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The Scattering of High Energy Neutrons by Protons

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The scattering has been calculated, using the exact formula, for neutron energies between 20 and 80 Mev, with ranges of 2.0 and 2.8×10^{-13} cm for the nuclear force and using ordinary, exchange, and symmetric forces. The maximum of the scattered intensity at 180° is sensitive to the range of the forces; experiments with neutrons from 30 to 60 Mev would permit a rather conclusive determination of the range. The total cross section for forces of the exchange type is much smaller than that obtained from the Born approximation, even at 80 Mev.

THE scattering of neutrons by protons at energies from 20 to 80 Mev has been calculated, making different assumptions about the nuclear forces. Throughout the calculation a pure central force was used and the tensor force neglected. A square well potential was assumed in each case.

The range of the nuclear forces was taken to be either 2.8 or 2.0×10^{-13} cm^2. The first of these values is the one commonly assumed and is in agreement with the results from proton-proton scattering at low energies.[1] The smaller range is suggested, especially for the triplet interaction, by the experiments on scattering of neutrons by para-hydrogen.[2] We realize that this smaller range may result in disagreement with the experimental total cross section in the range from 2–5 Mev.

The three customary[3] types of forces were assumed, namely: ordinary forces, exchange forces, and "symmetric" forces, i.e., those derived from the symmetric meson theory which are proportional to the operator $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ ($\boldsymbol{\sigma}$ = spin, $\boldsymbol{\tau}$ = isotopic spin). On the basis of the recent experiments made with the Berkeley[4] cyclotron, ordinary forces may be considered as excluded and symmetric forces as the most likely.

The exact theory of scattering[5] was used throughout. For the determination of the phase shifts δ_l, a convenient method was developed which is described in the appendix. The depth of the potential well for the triplet interaction was determined to fit the binding energy of the deuteron, 2.19 Mev. The depth of the singlet interaction was determined to give the cross section of 20 barns for the scattering of slow neutrons by protons. The well depths, for the triplet and singlet interaction respectively, are then 21.3 and 11.9 Mev for the longer range, and 36.6 and 24.0 Mev for the shorter range of the forces.

The results for the phase shifts of the triplet and singlet states are given in Table I. As is to

[1] G. Breit, H. M. Thaston, and L. Eisenbud, Phys. Rev. 55, 1018 (1939).
[2] R. B. Sutton, T. Hall, E. E. Anderson, H. S. Bridge, J. W. DeWire, L. S. Lavatelli, E. A. Long, T. Snyder, and R. W. Williams, Phys. Rev. 72, 1147 (1947); See also C. S. Wu, L. J. Rainwater, W. W. Havens and J. R. Dunning, Phys. Rev. 69, 236 (1946).
[3] W. Rarita and J. Schwinger, Phys. Rev. 59, 556 (1941).
[4] E. M. MacMillan, private communication.
[5] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, 1933), p. 24.

be expected, the phase shifts for ordinary forces are all positive, while for the two other types, the phase shifts for odd l are negative (repulsive force). For a given energy and given l, the negative phase shift for the exchange force is smaller than the corresponding positive one for the ordinary force, because a repulsive potential is less effective than an attractive potential of the same magnitude. For the symmetric theory, the phase shifts δ_l are smaller than the corresponding ones for exchange forces in the triplet states of odd l, but larger in the singlet states, because the forces are multiplied by factors $\frac{1}{3}$ and 3, respectively. For the ordinary force, the phase shift generally decreases with increasing l for a given energy; however, this is reversed for $l=0$ and 1 when the wave-length inside the potential well becomes less than twice the range of the nuclear forces: this is a peculiar effect for a square well which can be verified directly from the formulas for the phase shifts.

The scattered intensity in the center-of-mass system, was calculated as a function of angle, and expressed in terms of a power series in $\cos\theta$, thus:

$$d\sigma = 2\pi C \sin\theta d\theta (A_0 + A_1 \cos\theta + \cdots + A_n \cos^n\theta + \cdots), \quad (1)$$

$$C = \lambda^2/16.$$

The coefficients A_n and C are given in Table II. It is interesting to note that the coefficient A_0 which gives directly the scattering through 90°, is determined entirely by the phase shifts for *even* l because the spherical harmonics of odd l vanish for $\theta=90°$. Therefore, the 90° scattering is independent of the type of forces assumed (always neglecting tensor forces).

