lambda_\mu^{-1}$, λ_μ, representing the mean free path for a collision of type μ, is the number of μ-encounters made by one electron of speed v_μ' per second, and that

$$Nq_\mu\lambda_\mu = 1 \quad (8)$$

[4] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1939), Chapter 3; P. M. Morse, W. P. Allis, and E. S. Lamar, Phys. Rev. **48**, 412 (1935).

[5] Collisions of unspecified type will be labeled by the index μ; elastic collisions by e, inelastic (but non-ionizing) by s, ionizing by i. Similar indices will be used on cross sections (q_e etc.) and mean free paths (λ_e etc.).

is the general relation connecting N, the number of gas molecules per cm^3, and q_μ, the cross section for a μ-encounter, with the mean free path. The rate of decrease of electrons at $\mathbf{v}$ is due to collisions which lead from $\mathbf{v}$ to some lower velocity, with cross section q_μ; it is given by an expression similar to (7). Hence

$$(\delta f/\delta t)d\mathbf{v} = N\sum_\mu v_\mu' q_\mu(\mathbf{v}_\mu')f(\mathbf{v}_\mu')d\mathbf{v}_\mu' - N\sum_\mu vq_\mu(\mathbf{v})f(\mathbf{v})d\mathbf{v}. \quad (9)$$

If the form of f which was assumed in Eq. (6) is substituted into this formula, the collisional rate of increase of f is obtained. But of the two terms in Eq. (6), the second is much the easier to compute and will be considered first.

We have

$$\frac{\delta}{\delta t}[v_x\phi(v)]d\mathbf{v} = N\sum_\mu v_\mu' q_\mu(\mathbf{v}_\mu')v_{\mu x}'\phi(v_\mu')d\mathbf{v}_\mu' - Nvv_x\phi(v)d\mathbf{v}\sum_\mu q_\mu(\mathbf{v}).$$

The last summation is the total collision cross section q, which by (8) is $(N\lambda)^{-1}$. The remaining summation over μ in the first term on the right includes an integration over all orientations of the vector $\mathbf{v}_\mu'$. This will not be zero if q_μ depends on the orientation of $\mathbf{v}_\mu'$. However, if this is not the case, that is, if the cross section is isotropic, the only term affected by this integration is $v_{\mu x}'$, and it yields zero. Isotropy of collisions will now be assumed, so that we may write

$$\frac{\delta}{\delta t}[v_x\phi(v)] = -\frac{vv_x}{\lambda}\phi(v). \quad (10)$$

We now turn to the calculation of $\delta f^{(0)}/\delta t$, which, in view of Eq. (9), is given by

$$\frac{\delta f^{(0)}}{\delta t}\cdot 4\pi v^2 dv = N\sum_\mu v_\mu' q_\mu f^{(0)}(v_\mu')4\pi v_\mu'^2 dv_\mu' - N\sum_\mu vq_\mu f^{(0)}(v)\cdot 4\pi v^2 dv. \quad (11)$$

III. ELASTIC COLLISIONS

The relation between v' and v, needed in evaluating (11), is different for elastic and inelastic collisions. The characteristic feature of an elastic one is that the electron loses, on the average over all encounters, a small and constant *fraction* of its original energy or speed, while in an inelastic collision the actual *amount* of energy lost is prescribed. Thus for an elastic encounter,

$$v' = (1+m/M)v \quad (12)$$

provided the molecules, whose mass is M, are effectively at rest, and m is the mass of an electron. The approximation of treating all elastic collisions as belonging to a single type, which involves always the average loss of speed, is made valid by the smallness of each loss.

Equation (11) now reduces to the form

$$v^2\left.\frac{\delta f^{(0)}}{\delta t}\right|_e = \frac{N}{v}[v'^4 q_e(v')f^{(0)}(v') - v^4 q_e(v)f^{(0)}(v)] = \frac{N}{v}\frac{d}{dv}(v^4 q_e f^{(0)})(v'-v).$$

On using (12), we thus obtain

$$\left.\frac{\delta f^{(0)}}{\delta t}\right|_e = \frac{m}{M}\cdot\frac{N}{v^2}\frac{d}{dv}[v^4 q_e f^{(0)}]. \quad (13)$$

If account is taken of the velocity distribution of the molecules, this relation is replaced by

$$\left.\frac{\delta f^{(0)}}{\delta t}\right|_e = \frac{m}{M}\cdot\frac{N}{v^2}\left\{\frac{d}{dv}[v^4 q_e f^{(0)}] + \frac{kT}{m}\frac{d}{dv}\left[v^3 q_e \frac{df^{(0)}}{dv}\right]\right\} \quad (13')$$

as is shown by Chapman and Cowling,[4] pp. 348 ff. The added term, which needs to be introduced in order that the vanishing of $\delta f^{(0)}/\delta t|_e$ shall yield a Boltzmann distribution for $f^{(0)}$, is not exact under discharge conditions. For the most part in this work, it will be regarded as correct in order of magnitude; it may usually be neglected.

