The Charge Distribution in Nuclei and the Scattering of High Energy Electrons

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It is pointed out that the finite size of the nucleus will give rise to large deviations from Mott scattering when the change in wave-length of the electrons is of order of the nuclear dimensions. This deviation from Mott scattering at large scattering angles therefore provides a possibility for determination of the shape of the charge distribution and size of nuclei. In the case of a spherically symmetric charge distribution the nuclear charge density is immediately obtained from the observed angular distribution by a Fourier transform. The effects of competing processes, inelastic collisions with nuclear excitation or disintegration, atomic excitation or ionization and bremstrahlung are considered. It is shown that the first two competing effects may be disregarded if the electron energy is in the neighborhood of 50 Mev, the angle of scattering large (but not near π) and if the scattered electron has an energy equal to or nearly equal to the primary energy. With the latter condition fulfilled the bremstrahlung is reduced by the same factor as the elastic scattering and the two processes are indistinguishable.

I. INTRODUCTION

RECENT developments in the direction of obtaining high energy electron beams, as in the betatron and synchrotron, raise the question of using these high energy electrons in scattering experiments to obtain some information regarding nuclear structure. It will be readily recognized that for large angle scattering of electrons of several Mev ($\sim$50 Mev) the intensity of the scattering will depend strongly on the interaction between charges at very small distances. Specifically, the scattering nucleus cannot be regarded as a point charge but must be represented by an extended charge distribution whose shape can be explored by the scattering of electrons which penetrate inside the nucleus. Thus, even if the electrostatic interaction between charged elements is Coulombian down to essentially zero separation, the angular distribution of the scattered electrons will deviate markedly from the so-called Mott scattering.[1]

Actually there are two questions of considerable importance which are involved in the interpretàtion of scattering experiments of the kind under discussion. These concern (1) the nature of the (electric) interaction between charged particles at very small distances of separation and (2) the charge distribution and size of nuclei. Clearly, at least in principle, one may obtain information as to either question if the answer to the other is known. Unfortunately there is no case in which the nuclear charge distribution is known sufficiently well to allow a determination of possible deviations from the Coulomb interaction. However it should be possible to obtain evidence on this point from the scattering of high energy electrons by protons. It is plausible that deviations from Coulomb interaction, if such exist, are too small to be significant from the point of view of the accuracy obtainable in scattering measurements and we omit consideration of such deviations in the following.

With the assumption of an electrostatic Coulomb interaction between charges[2] it is possible to determine the charge density explicitly in terms of the nuclear form factor or observed scattering intensity. Applications to two important cases are immediately apparent: (1) Scattering in deuterium from which one may hope to obtain, for the first time, detailed information as to the deuteron wave function thereby establishing a criterion for the validity of nuclear force models. Here the effect of the non-central forces are unimportant unless the electron energy is several hundred Mev; i.e., the square of the reduced deBroglie wave-length is about equal to the quadrupole moment of the deuteron. (2) Scattering in heavier nuclei. Here the charge density is uniform, or nearly so, so that the

[1] E.g., see H. A. Bethe, *Handbuch der Physik*, XXIV/1, p. 495f.

[2] Effects of magnetic interaction with the nuclear spin are negligible.

primary information to be obtained is a measurement of nuclear radii.[3]

In the following these considerations are given quantitative formulation. Results for the angular distribution in the cases mentioned above are given as an indication of the magnitude of the effect. Finally, consideration of the feasibility of the measurements from the point of view of competing processes is presented.