In Table III, we have given the differential scattering cross sections at 180°, 90° and 0° corresponding to protons projected forward, at 45° and at 90° in the laboratory system. We have also given the ratio of the scattered intensities at 180° and 90°. It is seen that this ratio increases rapidly with energy for exchange and symmetric forces as is to be expected. For ordinary forces, there is first a decrease which again is expected, but then, at the higher energies, an increase; this is due to the well-known phenomenon[6] that the scattered intensity has a secondary maximum in the backward direction if the exact scattering formula is used. This maximum does not appear in the Born approximation and has been investigated in some detail for proton-neutron scattering by Marshak and Ashkin.[7] An illustration of the backward maximum is given in Fig. 1.

The maximum of the intensity at 180° for

TABLE I. Phase shifts (in degrees).

	Range 2.0×10^{-13} cm						Range 2.8×10^{-13} cm					
	Triplet			Singlet			Triplet			Singlet		
	Ord.	Exch.	Sym.	Ord.	Exch.	Sym.	Ord.	Exch.	Sym.	Ord.	Exch.	Sym.
20 Mev												
0	86.5			57.9			75.4			47.8		
1	6.4	−2.6	−1.07	3.35	−1.9	−4.0	20.5	−6.7	−2.9	8.0	−4.4	−9.1
2	0.13			0.08			0.75			0.4		
30 Mev												
0	76.1			53.0			63.2			41.0		
1	11.3	−4.4	−1.85	5.8	−3.2	−6.7	31.0	−10.6	−4.6	12.6	−7.0	−14.4
2	0.22			0.13			1.9			0.9		
3							0.08	−0.08	−0.03	0.04	−0.04	−0.17
40 Mev												
0	68.5			48.6			54.5			35.7		
1	16.3	−6.3	−2.7	8.4	−4.6	−9.5	37.5	−14.4	−6.2	16.4	−9.5	−19.5
2	0.65			0.4			3.6			1.75		
3	0.02	−0.02	−0.01	0	0	−0.03	0.20	−0.17	−0.05	0.10	−0.08	−0.27
80 Mev												
0	49.4			36.0			34.8			22.1		
1	31.0	−13.4	−5.6	17.2	−9.8	−20.1	40.2	−26.1	−10.5	22.0	−16.5	−36.2
2	3.1			1.9			13.8			6.5		
3	0.18	−0.17	−0.05	0.12	−0.08	−0.25	1.6	−1.2	−0.4	0.8	−0.7	−2.1
4							0.1			0.05		

[6] The phenomenon is due to the fact that the Legendre polynomials P_l alternate between +1 and −1 at 180° which permits only very incomplete destructive interference between the contributions of various values of l, whereas at other angles the values of P_l are more erratic and destructive interference more effective.

[7] Marshak and Ashkin, Phys. Rev., to be published.

TABLE II. Coefficients in Fourier expansion of differential cross section. See Eq. (1).