IV. NON-ELASTIC COLLISIONS; INTEGRAL REPRESENTATION

This section contains some modifications and a slight extension of the work of Smit[6] and of Chapman and Cowling,[4] Chapter 18. Consideration will be given not only to inelastic collisions, but also to impacts which result in ionization, capture, and so forth. Assume that an ordinary inelastic collision of type s causes the electron

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[6] J. A. Smit, Physica **3**, 543 (1937).

to lose an amount of energy equal to $\epsilon_s = \frac{1}{2}mu_s$. In the evaluation of the terms of Eq. (11) which correspond to this collision, we must then put

$$v_s'^2 = v^2 + u_s, \quad v_s' dv_s' = v dv,$$

and the contribution of the collision to that equation is

$$4\pi v dv \left[\frac{f^{(0)}(v_s^0) v_s'^2}{\lambda_s(v_s')} - \frac{f^{(0)}(v) v^2}{\lambda_s(v)} \right]$$

again in view of Eq. (8). But this expression can be written as $(dS_s/dv)dv$ provided the function S_s is defined by

$$S_s(v) = \int_v^{v_s'} \frac{v}{\lambda_s(v)} f^{(0)}(v) \cdot 4\pi v^2 dv. \quad (14)$$

It represents the rate of increase in the concentration of electrons having speeds below v. In terms of the variable $u = v^2$

$$S_s(u) = 2\pi N \int_u^{u+u_s} f^{(0)}(u) \cdot q_s(u) \cdot u du. \quad (14')$$

We note that $S_s(\infty) = 0$. If $S = \sum_s S_s$, Eq. (11) can be written

$$\frac{\delta f^{(0)}}{\delta t} = \frac{1}{4\pi v^2} \frac{dS}{dv} + \left.\frac{\delta f^{(0)}}{\delta t}\right|_e.$$

To calculate S, one would have to know the function $f^{(0)}$ in advance; i.e., the use of Eq. (14) would convert the transfer equation into an integro-differential equation. There may be situations, however, in which the physical significance of S leads to a fairly reasonable conjecture as to the form of this function and where this may be used as a starting approximation in the calculation of $f^{(0)}$. A way of doing this is suggested by Chapman and Cowling. An attempt was made in the present investigation to assume a flexible form for $f^{(0)}$ which contained two adjustable parameters, then to compute S, and finally to fix the parameters so as to achieve self-consistency for the solutions of the transfer equation. But the results were not encouraging, even when ionizing collisions were neglected.

A different situation arises when collisions are included in which electrons are produced or captured. To take account of ionization, for example, the first summation on the right of Eq. (11) must be doubled because every ionizing electron of speed v' produces two electrons. Furthermore, to specify v', an assumption regarding the division of energy available to the two electrons after ionization must be made. To say that it is divided equally seems not far from the truth. We may therefore take

$$v_i'^2 = 2v^2 + u_i, \quad v_i' dv_i' = 2v dv,$$

u_i being the ionization "energy" of the atom or molecule. Instead of (11) we then have

$$\left.\frac{\delta f^{(0)}}{\delta t}\right|_i \cdot 4\pi v^2 dv = 2\pi du [4Nq_i(u_i') f^{(0)}(u_i') \cdot u_i' - Nq_i(u) f^{(0)}(u) \cdot u]$$

provided q_i is the ionization cross section. If this is to be written in the form $dS_i(u)$, S_i must have the form

$$S_i(u) = 2\pi N \left\{ 2\int_u^{u_i'} f^{(0)} \cdot q_i \cdot u du + \int_0^u f^{(0)} q_i u du \right\}. \quad (15)$$

The significance of the two integrals occurring here is clear: the first accounts for the two electrons produced at energies below u by ionizations with the energy of the ionizing electron above u; the second for the one additional electron created by ionizations below u. Thus, $S_i(\infty)$ is not zero; it becomes in fact

$$2\pi N \int_0^\infty f^{(0)} q_i u du.$$

The case of capture can be dealt with in a similar way. The total S, defined as the sum of all S_s, S_i, and S_c, must vanish at infinity if the number of electrons in the discharge is to be constant. In Article III, Mr. L. M. Hartman has applied the analysis of this section to the calculation of breakdown potentials in He and Ne under certain conditions.

V. NON-ELASTIC COLLISIONS; DIFFERENTIAL REPRESENTATION

Instead of dealing with each type of non-elastic collision individually, as in the preceding section, we now treat the extreme case in which there are so many different kinds of inelastic encounters that the index s in Eq. (11) may be regarded as a continuous variable. To simplify matters further, ionization and electron removal will be included among the ordinary inelastic collisions, no account being taken of the effect of "number" production and destruction upon the distribution function.