II. ELASTIC SCATTERING BY EXTENDED CHARGE DISTRIBUTIONS

In the following the electron energy is sufficiently large to make the effect of nuclear penetration important, which implies extreme relativistic energies, but not so large that magnetic effects from nuclear recoil need be considered. For this reason and because of the considerations presented below our considerations are restricted to electron energies of order 50 Mev. The differential cross section for scattering electrons of total energy W into solid angle $d\Omega$ is

$$\sigma d\Omega = \frac{d\Omega}{4\pi^2}\frac{W^2}{(\hbar c)^4}|V_{if}|^2 \tag{1}$$

where

$$V_{if} = Ze^2\int\int d\tau_e d\tau_p \Psi^*(\mathbf{r}_p)\psi_f^*(\mathbf{r}_e) \times V(|\mathbf{r}_e - \mathbf{r}_p|)\Psi(\mathbf{r}_p)\psi_i(\mathbf{r}_e). \tag{2}$$

Here $\mathbf{r}_e$ and $\mathbf{r}_p$ are vectors defining the position of the electron and a volume element of protonic charge while the interaction energy between the electron and unit charge of the latter is represented by eV. The charge density in the nucleus is

$$\rho(\mathbf{r}_p) = |\Psi(\mathbf{r}_p)|^2$$

which is normalized to unity

$$\int \rho(\mathbf{r}_p)d\tau_p = 1. \tag{3}$$

The subscripts i and f on the electron wave functions refer to initial and final states and are taken to be plane waves. Thus

$$\psi_i = a_i(\mathbf{P})\exp(i\mathbf{P}\cdot\mathbf{r}_e/\hbar),$$
$$\psi_f = a_f(\mathbf{P}')\exp(i\mathbf{P}'\cdot\mathbf{r}_e/\hbar),$$

where $\mathbf{P}$ and $\mathbf{P}'$ are the initial and final momenta while a_i and a_f are Dirac amplitudes for the plane wave.

We allow for non-central fields (quadrupole moment!) by writing

$$\int V(|\mathbf{r}-\mathbf{r}_p|)\rho(\mathbf{r}_p)d\tau_p = \sum_0^\infty (2l+1)v_l(r)P_l(\cos\beta) \tag{4}$$

in which the subscript e has been dropped. In (4) the polar axis ($\beta=0$) is the direction of quantization for the nuclear spin. We introduce

$$\psi_f^*\psi_i = a_f^*a_i e^{i\mathbf{q}\cdot\mathbf{r}} = a_f^*a_i\sum_0^\infty(2l+1)j_l(qr)P_l(\cos\Theta), \tag{5}$$

$$j_l = i^l(\pi/2qr)^{\frac{1}{2}}J_{l+\frac{1}{2}}(qr) \tag{5a}$$

where Θ is the angle between $\mathbf{q}$ and $\mathbf{r}$, J is the Bessel function and $\hbar\mathbf{q}$ is the change of momentum. In terms of the scattering angle ϑ we have

$$q = 2P/\hbar \sin\vartheta/2.$$

After integration over the angular coordinates, the matrix element becomes

$$V_{if} = 4\pi Ze^2(a_f^*a_i)\sum_0^\infty(2l+1)K_l(q)P_l(\cos\theta)$$

where

$$K_l = \int_0^\infty v_l(r)j_l(qr)r^2dr \tag{6}$$

and θ is the angle between $\mathbf{q}$ and the spin axis. Averaging over all directions of the nuclear spin, we get

$$|V_{if}|^2 = 16\pi^2Z^2e^4|a_f^*a_i|^2\sum_0^\infty(2l+1)K_l^2. \tag{7}$$

Summing over final spin states of the electron and averaging over initial spin states we obtain

$$|a_f^*a_i|^2_{\text{Av}} = c^2/W^2(m^2c^2+P^2\cos^2\vartheta/2). \tag{8}$$

From (1), (7) and (8) the angular distribution is

$$\sigma(\vartheta) = \left(\frac{Ze^2m}{2P^2\sin^2\theta/2}\right)^2 \times(1+(P/mc)^2\cos^2\vartheta/2)\sum_0^\infty(2l+1)f_l^2 \tag{9}$$

[3] Such deviations from the uniform distribution as may arise from the electrostatic repulsion between the protons are sufficiently small to be ignored.

where

$$f_l(q)=q^2K_l(q). \tag{9a}$$

The ratio of expected scattering to Mott scattering[4] is, therefore,

$$\sigma/\sigma_M=\sum_0^\infty(2l+1)f_l^2. \tag{10}$$

It may be noted that the small angle scattering is determined chiefly by the isotropic term in (10) and for $q\approx 0$ the scattering is unchanged. Therefore the total cross section for scattering will be affected very slightly, in agreement with observations of cascade showers in the cosmic radiation, whereas the large angle scattering will be materially reduced by the penetration effect.