Energy	Range in 10^{-13} cm	Force	C in 10^{-28} cm^2	A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7
20 Mev	2.0	ord.	26.0	14.8	+2.0	+1.5	+0.16				
	2.0	exch.	26.0	14.8	−0.36	+0.32	−0.07				
	2.0	sym.	26.0	14.8	−0.71	+0.27	−0.04				
	2.8	ord.	26.0	13.2	+15.1	+14.5	+2.5	+0.29			
	2.8	exch.	26.0	13.2	−1.6	+2.4	−0.95	+0.19			
	2.8	sym.	26.0	13.2	−2.0	+1.9	−0.60	+0.28			
30 Mev	2.0	ord.	17.3	13.8	+7.0	+4.7	+0.44	+0.04			
	2.0	exch.	17.3	13.8	−1.4	+0.95	−0.19	+0.02			
	2.0	sym.	17.3	13.8	−2.1	+0.81	−0.12				
	2.8	ord.	17.3	10.4	+28.0	+32.4	+8.9	+1.7	+0.17		
	2.8	exch.	17.3	10.4	−3.2	+6.5	−3.7	+1.2	−0.17		
	2.8	sym.	17.3	10.4	−3.3	+5.2	−2.3	+1.7	−0.09		
40 Mev	2.0	ord.	13.0	+12.2	+12.9	+10.1	+1.9	+0.3			
	2.0	exch.	13.0	12.2	−2.5	+2.3	−0.8	+0.2			
	2.0	sym.	13.0	12.2	−2.4	+2.0	−0.5	+0.2			
	2.8	ord.	13.0	7.4	+31.2	+46.3	+18.8	+5.2	+0.9	+0.2	
	2.8	exch.	13.0	7.4	−3.6	+11.8	−9.0	+3.8	−0.7	+0.1	
	2.8	sym.	13.0	7.4	−3.8	+9.6	−5.5	+3.6	−0.4	+0.1	
80 Mev	2.0	ord.	6.49	6.5	+25.3	+38.0	+15.2	+3.1	+0.8	+0.1	
	2.0	exch.	6.49	6.5	−4.7	+10.9	−7.6	+3.0	−0.5	+0.1	
	2.0	sym.	6.49	6.5	−4.0	+9.5	−4.9	+3.1	−0.35	+0.1	
	2.8	ord.	6.49	0.59	+3.3	+37.2	+65.2	+54.5	+24.2	+7.2	+1.4
	2.8	exch.	6.49	0.59	+4.8	+15.5	−36.4	+47.7	−17.5	+5.6	−1.2
	2.8	sym.	6.49	0.59	+2.0	+9.5	−23.5	+45.2	−10.5	+5.4	−0.6

exchange and symmetric forces should be observable already at 20 Mev neutron energy (in the laboratory system) if the nuclear forces have a range of 2.8×10^{-13} cm; it becomes quite marked at 30 and very pronounced at 40 Mev. For the shorter range, nearly twice as much energy is required to produce the same asymmetry in the scattering; this is a direct consequence of the fact that the depth of the potential wells is nearly twice as great as for the longer range. Because of this great sensitivity of the angular distribution to the range of the forces, it seems very much worth while to make scattering experiments in the intermediate energy range, from about 20 to 60 Mev; such experiments would be especially conclusive after the symmetry properties of the forces have been established by experiments at higher energy. Intermediate energies have also the advantage that relativistic corrections will be unimportant, while they may be appreciable at 100 Mev. Also, probably the influence of tensor forces is relatively smaller at intermediate energies.

Figure 1 gives the angular distribution of the scattered neutrons in the center-of-mass system at 80 Mev neutron energy, for the various assumptions about the forces. The very deep minimum near 90° for $a=2.8\times10^{-13}$ is largely an accidental effect; there is near cancellation of the contributions from $l=0$ and $l=2$ to the

TABLE III.

Energy in	Range forces	Differential cross section in c.m. system per unit solid angle in 10^{-26} cm^2: at $\theta=0°$	at $\theta=90°$	at $\theta=180°$	Ratio 180/90	Total scattering cross section in barns (10^{-24} cm^2): Exact formula	Born approximation
20 Mev	2.0 ord	4.81	3.85	3.69	0.96	0.500	0.410
	2.0 exch	3.82	3.85	4.05	1.05	0.486	0.410
	2.0 sym	3.73	3.85	4.12	1.07	0.486	*
	2.8 ord	11.88	3.43	2.72	0.79	0.590	0.714
	2.8 exch	3.42	3.43	4.75	1.38	0.456	0.714
	2.8 sym	3.30	3.43	4.67	1.36	0.451	*
30 Mev	2.0 ord	4.48	2.38	1.92	0.81	0.333	0.336
	2.0 exch	2.27	2.38	2.83	1.19	0.306	0.336
	2.0 sym	2.14	2.38	2.91	1.22	0.305	*
	2.8 ord	14.08	1.79	1.28	0.72	0.466	0.529
	2.8 exch	1.89	1.79	4.37	2.44	0.277	0.529
	2.8 sym	2.00	1.79	3.97	2.22	0.268	*
40 Mev	2.0 ord	4.87	1.59	1.01	0.64	0.256	0.290
	2.0 exch	1.48	1.59	2.34	1.47	0.214	0.290
	2.0 sym	1.49	1.59	2.25	1.42	0.213	*
	2.8 ord	14.22	0.96	1.05	1.09	0.388	0.408
	2.8 exch	1.28	0.96	4.72	4.92	0.201	0.408
	2.8 sym	1.44	0.96	3.94	4.10	0.184	*
80 Mev	2.0 ord	5.78	0.42	0.42	1.00	0.155	0.165
	2.0 exch	0.50	0.42	2.16	5.14	0.0880	0.165
	2.0 sym	0.65	0.42	1.85	4.40	0.0837	*
	2.8 ord	12.57	0.038	3.52	92.6	0.203	0.206
	2.8 exch	1.24	0.038	7.75	204	0.131	0.206
	2.8 sym	4.32	0.038	6.05	159	0.111	*