As before, $v_s'dv_s'=vdv$, and Eq. (11) takes the form

$$\frac{\delta f^{(0)}}{\delta t}=\frac{N}{v}\left\{\sum_s u_s'q_s(u_s')f^{(0)}(u_s') -uf^{(0)}(u)\sum_s q_s(u)\right\} \quad (16)$$

so far as *inelastic* collisions are concerned. As will be clear later (see Eq. (28)), if the transfer equation is to reduce to an ordinary second-order differential equation, the right-hand side of (16) must be transformable to

$$\frac{N}{v}\frac{d}{du}\left\{X(u)\int_u^\infty Y(u,t)f^{(0)}(t)dt\right\}. \quad (17)$$

Here X and Y are arbitrary functions, at present unknown. But (16) is thus transformable only if certain rather special assumptions are made concerning the cross sections q_s. Two fairly obvious assumptions leading to solutions will here be studied.

The cross section $q_s(u_s')$ is a function of two variables: the initial "energy" u_s' and the final energy u. We therefore put

$$q_s(u_s')=q(u_s',u). \quad (18)$$

But initial and final energy of the electron are not useful variables in terms of which to specify the probabilities of transitions; it seems preferable to use the energy *loss* together with the final (or the initial) energy, because energy loss, i.e., the difference between two molecular energy levels, is more significant in the description of transitions.

Let us therefore characterize a process s by the loss of "speed" $u_s'-u\equiv p$, and let p be a continuous variable. Then

$$q(u_s',u)=A(p,u)\delta(u_s',u+p)du_s'dp, \quad (19)$$

where δ is Dirac's singular function. The summation over s in Eq. (16) then turns into an integration over u_s' and p.

The function $q_s(u)$ is constructed by analogy with (19);

$$q_s(u)=q(u,u_s'') =A(p,u_s'')\delta(u_s'',u-p)du_s''dp. \quad (20)$$

One may then write in place of (16), after integrating over u_s' and u_s'',

$$\frac{\delta f^{(0)}}{\delta t}=\frac{N}{v}\left\{\int_0^\infty (u+p)A(p,u)f^{(0)}(u+p)dp -uf^{(0)}(u)\int_0^u A(p,u-p)dp\right\}, \quad (21)$$

if it is remembered that A vanishes when its second argument is negative.

A given value of p characterizes a molecular transition; the probability of its occurrence is a function of u. Hence $A(p,u)$ is factorable,

$$A(p,u)=A_1(p)A_2(u),$$

and

$$\frac{\delta f^{(0)}}{\delta t}=\frac{N}{v}\left\{A_2(u)\int_u^\infty tA_1(t-u)f^{(0)}(t)dt -uf^{(0)}(u)\int_0^u A_1(t)A_2(u-t)dt\right\}. \quad (22)$$

To see under what conditions this can be expressed in the form (17) we expand that quantity, taking Y to be $tA_1(t-u)$, and obtaining

$$\frac{N}{v}\left\{X'(u)\int_u^\infty tA_1(t-u)f^{(0)}(t)dt -X(u)uA_1(0)f^{(0)}(u) -X(u)\int_u^\infty tA_1'(t-u)f^{(0)}(t)dt\right\}. \quad (23)$$

If this is to be identical with (22), it is seen, first of all, that A_1' must be zero. Therefore, the scheme works only if we take $A_1(p)$ to be a constant, which would imply that every pair of molecular levels has the same intrinsic transition probability. The present method thus limits our consideration to artificial physical situations, which are perhaps approximated in complicated molecules having a great number of excitable levels.

To make Eqs. (22) and (23) agree no further conditions need be imposed, but one must take

$$X'(u)=A_2(u).$$

This causes both the remaining terms in (22) and (23) to become identical. The result is

$$\frac{\delta f^{(0)}}{\delta t}=\frac{N}{v}\frac{d}{du}\left\{\int_0^u A(t)dt\int_u^\infty tf^{(0)}(t)dt\right\},$$

where $A(t)$ is now written for $A_1A_2(t)$. But it is easily shown that $\int_0^u A(t)dt$ is the inelastic collision cross section which will here be written q since we are concerned only with inelastic encounters in this section of the paper. Hence,

$$\frac{\delta f^{(0)}}{\delta t}=\frac{1}{v}\frac{d}{du}\left\{\frac{1}{\lambda_s(u)}\int_u^\infty tf^{(0)}(t)dt\right\}. \qquad (24)$$

Another workable method for evaluating the inelastic collision rate has been suggested by Bennett and Thomas.[7] Starting again with Eq. (18) we now put

$$q(u_s', u)=B(u', u)\delta(u_s', u')du_s'du'$$

and use a product function for $B(u', u)$:

$$B(u', u)=B_1(u')B_2(u), \quad u'\geqslant u.$$

Evidently this implies that the collision probability is composed independently of a probability associated with the initial speed and one associated with the final speed. This hypothesis is less reasonable than (19), but it has greater mathematical fertility, for it permits transformation to the form (17) without further restrictions.