a. Scattering by Central Fields

It is clear that deviations from central symmetry make a non-vanishing contribution to the scattering so that in principle one might use such measurements for the determination of nuclear quadrupole moments. However, at energies for which such quadrupole contributions are appreciable the penetration effect arising from the spherically symmetric part of the charge distribution would be important and the two effects would have to be disentangled. For the sake of simplicity we consider only those cases wherein deviations from central symmetry produce negligible or vanishing effects; that is, the quadrupole moment may vanish or be small compared to 10^{-24} cm², or the electron energy may have some intermediate value for which the monopole effect is appreciable and the quadrupole effect is very small.[5]

For the monopole term we have from (6) and (5a)

$$K_0=f_0/q^2=\int_0^\infty\frac{\sin qr}{qr}v_0(r)r^2dr$$

so that K_0 is proportional to the Fourier transform of v_0. Inverting we get

$$v_0=\frac{2}{\pi}\int_0^\infty f_0(q)\frac{\sin qr}{qr}dq.$$

For the Coulomb interaction the charge density in the nucleus is given by

$$\rho(r)=-\frac{1}{4\pi r^2}\frac{d}{dr}r^2\frac{dv_0}{dr}$$

$$=\frac{1}{2\pi^2 r}\int_0^\infty f_0(q)q\sin qr dq. \tag{11}$$

Since f_0 is real and for a point charge

$$f_0(q)=1$$

it follows that

$$f_0(q)=(\sigma/\sigma_M)^{\frac{1}{2}}$$

and

$$\rho(r)=\frac{1}{2\pi^2 r}\int_0^\infty(\sigma/\sigma_M)^{\frac{1}{2}}q\sin qr dq. \tag{12}$$

Alternatively the deviation from a point charge distribution is expressed by

$$\rho(r)-\frac{\delta(r)}{4\pi r^2}=\frac{1}{2\pi^2 r}\int_0^\infty[(\sigma/\sigma_M)^{\frac{1}{2}}-1]q\sin qr dq. \tag{13}$$

Either (12) or (13) permit the determination of the shape of nuclear charge distributions directly from experimental data.

b. Scattering by Deuterium Nuclei

For the purpose of illustration we consider the example of electron scattering by a relatively extended nuclear charge distribution, viz: the deuteron for which a reasonable estimate of the nuclear wave function can be made. From (11) we find in general

$$f_0=4\pi\int_0^\infty\rho(r)\frac{\sin qr}{qr}r^2dr. \tag{14}$$

For not too large q this may be written

$$f_0=1-\frac{1}{3!}q^2(r^2)_{\text{Av}}+\frac{1}{5!}q^4(r^4)_{\text{Av}}-\cdots. \tag{14a}$$

Appreciable penetration effects may therefore be expected for scattering angles as small as $\hbar/PR$

[4] The first two factors in (9) give, of course, the Born approximation to the scattering by a point charge. A first order correction to the Born approximation consists in replacing the second factor by

$$1+(P/mc)^2\cos^2\vartheta/2+(\pi e^2ZPW/\hbar m^2c^4)\sin\vartheta/2,$$

cf. P. Urban, Zeits. f. Physik 119, 67 (1942).

[5] For small q it follows from (6) that

$$f_l\approx\delta_{l0}+\text{const.}(qR)^{l+2}$$

where R is a length of order of nuclear dimensions, δ_{l0} is the Kronecker symbol.

