Second Sound and the Heat Conductivity in Helium II

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By introducing a time of relaxation in a wave equation for second sound (1) relations between the dispersion of second sound and ordinary sound are derived; (2) the high heat conductivity of He II is related to second sound in the same way as the normal heat conductivity is related to ordinary sound on Debye's theory; (3) values for the time of relaxation and the velocity of of the heat flow are computed which are practically the same as for the flow of liquid He II through narrow slits above the critical velocity, indicating that in both cases the restriction of the flow is due to the same mechanism.

1. INTRODUCTION

L. TISZA[1,2] and L. Landau[3,4] independently derived a wave propagation in helium II generally called "second sound," by considering helium II as a mixture of two sorts of particles with different energy contents. These waves consist of fluctuations of the relative density of the two sorts of particles, the total density remaining constant in first approximation. In their derivation of the wave equation both authors use separate and independent continuity equations for the motions of the different sorts of particles. This procedure implies that no interchanges take place between the two sorts of particles during the time of vibration.

A similar condition has been investigated for the transmission of ordinary sound in a dissociating gas by A. Einstein.[5] Here also two different sorts of particles with different energy contents take part in the vibrations. Usually the time of a cycle in ordinary sound is so long that the two sorts of particles can interact with each other sufficiently to maintain the state of equilibrium during all phases of the vibration. But if the frequency is increased sufficiently the time necessary to establish equilibrium (generally called the time of relaxation) becomes larger than the time of one cycle. Einstein showed that this causes a change in the velocity of sound, with a region of dispersion and absorption in the range of frequencies where the time of relaxation is comparable with the time of one cycle.

[1] L. Tisza, C. R. Acad. Sc. **207**, 1035, 1186 (1938).
[2] L. Tisza, J. de phys. et rad. **VIII**, 1, 350 (1940).
[3] L. D. Landau, J. Phys. U.S.S.R. **5**, 71 (1941).
[4] L. D. Landau, J. Phys. U.S.S.R. **11**, 91 (1947).
[5] A. Einstein, Sitzungsberichte Berl. Akad. **1920**, 380.

2. THE DISPERSION OF SECOND SOUND AND OF ORDINARY SOUND

It is the aim of this paper to show that by introducing the same conception of a time of relaxation in the equations for second sound, relations between the dispersion of second sound and of normal sound can be obtained, and furthermore the extremely high heat conductivity of helium II can be associated with second sound in practically the same way as the normal heat conductivity accompanies ordinary sound on Debye's theory,[6] yielding simultaneously very reasonable values for the time of relaxation.

In order to simplify the discussion we assume, like Tisza and Landau, that the energy content of the particles in the lower state of helium II is negligible compared with that of the others,* so that the density of these latter so-called "excited" particles alone determines the thermodynamic properties in first approximation. We therefore write the wave equation for fluctuations of the concentration of the excited particles ρ_e, including a relaxation term:

$$\nabla\cdot\nabla\rho_e=\frac{1}{v^2}\left[(\partial^2\rho_e/\partial t^2)+\frac{1}{\tau}\dot{\rho}_e\right] \qquad (1)$$

v being the velocity of second sound, and τ the time of relaxation. Physically this relation implies that as long as τ is great compared with the period of fluctuations of ρ_e, such a fluctuation

[6] P. Debye, *Vortraege ueber die kinetische Theorie der Materie* (Leipzig, 1914) p. 43.

* This assumption does not mean that we assume a Bose-Einstein condensation. It would hold equally well also for a hydrodynamical model of the Landau type, or even for an order-disorder model of the type suggested by H. Frohlich (Physica 's Gravenhage **4**, 6391 (1937)) and H. Jones (Proc. Camb. Phil. Soc. **34**, 253 (1938)).

can be propagated as a wave, but when τ is small compared with this period, a local change of ρ_e is completely dissipated by a diffusion process and any "second sound" wave would suffer immediate damping.