Since

$$q_s(u)=q(u,u_s'')=B_1(u)B_2(u'')\delta(u_s'',u'')du_s''du'', \quad u\geqslant u'',$$

Eq. (16) becomes, on combining the two integrals,

$$\frac{\delta f^{(0)}}{\delta t}=\frac{N}{v}\frac{d}{du}\left\{\int_0^u B_2(t)dt\int_u^\infty u'B_1(u')f^{(0)}(u')du'\right\}.$$

In this case

$$q(u)=B_1(u)\int_0^u B_2(t)dt$$

so that, finally,

$$\frac{\delta f^{(0)}}{\delta t}=\frac{1}{v}\frac{d}{du}\left\{\frac{1}{B_1(u)\lambda(u)}\int_u^\infty tB_1(t)f^{(0)}(t)dt\right\}. \qquad (25)$$

The result is more general than (24); it is identical with it when B_1 is constant. In the following sections various suppositions will be made regarding $B_1(u)$, and for each of them the distribution function will be computed.

VI. DISTRIBUTION FUNCTION OF ELECTRONS IN AN A.C. FIELD

Having calculated the right-hand side of Eq. (5), we now turn our attention to the terms of the left. Ignoring the diffusion of electrons, we set $\nabla f=0$. We choose the field strength to be of the form $E\cdot\cos\omega t$ along the x-axis, so that $a=(eE/m)\cos\omega t\equiv\gamma\cos\omega t$. The distribution function is given by (6). But ϕ will have one part in phase with the field and another part out of phase. Thus,

$$f(\mathbf{v})=f^{(0)}(v)+(v_x/v)(f^{(1)}\cos\omega t+g^{(1)}\sin\omega t). \qquad (26)$$

Substitution of f in Eq. (5) and subsequent reduction in a way similar to that used in a previous publication[8] leads to the following three equations:

$$\frac{\gamma}{6v^2}\frac{\partial}{\partial v}(v^2f^{(1)})=\left.\frac{\delta f^{(0)}}{\delta t}\right|_e+\left.\frac{\delta f^{(0)}}{\delta t}\right|_s \qquad \text{(a)}$$

$$\gamma\lambda\frac{df^{(0)}}{dv}+\omega\lambda g^{(1)}=-vf^{(1)} \qquad \text{(b)} \qquad (27)$$

$$\omega\lambda f^{(1)}=vg^{(1)}. \qquad \text{(c)}$$

If we use Eq. (13), they combine to give

$$-\tfrac{2}{3}\gamma^2\frac{1}{v}\frac{d}{du}\left(\frac{\lambda u^2}{u+\omega^2\lambda^2}\frac{df^{(0)}}{du}\right)=\frac{2m}{Mv}\frac{d}{du}\left[\frac{u^2}{\lambda_e}f^{(0)}(u)\right]+\left.\frac{\delta f^{(0)}}{\delta t}\right|_s. \qquad (28)$$

Inspection of this equation shows that $\delta f/\delta t$ must be of the form (17) if $f^{(0)}$ is to be the solution of a second-order differential equation. Using the integral representation for the inelastic collision,

[7] W. H. Bennett and L. H. Thomas, Phys. Rev. 62, 41 (1942).

[8] H. Margenau, Phys. Rev. 69, 508 (1946).

Eq. (14) *et. seq.*, we find

$$\delta f^{(0)}/\delta t|_s=(1/2\pi v)(dS/du).$$

Integration of Eq. (28) yields

$$-\frac{4\pi}{3}\gamma^2\frac{\lambda u^2}{u+(\omega\lambda)^2}\frac{df^{(0)}}{du}=2\pi\eta\frac{u^2}{\lambda_e}f^{(0)}+S. \quad (29)$$

Here λ, the gas-kinetic mean free path, is a function of u depending on the gas. It satisfies the relation

$$1/\lambda=1/\lambda_e+1/\lambda_s.$$

If nearly all collisions are elastic, which is the case for small u, λ and λ_e may be identified. This will now be done. Let $F(u)$ be the distribution function calculated without considering inelastic collisions, that is, the solution of Eq. (29) with $S=0$. It was derived and discussed in reference 8. In terms of it, Eq. (29) can be written

$$\frac{d}{du}\left(\frac{f^{(0)}}{F}\right)=-\frac{3\lambda}{\pi}\cdot\frac{u+u_1}{u^2u_2^2}\cdot\frac{S(u)}{F}$$

if we use the abbreviations

$$u_1=(\omega\lambda)^2,\quad u_2=2\gamma\lambda. \quad (30)$$

Since $\lim(f^{(0)}/F)=0$ for $u\rightarrow\infty$, this equation has the solution

$$f^{(0)}=\frac{3}{\pi}F\int_u^\infty\lambda\frac{u+u_1}{u_2^2u^2}\frac{S(u)}{F}du. \quad (31)$$

This formula, with refinements, will be used in the work presented in the third paper by Mr. Hartman.