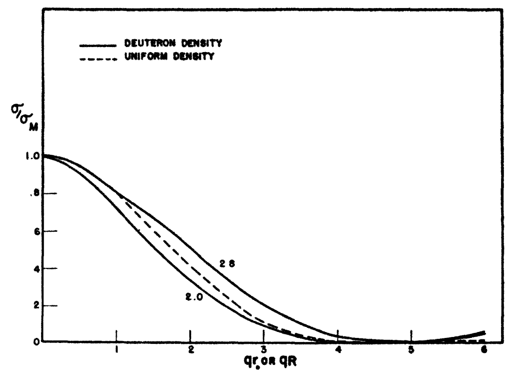

FIG. 1. Full curves give the ratio of expected elastic scattering to Mott scattering in deuterium as a function of qr_0 with $\hbar q$ the change in momentum and r_0 the range for a central square well. Numbers affixed to the curves give r_0 in units 10^{-13} cm. The dashed curve shows the same scattering ratio for a nucleus with constant charge density and radius R as a function of qR.

where again R is of order of nuclear dimensions. For 50 Mev electrons this is an angle of about 10°.

Using the wave function of the deuteron corresponding to a central square well potential of range r_0 and depth V_0 the scattering can be easily calculated from (14). We find

$$f_0=\frac{\cos^2 b}{1+a}\cos\frac{\xi}{2}+\frac{a}{1+a}\left\{4e^{2a}(G/\xi)\sin^2 b+\frac{2}{\xi}Si\left(\frac{\xi}{2}\right)+\frac{1}{\xi}\left[Si(2b-\tfrac{1}{2}\xi)-Si(2b+\tfrac{1}{2}\xi)\right]+\frac{1}{2}\left(1-\frac{\xi}{b}\right)\frac{\sin(2b-\frac{1}{2}\xi)}{2b-\frac{1}{2}\xi}+\frac{1}{2}\left(1+\frac{\xi}{b}\right)\frac{\sin(2b+\frac{1}{2}\xi)}{2b+\frac{1}{2}\xi}\right\} \quad (15)$$

where

$$Si(y)=\int_0^y \sin t\,dt/t, \quad \xi=qr_0,$$
$$a=(M\epsilon)^{\frac{1}{2}}r_0/\hbar, \quad b=[M(V_0-\epsilon)]^{\frac{1}{2}}r_0/\hbar \quad (15a)$$

and ϵ is the binding energy $=2.17$ Mev. The quantity

$$G=\int_{\frac{1}{2}}^{\infty} e^{-4ax}\sin\xi x\,dx/x =\arctan\xi/4a-\int_0^{\frac{1}{2}} e^{-4ax}\sin\xi x\,dx/x$$

is computed numerically.

The ratio to Mott scattering, that is, f_0^2, is given as the full curves in Fig. 1 for ranges $r_0=2.0$ and 2.8×10^{-13} cm. For the scattering of 50 Mev electrons at $\vartheta=\pi/2$ the scattering is reduced to 23 percent of the Mott scattering in the case of the larger range. At a larger scattering angle, say $\vartheta=\pi$ the deviation from Mott scattering is, of course, even more striking; but of course the cross section becomes smaller and the measurements more difficult (cf. further Section IIIa and b below).

c. Scattering by Heavier Nuclei

As a second example we consider heavier nuclei for which it is reasonable to assume a constant charge density. We find directly from (14)

$$f_0=\frac{3}{\eta^2}\left(\frac{\sin\eta}{\eta}-\cos\eta\right) \quad (16)$$

where $\eta=qR$ and R is the nuclear radius. The scattering ratio f_0^2 is given as the dashed curve in Fig. 1. For typical values of the nuclear radius the large angle scattering is again reduced from the Mott value by a large amount.

III. COMPETING PROCESSES

In order to form some idea as to the feasibility of the proposed scattering experiments we consider what might be expected from concomitant processes. These are (1) excitation and disintegration of the nucleus, (2) inelastic scattering involving atomic excitation and ionization, and (3) bremsstrahlung.

a. Nuclear Excitation and Disintegration

While it is rather difficult to make a quantitative estimate of the angular distribution of scattering with nuclear excitation or disintegration, the total cross section for this process may be evaluated with sufficient accuracy. Since the collision considered takes place mainly through the virtual quanta emitted by the deflected electron and the consequent photo-effect of these quanta, the angular distribution of the scattered electrons will show a strong forward peak very much like that exhibited by the elastic scattering. Therefore, under conditions which make the total cross section for scattering with nuclear excitation small compared to the total elastic scattering cross section, the former may be disregarded.