Because ρ_e is temperature dependent, fluctuations of ρ_e will occur also in ordinary sound waves, in complete analogy to the fluctuations in concentration in a dissociating gas considered by Einstein. The time of relaxation introduced into (1) is then essentially the mean time between exchanges of excitation energy, just as in Einstein's theory (see, e.g., Rutgers[7]). This leads to the expectation that ordinary sound should show dispersion and absorption in the same frequency range as second sound.** The main difference between the two sets of phenomena would be that while ordinary sound should only change its velocity when passing through the critical frequency, showing an abcorption maximum there, second sound can exist only at higher frequencies, and will be entirely damped out at all frequencies lower than the critical value.

3. THERMAL RESISTANCE

Second sound waves imply a transport of heat,*** just as ordinary sound waves do in Debye's theory;[6] the damping of the waves limits the conductivity to finite values. The following discussion of thermal resistance in terms of second sound relaxation therefore represents a complete analogy to Debye's theory in terms of ordinary sound. Its validity depends on the assumption that the resistance accompanying ordinary sound is very large compared with that accompanying second sound; the two resistances being effectively in parallel, so that only the latter is important.

To obtain the thermal resistance we transform the fluctuations of concentration (1) into fluctuations of temperature. Under completely relaxed conditions ρ_e is a function of the equilibrium temperature T of the system:

$$\rho_e=\rho_e(T). \tag{2}$$

When the system is not completely relaxed we may still use this equation to define a temperature T_e of the excited particles:

$$\rho_e=\rho_e(T_e) \tag{2'}$$

where in general T_e will differ from a possibly measured temperature T by a certain quantity which depends on frequency. Putting (2') into (1) we obtain a temperature wave:

$$\nabla\cdot\nabla T_e=\frac{1}{v^2}\left[(\partial^2 T_e/\partial t^2)+\frac{1}{\tau}\dot{T}_e\right]. \tag{3}$$

It must be emphasized that T_e in this wave equation is not the measurable thermodynamic-equilibrium temperature, but the temperature of the excited particles as defined by (2').† To obtain now†† the relation between the relaxation damping according to (3) and the heat resistance, we use the equation for the conservation of heat under completely relaxed conditions which apply to actual measurements of heat conductivity:

$$\nabla\cdot h=-\rho C_p\dot{T} \tag{4}$$

where C_p is the true heat capacity, and T the true equilibrium temperature, h the heat flow. Under general conditions we must write T_e in place of T, and by using instead of C_p a different, suitably defined frequency-dependent value

[7] A. J. Rutgers, Ann. d. Physik **16**, 350 (1933).

** A difference in the two regions of dispersion could mean only that the damping of second sound is due also to other influences not present in ordinary sound, e.g., a mechanical viscosity between the two sorts of particles when vibrating in opposite directions. In helium II the exchanges of momentum and of excitation energy have usually been considered as the same process, but the experimental determination of the dispersion has to decide whether this assumption is true.

*** This is also consistent with Tisza's model, which leads to a simple classical picture of the nature of second sound: Helium II consists of two parts which may each transmit sound waves, (a) which are in phase with each other—ordinary sound, (b) in opposite phase to each other—second sound. Only so long as there is no appreciable relaxation between the two components can the second mode of vibration persist.

† The lack of temperature equilibrium between the two sorts of particles in second sound necessary to ensure propagation and not dissipation of a local fluctuation is overlooked in some derivations by introducing calorically measured thermodynamic quantities such as entropy, etc., which by definition are only valid for the state of equilibrium. (See, e.g., L. Landau,[3] page 85 and D. V. Gogate and P. D. Pathak, Proc. Phys. Soc. London **59**, 457 (1947).) The use of separate and independent continuity equations for the relative motion of the two sorts of particles is inconsistent with the introduction of statically measured quantities as entropy and temperature which are only valid for the completely relaxed system.

†† This argument is similar to that of H. Jones (Proc. Camb. Phil. Soc. **34**, 253 (1938)) in terms of the flow of disorder.