In the remainder of this section we pursue the consequences of the differential representation. If Eq. (28) is integrated with the use of Eq. (25), one obtains

$$-\tfrac{2}{3}\gamma^2\frac{\lambda u^2}{u+u_1}\frac{df^{(0)}}{du}=\frac{2m}{M}\cdot\frac{u^2}{\lambda_e}f^{(0)}$$
$$+\frac{1}{B_1(u)\lambda_s}\int_u^\infty tB_1(t)f^{(0)}(t)dt, \quad (32)$$

and all three mean-free-paths appear. This equation will be solved for two ranges of u: the "elastic-impact" region where u lies below some critical value and $\lambda_s=\infty$, and the "inelastic-impact" region. The solution for the first region is F and is known. In the second region λ_s is finite and depends on u. We shall assume it to be constant and put

$$\lambda_s=K\lambda$$

so that $\lambda_e=K\lambda/(K-1)$. Moreover, let us put the probability of an inelastic encounter

$$B_1(u)=\text{const.}u^r$$

and see how the distribution function in the inelastic region depends on r. More refined assumptions require numerical integrations of the resulting equations and do not seem worth while at the present time. Equation (32) becomes, after differentiation,

$$\frac{d^2f^{(0)}}{du^2}+\left(\frac{r+2}{u}-\frac{1}{u+u_1}+\frac{12m}{M}\cdot\frac{K-1}{K^\mu}\cdot\frac{u+u_1}{u_2^2}\right)\frac{df^{(0)}}{du}$$
$$-\frac{6}{u_2^2}\cdot\frac{u+u_1}{Ku}\left[1-2(r+2)\frac{m}{M}(K-1)\right]f^{(0)}=0.$$

Now m/M is of the order 10^{-4}, K and r presumably not far from 1 in the region of inelastic impacts; hence, neglect of the last term in the preceding equation is always legitimate. In the parentheses, u must attain a value of $(12m/M)^{-\frac{1}{2}}u_2$, if its last member is to be considerable. For such high energies the distribution function is usually small and the term in question has no appreciable effect. It is a permissible approximation, therefore, to reduce Eq. (32) to the form

$$\frac{d}{du}\left(\frac{u^{r+2}}{u+u_1}\frac{df^{(0)}}{du}\right)=\frac{6}{K}\cdot\frac{u^{r+1}}{u_2^2}f^{(0)} \quad (33)$$

when dealing with the region of inelastic collisions. In solving this equation we consider two instances: the d.c. case ($u_1=0$) and the high frequency case.

D.C. Case

Equation (33) can be solved exactly. If we put

$$k^2=\frac{6}{Ku_2^2}=\tfrac{3}{2}\frac{m^2}{K(eE\lambda)^2},\quad f^{(0)}=u^{-\frac{1}{2}r}Z_{-(r/2)}(iku).$$

Since $f^{(0)}$ must vanish at infinity, the Bessel function Z must be identified with the Hankel function $H^{(1)}$, which has the asymptotic expansion[9]

$$H_{-\frac{1}{2}r}(iku) = \text{const.}\, u^{-\frac{1}{2}} e^{-ku} S_{\frac{1}{2}r}(2ku)$$

with

$$S_{\frac{1}{2}r}(2ku) = 1 + \frac{r^2-1}{8ku} + \frac{(r^2-1)(r^2-9)}{128(ku)^2} + \frac{(r^2-1)(r^2-9)(r^2-25)}{3072(ku)^3} + \cdots. \quad (34)$$

Hence

$$f^{(0)} = Au^{-(r+1)/2} e^{-ku} S_{\frac{1}{2}r}(2ku). \quad (35)$$

A.C. Case

It is convenient to write

$$x = \frac{u}{u_1}, \quad q^2 = \frac{6}{K}\left(\frac{u_1}{u_2}\right)^2.$$

The equation to be solved is

$$\frac{d^2 f^{(0)}}{dx^2} + \left(\frac{r+2}{x} - \frac{1}{x+1}\right)\frac{df^{(0)}}{dx} = \left(1 + \frac{1}{x}\right) q^2 f^{(0)}. \quad (36)$$

Since it is not valid for small x, and our interest is in the region of large x, we endeavor to find a solution valid to terms in x^{-1} and expand the equation with retention of terms in x^{-2}. On writing

$$f^{(0)} = \exp\left[-\int_a^x y(x)dx\right]$$

and employing what is known as the J-W-K-B-method, one finds

$$y = q + \frac{q+r+1}{2x} + \frac{1 + (r^2-1)/2q - q/2}{4x^2} + \cdots$$

and therefore

$$f^{(0)} = Ax^{-(q+r+1)/2} \times \exp\left[-qx + \frac{1 + (r^2-1)/2q - q/2}{4x}\right]. \quad (37)$$

To facilitate interpretation a summary of the meaning of symbols employed thus far and of the results is given below.