The total cross section for nuclear disintegration may be calculated by the Williams-Weiszäcker method.[6] For the ejection of a single particle from the nucleus the total cross section is approximately

$$\sigma_{\text{dis}} \sim \frac{\pi e^4}{Mc^2 E_0} \log \frac{MW}{mE_0} \tag{17}$$

where E_0 is the threshold energy and M the mass of the ejected particle. In (17) some numerical factors of order unity have been omitted.[7] For the ejection of a single particle (neutron, or proton) with $E_0 = 6$ Mev and $W = 50$ Mev we have $\sigma_{\text{dis}} \sim 10^{-4}$ barn. The total elastic scattering cross section can be evaluated quite easily. Since the major contribution comes from small angles we have for the differential cross section at high energies (cf. reference 9)

$$\sigma_{\text{el}}(\vartheta) = \left(\frac{Ze^2 \cos\vartheta/2}{2cP \sin^2\vartheta/2}\right)^2 (1-F)^2 \tag{18}$$

where F is the atomic form factor arising from the scattering by orbital electrons.[8] We find for the total elastic scattering cross section

$$\sigma_{\text{el}} \approx 6Z^{4/3}(\hbar/Mc)^2 \tag{19}$$

which is enormously greater than σ_{dis}. While (19) includes essentially unobservable scattering at $\vartheta = 0$ the same is true of the cross section for disintegration, Eq. (17).

In order to be more certain that the disintegration cross section is also negligible at large scattering angles a comparison of the angular distributions for elastic scattering and inelastic scattering with disintegration may be made in the one case where the latter can be readily calculated, *viz*; disintegration of the deuteron.[9] The differential cross section for high energy electrons is approximately

$$\sigma_{\text{dis}}(\vartheta) = \frac{8}{3\pi}\frac{m}{M}\left(\frac{e^2}{Mc^2}\right)^2 \epsilon^{\frac{1}{2}} \int_0^{W-mc^2-\epsilon} \frac{dE E^{\frac{1}{2}}}{(E+\epsilon)^4} \times \left[\frac{W^2+W'^2}{(\hbar cq)^2-(W-W')^2} - \frac{1}{2}\right] \tag{20}$$

where W and W' are the initial and final electron energies and $E = W - W' - \epsilon$ is the kinetic energy of the nucleons. The evaluation of (20) to give the angular distribution leads to the following conclusions: (1) The comparison of total cross sections as given above is somewhat too optimistic insofar as the decrease of $\sigma_{\text{dis}}(\vartheta)$ with the angle of deflection is not nearly so rapid as is the decrease of the elastic differential cross section $\sigma_{\text{el}}(\vartheta)$. (2) Since $\sigma_{\text{dis}}(\vartheta)$ varies only slowly with energy (cf. Eq. (17)), and the differential cross section $\sigma_{\text{el}}(\vartheta)$ varies as W^{-2} (except in the backward direction), the disintegration effect can be neglected only if the electron energy is not greatly in excess of 50 Mev. (3) The scattering angle at which the electrons are observed should not be too near π since in this case elastic scattering is very much reduced (because of factor $P^2 \cos^2\vartheta/2$) and will be considerably larger than $\sigma_{\text{dis}}(\vartheta)$ only for energies so low that the effect of electron penetration into the nucleus is negligible. With $W = 50$ Mev and $\vartheta = \pi/2$, which values represent favorable conditions, $\sigma_{\text{dis}} = 6 \times 10^{-7}$ barn and $\sigma_{\text{el}} = 1.6 \times 10^{-5}$ barn.