C_p^*,††† we may retain (4) in the form:

$$\nabla \cdot h = -\rho C_p^* \dot{T}_e. \tag{4'}$$

As relaxation increases C_p^* approaches C_p, and $T_e \rightarrow T$. We may neglect the second derivative on the right side of (3) under partially relaxed conditions if the rate of change of T is small enough:

$$\nabla \cdot \nabla T_e = \dot{T}_e/\tau v^2. \tag{5}$$

Comparing this with (4') we obtain

$$h = -\rho C_p^* \tau v^2 \nabla T_e \tag{6}$$

which becomes, for complete relaxation as present in the irreversible heat flow experiment:

$$h = -\rho C_p \tau v^2 \nabla T. \tag{7}$$

The wave of concentration (1) together with partial relaxation implies a motion of excited particles. We may therefore regard the heat flow as a transport of excitation energy by excited particles diffusing with an average velocity u:

$$h = (\epsilon/\rho)\rho_e u \tag{8}$$

where ϵ/ρ means the excitation energy per unit mass associated with the excited particles. Under the simplifying assumptions made at the beginning, the energy content of the excited particles represents practically the total heat content ρQ_T of helium II, so we may write

$$(\epsilon/\rho)\rho_e = \rho Q_T = \rho \int_0^T C_v dT \tag{9}$$

since ρ is practically independent of T. Therefore (8) becomes

$$h = \rho Q_T u. \tag{10}$$††††

Using the measured values of Q_T and h,[8,9,10] we can compute from (10) the velocity u as a function of T. Table I contains these values for a temperature gradient of 0.001°K per cm. This table also shows the values of τ calculated from (7) and the same experimental data on h and the observed values of second sound velocity.[11]

From the experimental fact that $h \propto (\nabla T)^{\frac{1}{2}}$ it follows that $\tau \propto 1/h^2$. (See Eq. (7).‡) Taking into account that the energy supplied to the system in second sound measurements is at least ten times smaller than for the heat conduction experiments used in Table I (1/25 watt over a cross section of several cm² in the sound tube compared with between 1 and 3 watts per cm² in the heat conductivity measurements), it must be expected that the time of relaxation is in this case at least a hundred times greater than the values given in Table I. Dispersion and absorption ought therefore to appear in both kinds of sound (at such an energy input) at around 1 to 10 cycles per second. This is not in conflict with Peshkov's[12] results, in which no dispersion was

††† Under the simplifying conditions mentioned above $C_p = \partial(\rho_e \epsilon)/\partial T$ where ϵ is the "excitation" energy per particle at the temperature T. For the completely unrelaxed system $\tau \rightarrow \infty$, $C_p^* = \epsilon \partial \rho / \partial T$ as without interaction, the excitation energy carried by the particles cannot adjust itself to the temperature.

†††† Equations (7), (8), (10) have been derived by us earlier (Phys. Rev. **71**, 828 (1947)) on the assumption that in the absence of a constriction the internal forces responsible for the fountain effect are still present, but produce an internal momentum density M according to the second law of motion:

$$\nabla p = \mu \nabla T = -dM/dt \tag{11}$$

where T is the temperature, p the fountain effect pressure difference and μ the fountain effect coefficient. L. Tisza (Phys. Rev. **72**, 353 (1947)) criticized this equation, stating that we had omitted a factor $\rho z/\rho e$ where ρz is the density of the particles in the lower state. This is due to a misunderstanding: The above Eq. (11) was written simply as a definition of the momentum in terms of the fountain effect, but is not the momentum appearing in Tisza's model. Incidentally the momentum cancels from the equations leading to (7), so that Tisza's factor makes no difference to our evaluation of τ and u from the observed values of k, C_p and v.