[9] E. Jahnke and F. Ende, ***Tables of Functions*** (Dover Publications, New York, 1945), p. 137f.

List of Symbols

v = speed of electrons
$u = v^2$
λ, λ_s = total and inelastic electronic mean free path
$K = \lambda_s/\lambda$, treated as constant
ω = radian frequency of field
$u_1 = (\omega\lambda)^2$
r defined by: cross section for inelastic collisions $\sim u^r$
E = amplitude of field strength
m = mass of electron
M = mass of molecule
$u_2 = 2eE\lambda/m$
$k^2 = 6/Ku_2^2$
$q^2 = 6/K(u_1/u_2)^2$

$$f_{\text{d.c.}}^{(0)} = Au^{-(r+1)/2} e^{-ku}(1 + (r^2-1)/8ku + \cdots)$$

$$f_{\text{a.c.}}^{(0)} = Au^{-(q+r+1)/2} e^{-ku} \times (1 + (r^2-1)/8ku + (2-q)u_1/8u + \cdots). \quad (38)$$

It may be noted that the d.c. distribution function (35) is Maxwellian when $r = -1$. It then corresponds to an electron temperature (for electrons capable of inelastic collisions)

$$\theta = (K/6)^{\frac{1}{2}} eE\lambda,$$

where θ is Boltzmann's constant times T. This implies that the mean energy of an electron is about equal to the energy it gains between collisions. The Maxwellian case, however, refers to the rather implausible situation where the collision cross section decreases with growing speed. In all other instances $f^{(0)}$ falls off more rapidly at large u than does e^{-ku}.

For very large frequencies the alternating field has no effect on the distribution function. This is seen from Eq. (28). If $\omega \to \infty$, the left-hand side vanishes, which is equivalent to equating γ to zero.

As an illustrative example we treat the following case. A discharge is excited by the passage of 10-cm waves through argon gas at a pressure of 6 mm of Hg. The resulting distribution function is to be compared with that for a similar d.c. discharge. Assume

$\omega = 1.88 \times 10^{10}$ sec.$^{-1}$; $\lambda = 7.2 \times 10^{-3}$ cm.; $u_1 = 1.80 \times 10^{16}$ cm^2 sec.$^{-2}$; $u_2 = 7.65 \times 10^{15} E$ cm^2 sec.$^{-2}$; $k = (3.20/E) \times 10^{-16}$ cm^{-2} sec.^{2}; $q = 5.80/E$; $m/M = 1.36 \times 10^{-5}$; E is in e.s.u. per cm.

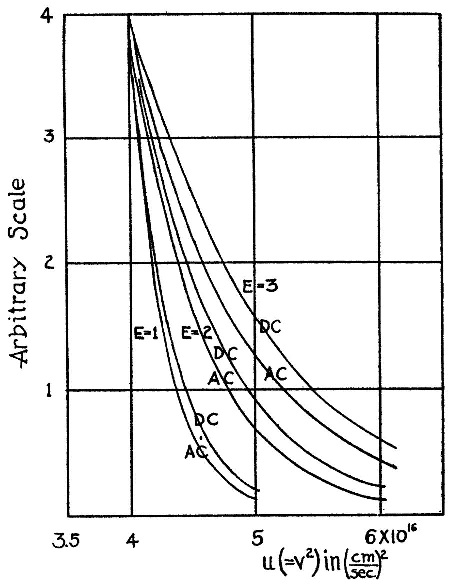

FIG. 1. Plot of $uf^{(0)}(u)$ as a function of u for argon at 6-mm pressure and wave-length 10 cm E is in electrostatic units.

Excitation in argon starts at $u=4.0\times 10^{16}$ cm^2 sec.$^{-2}$.

Below this value of u the distribution function is F and this turns out to be essentially constant for field strengths high enough to cause a discharge. Hence, there arises no distinction between the d.c. and the a.c. distribution in this region. At higher energies, however, there is an important difference which is illustrated in Fig. 1 by plotting $uf^{(0)}$ against u. These curves were drawn for the case $B_1(u)=\text{constant}$; they are not normalized, and E for the a.c. curves is the amplitude, not the r.m.s. value of the field strength.

In Table I are collected the results of numerical integrations yielding the relative numbers of electrons capable of inelastic collisions at different field strengths. In the a.c. case these numbers rise faster with the root mean square value of E than the d.c. values rise with E.

VII. DIFFUSION OF ELECTRONS

In this section it is desired to show what modifications are required in the present formalism when diffusion of electrons is to be included. A brief indication will suffice, since the results are not new.