b. Ionizing Collisions

The cross section for inelastic collisions with the orbital electrons is not negligible compared to the elastic scattering. However, competition due to such collisions is unimportant if it is arranged to observe electrons which have energies equal or nearly equal to the primary energy and are scattered through angles other than 0 or π. This follows from simple energy and momentum considerations which show that the primary electron scattered through an angle $\mu = \arccos\vartheta$ or a secondary traveling in the same direction has a total energy given by

$$\frac{W'}{mc^2} = \frac{W + mc^2 + \mu^2(W - mc^2)}{W + mc^2 - \mu^2(W - mc^2)}. \tag{21}$$

Here the binding of the secondary in the initial state is neglected. Therefore, for $W \gg mc^2$ the secondary energy W' cannot be large unless $\mu^2 \approx 1$. Therefore, under the conditions $W \gg mc^2$ and ϑ in the range $\pi/2 \pm \pi/4$ say, there will be no fast electrons which were not elastically scattered. These conditions are the same as those providing a large ratio of elastic scattering to nuclear disintegration.

[6] E. J. Williams, K. Danske Vidensk. Selskab. **13**, no. 17 (1934–36).

[7] For energies such that $\hbar c/W$ is larger than nuclear dimensions the argument of the *log* in (17) is multiplied by W/Mc^2 which reduces the cross section. Cf. further reference 9.

[8] Cf. H. A. Bethe, Ann. d. Physik **5**, 325 (1930).

[9] H. A. Bethe and R. Peierls, Proc. Roy. Soc. **148**, 146 (1935).

c. Bremsstrahlung

The bremsstrahlung competes with the elastic scattering in a way which would make interpretation of measurements ambiguous except when the deflected electrons have energies about equal to the primary energy. In this case, the quanta emitted are soft and, as is well known, the angular distribution of the scattered electrons is precisely the same as the distribution of elastically scattered electrons.[10] Obviously the bremsstrahlung will also be reduced by the same factor f_0^2 due to nuclear penetration and thus, under the conditions cited, one need not distinguish between elastic scattering and bremstrahlung.

[10] F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).

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Ionization Currents in Divergent Fields in Hydrogen and in Air

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Ionization currents were measured in hydrogen and in dry, mercury-free air between concentric cylinders for pressures from 0.01 mm to 760 mm. Above the pressures corresponding to the minimum sparking potentials, $(1/r)(1/p)\log i/i_0$ was a single-valued, continuous function of $(X/p)_{max}$. Calculated values of apparent α for air agreed with uniform field values at low X/p but were high at higher X/p. In hydrogen apparent α was high over the full range of variables studied. Auxiliary anodes permitted estimates of the location of the ionization which was found to agree with the spatial distribution predicted by Morton. Electrons released in the low and collected in the high field produced the same ionization as for the reverse case except at low pressure in hydrogen where about 10 percent less ionization occurred for electrons released in the low field. The peaks of Morton's i/i_0 *vs.* p curves were identified with the minimum sparking potential. Application of the back diffusion equation of Rice permitted the evaluation of i_0 at higher pressures than hitherto has been possible. Comparison of the present results with those reported by Fisher and Weissler show that the present results can be directly applied if the focal length of the point parabola is substituted for r in the electron multiplication parameter. On the basis of this agreement, it is expected that the present values of the parameter may be applied to any geometry in which the field varies inversely as the distance from a fixed point. A necessary condition for the application of the observed values of the parameter to any system is that the electrons give up all of their ionizing energy to the gas before being collected.

INTRODUCTION

IONIZATION by collision by electrons in a gas has been under experimental and theoretical investigation since about 1900. Since that time, the practical applications of the basic phenomena have assumed great proportions in industry. Yet, at this time, one cannot with assurance predict what the electron multiplication will be, except in plane parallel gaps or in other special geometries which have been given particular study.

The early work of Townsend beginning about 1900 established for plane parallel gaps the well known law,

$$i = i_0 e^{\alpha d}, \qquad (1)$$

where i is the total current collected at the anode, i_0 is the primary current, d is the distance between the electrodes, and α, the First Townsend Coefficient, is the number of new electrons produced by an electron in moving one centimeter through the gas in the field direction. In addition to the relationship given in Eq. (1), it was shown that α/p was a single-valued and continuous function of X/p, where X is the field strength in volts/cm and p is the pressure in mm

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