The equivalence of both derivations can be shown as follows: Comparing (8) with (6) yields:

$$-(\rho/\epsilon)\rho C_p^* v^2 \nabla T_e = \rho_e u/\tau \tag{12}$$

where $\rho_e u$ represents a momentum density. Equation (12) has the form of the second law of motion under conditions of considerable relaxation:

$$-\nabla p = \dot{M} + M/\tau \doteqdot M/\tau. \tag{13}$$

Comparing (12) with (13) gives:

$$\nabla p = (\rho/\epsilon)\rho C_p^* v^2 \nabla T_e \tag{14}$$

which represents a fountain pressure relation:

$$\nabla p = \mu \nabla T_e. \tag{14'}$$

[8] W. H. Keesom, B. F. Saris, L. Meyer, Physica 's Gravenhage **7**, 870 (1940).

[9] L. Meyer and J. H. Mellink, Physica 's Gravenhage **13**, 197 (1947).

[10] W. H. Keesom, *Helium* (Elzevier, Amsterdam–New York, 1942).

[11] C. T. Lane, H. A. Fairbank, W. M. Fairbank, Phys. Rev. **71**, 600 (1947).

‡ The derivation of Eq. (7) is, at least for one-dimensional flow as discussed here, practically unaffected by the fact that τ is a function of ∇T: Introducing into (5) instead of a constant τ, the expression

$$\tau = \gamma(\nabla T)^{-\frac{1}{2}}$$

leads to

$$h = 3\rho C_p v^2 \gamma (\nabla T)^{\frac{1}{2}} = 3\rho C_p v^2 \tau \nabla T.$$

[12] V. P. Peshkov, Vestnik Akademii Nauk **4**, 117 (1945); Nature **157**, 200 (1946); J. Phys. U.S.S.R. **10**, 389 (1946).

found in the frequency range from 100 to 10,000 cycles per second.

The fact that the compressibility of helium II derived from measurements by Burton,[13] and by Findlay, Pitt, Grayson-Smith and Wilhelm,[14] on ordinary sound in helium II differs from the statically measured values at Leiden[10] by about 15 percent, may be due to the dispersion effect; the frequencies used (e.g., 1338 kc by Burton) were certainly higher than the critical value to be expected by our estimates of the relaxation time.

The sudden change in velocity of ordinary sound at the lambda temperature may also at least partly be due to this same dispersion. Above the transition, the time of relaxation has vanished, whereas below, it has already become long compared with the period of the waves.

The velocities u calculated from (10) are of the same order of magnitude as the critical velocities u_c for the frictionless flow through narrow slits[9] given in column 3.‡‡ It is very interesting that in the supercritical region—i.e., above the critical velocity—the velocity of flow through narrow slits is approximately proportional to $(\nabla T)^{\frac{1}{2}}$,‡‡‡ just as (10) demands for u. This leads to the conclusion that the restriction of the flow of particles in the heat current is the same as in the supercritical flow through slits, both being controlled by the time of relaxation.‡‡‡‡

TABLE I. Values of u and of τ.

T °K	u cm/sec.	u_c cm/sec.	τ sec.
1.2	19	25	$9.\ 10^{-4}$
1.5	15	20	$7.\ 10^{-4}$
1.8	11	13	$6.\ 10^{-4}$
2.0	5	8	$5.\ 10^{-4}$
2.1	2	3	$4.\ 10^{-4}$

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[13] E. F. Burton, Nature **141**, 970 (1938).

[14] J. C. Findlay, A. Pitt, H. Grayson-Smith and J. O. Wilhelm, Phys. Rev. **54**, 506 (1938); Phys. Rev. **56**, 122 (1939).

‡‡ The values of u_c contain with regard to their absolute magnitude an uncertainty up to about 25 percent due to the difficulties in measuring the exact width of the slit.

‡‡‡ Reevaluation of the Leyden Measurements by L. Meyer and J. H. Mellink[9].

‡‡‡‡ Experimental evidence (see reference 9) already lead to the conclusion that the friction in the supercritical flow is not produced in the slit itself but at its ends where the jet of "film" leaves or enters the bulk of the liquid.