First of all, it is necessary to include the term $v_x\partial f/\partial x$ in the transfer equation (assuming that diffusion occurs in the x-direction). Also, to obtain a steady state, ionization must be included explicitly in the formalism. This is best done by writing

$$\frac{\delta f^{(0)}}{\delta t}=\frac{1}{4\pi v^2}\frac{d}{dv}(S_i+S_h),$$

where S_i refers to ionization, S_h to all other collisions. Finally, Eq. (26) is replaced by

$$f=f^{(0)}(v,x)+\frac{v_x}{v}\times[h^{(1)}(v,x)+f^{(1)}(v)\cos\omega t+g^{(1)}(v)\sin\omega t].$$

On separation one then obtains, instead of Eq. (27), the following set

$$\frac{\gamma}{6v^2}\frac{d}{dv}(v^2f^{(1)})+\frac{v}{3}\frac{\partial h^{(1)}}{\partial x}=\frac{1}{4\pi v^2}\frac{d}{dv}(S_i+S_h)$$

$$\gamma\lambda\partial f^{(0)}/\partial v+\omega\lambda g^{(1)}=-vf^{(1)}$$

$$\omega\lambda f^{(1)}=vg^{(1)}$$

$$\partial f^{(0)}/\partial x=-h^{(1)}/\lambda_d,$$

where λ_d is a mean free path characteristic of the diffusion of electrons. In microwave discharges it corresponds to ambipolar diffusion. When the last of these relations is introduced in the first and the result is multiplied by $4\pi v^2dv$ and integrated over the entire range of v, there results

$$\frac{d^2}{dx^2}\int_0^\infty\frac{v\lambda_d}{3}\cdot f^{(0)}\cdot 4\pi v^2dv+S_i(\infty)=0 \qquad (39)$$

since $S_h(\infty)=0$. Now in the kinetic theory of gases the diffusion coefficient, $D(v)$, is defined as $v\lambda_d/3$. The integral in (39) is therefore n times

TABLE I. Values of the relative numbers, ν, of electrons having speeds greater than 2×10^8 cm sec.$^{-1}$.

$$\nu=\int_{4\times 10^{16}}^\infty u^{\frac{1}{2}}f^{(0)}du\Big/\int_0^\infty u^{\frac{1}{2}}f^{(0)}du.$$

E (volts cm^{-1})	d.c.	a.c.
300	1.09×10^{-2}	1.08×10^{-2}
600	2.31×10^{-2}	1.99×10^{-2}
900	3.43×10^{-2}	2.90×10^{-2}

the mean of $D(v)$ or nD, and

$$d^2(nD)/dx^2+S_i(\infty)=0.$$

This result implies equality of the rate of removal and the rate of production of electrons.

To proceed from here on, special assumptions have to be made concerning the dependence of $f^{(0)}$ on v and x. The simplest is to say that $f^{(0)}(v, x)=V(v)\cdot X(x)$ so that

$$D=\int_0^\infty \frac{v\lambda_d}{3}\cdot V(v)v^2dv \bigg/ \int_0^\infty Vv^2dv,$$

a quantity independent of x but through V a function of E, ω, and λ. Holstein's results are obtained when V is taken to be $f^{(0)}$, computed without taking account of diffusion, as in the previous section.

VIII. MAINTENANCE POTENTIAL IN HIGH FREQUENCY DISCHARGES

The experimental evidence concerning current-voltage characteristics of a.c. discharges is somewhat confused. For a discharge at a given frequency and pressure there is usually observed a "least maintaining potential," a potential below which the discharge is extinguished. To what extent it is a function of the circuit parameters is difficult to say.

When the l.m.p. is plotted against the pressure of the gas, the result is a curve either of type (a) or type (b), Fig. 2. Curve (a) has a minimum and is found in the work of Kirchner.[10] Rohde[11] finds graphs with a horizontal slope like (b). In both of these the current varies from point to point of the curve.

Brasefield's[12] measurements are in one respect more definite, for he determines the potential required to maintain a constant current, as a function of gas pressure and of the frequency. He definitely obtains curves of type (a). His results invite a theoretical interpretation which will be given in the sequel.

Other investigations[13] designed to determine the dependence of voltage on current for a given frequency and pressure, while interesting in themselves, have an indirect bearing upon the l.m.p. They indicate for the graph of V against I a curve like (b), with a horizontal slope. The current is therefore independent of the voltage over a very wide range of values. If this result is accepted and coupled with Brasefield's findings, it would appear that the curves of l.m.p. *vs.* gas pressure should also show a minimum in accordance with (a), since the current corresponding to the l.m.p. must lie on the horizontal part of the characteristic. Deviations from this behavior may well be caused by circuit peculiarities.

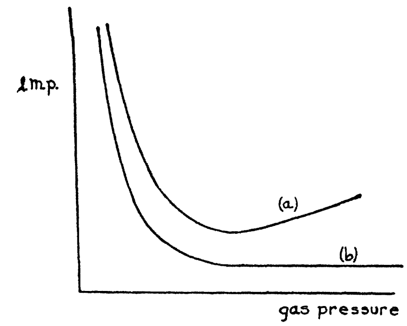

FIG. 2. Least maintaining potential *vs.* gas pressure.

Brasefield has given a qualitative explanation for the occurrence of minima, and the following considerations show how his results are related to the distribution functions computed in this paper.

If, for a given frequency, the pressure is too low, a high field is required to produce ionization, since the electrons are essentially free and have to draw all their energy from the alternating field in a single period, as described in Section I. On the other hand, if the pressure is too high, frequent collisions interfere with the transfer of energy from the field to the electrons.

These facts can be recognized from the following simplified and non-rigorous considerations. The current density across the discharge is given by

$$J=ne\bar{v}_x=e\int_0^\infty v_x^2(f^{(1)}\cos\omega t+g^{(1)}\sin\omega t)\frac{d\mathbf{v}}{v}. \quad (40)$$

The first term under the integral represents the dissipative current upon which attention will be

[10] F. Kirchner, Ann. d. Physik (4), **77**, 287 (1925).
[11] L. Rohde, Ann. d. Physik (5), **12**, 569 (1932).
[12] C. F. Brasefield, Phys. Rev. **35**, 1073 (1930).
[13] M. A. Herlin and S. C. Brown, Phys. Rev. **69**, 696 (1946); L. D. Smullin and C. G. Montgomery, *Microwave Duplexers*, Radiation Laboratory Series (McGraw-Hill Book Company, Inc., New York (in press)), Vol. 14.

focused here. From Eqs. (27b, c),

$$f^{(1)}=-\frac{u_2}{1+\frac{u_1}{u}}\frac{\partial f^{(0)}}{\partial u}. \qquad (41)$$

For $f^{(0)}$ we shall use the approximate formula (37) which may be written

$$f^{(0)}=A'u^{-(q+r+1)/2}\times\exp\left\{-ku+\frac{2-q}{8u}u_1+0\left(\frac{1}{ku}\right)\right\} \qquad (42)$$

and drop terms of order $1/ku$ and its higher powers. The examples discussed in Section VI show this to be reasonable for microwaves, although it is detrimental to quantitative accuracy. Equation (42) holds above some critical value u'; below u', $f^{(0)}$ is nearly constant and makes a small contribution to $f^{(1)}$. Within this set of simplifications, the integral in (40) is to be extended over an algebraic function of u, u_1, and u_2 times the exponential

$$\phi(u, u_1, u_2)=\exp\left\{-ku+\frac{2-q}{8u}u_1\right\}$$

and between the fixed limits u' and ∞.

The condition that the current be constant under a variation of the independent parameters u_1 and u_2 is that the differential of this integral with respect to these two parameters be zero. The major change in the integral is expected to come from the exponential factor in the integrand, hence we require that

$$\int\left[\frac{\partial\phi}{\partial u_1}du_1+\frac{\partial\phi}{\partial u_2}du_2\right]h(u, u_1, u_2)du=0, \qquad (43)$$

where h is an algebraic function. But this leads at once to

$$\left.\frac{\partial u_2}{\partial u_1}\right|_{J=\text{const.}}=\left(\frac{u_1}{u_2}-\left(\frac{K}{6}\right)^{\frac{1}{2}}\right)\frac{\int\frac{h\phi}{4u}du}{\int\frac{h\phi}{u_2^2}\left(u+\frac{u_1^2}{8u}\right)du}$$

(cf. Table I, Section VI, for the meaning of q, k). The ratio of the remaining integrals is positive, hence the graph of u_2 *vs.* u_1 has a *minimum* at

$$u_1=\left(\frac{K}{6}\right)^{\frac{1}{2}}u_2. \qquad (44)$$

It will be seen that a similar minimum occurs when E is plotted against λ (or pressure) at constant frequency, or E against ω at constant pressure.

Equation (44) is equivalent to

$$\frac{2eE\lambda}{m}=\left(\frac{6}{K}\right)^{\frac{1}{2}}\omega^2\lambda^2. \qquad (45)$$

Now $eE\lambda$ is approximately the energy acquired by an electron between collisions, $\frac{1}{2}mV^2$. Hence the minimum occurs when

$$\omega\approx\frac{V}{\lambda},$$

that is, when the frequency of the field is about equal to the collision frequency.

Equation (45) is in accord with the main results of Brasefield's work, giving correctly both the dependence of E on λ and on ω. As to numerical agreement, we note that in a typical case he finds

$$E\approx 10 \text{ volts cm}^{-1};\quad \omega=2\pi\times 1.5\times 10^7 \text{ sec.}^{-1};\quad \lambda\approx 5 \text{ cm}.$$

Using formula (45) we find for K the value 10, which is not unreasonable.