* Very close to value for ordinary and exchanges forces.

scattered amplitude, and higher l's do not yet contribute appreciably. (At higher energies, this accidental cancellation disappears but there is a general tendency for the cross section to decrease, so that $\sigma(90°)$ actually stays low.) But even with $a=2\times10^{-13}$, the ratio between 180° and 90° is 5.1 and 4.4, respectively, for exchange and symmetric forces. It is apparent that the range of the forces is more important for the ratio 180°/90° than their type, i.e., whether they are pure exchange or symmetric, and this is even true for the 0° scattering. Moreover, the tensor forces will greatly increase the scattering at 90° while not affecting it appreciably at 180°; therefore, it would be premature to draw conclusions on the range or the type of nuclear forces by comparing the present calculations with experiment.

In Table III we have also given the total cross section. It is striking that at the higher energies the total cross section is much smaller for exchange than for ordinary forces. This is due to the fact that $|\delta_l|$ is much smaller for exchange than for ordinary forces for any odd value of l. We mentioned before that this is caused by the repulsive character of the exchange force in states of odd l. The main contribution to the total cross section at 80 Mev for ordinary forces comes from $l=1$; this contribution is very much reduced for exchange forces. The difference is more pronounced for the shorter range, because in this case the potential has a greater magnitude. This effect will gradually decrease with further increase in energy..

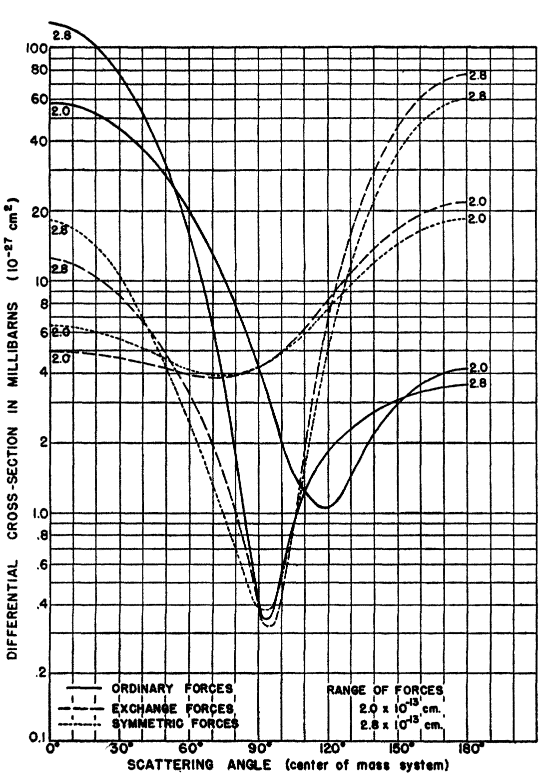

FIG. 1. Angular distribution of scattering of 80 Mev neutrons by protons.

We have also listed the cross sections according to the Born approximation. At 80 Mev there is quite good agreement between the Born approximation and the exact cross section in the case of ordinary forces. For the exchange types, the exact calculation gives much lower cross sections. The agreement in the case of ordinary forces is due to a compensation between two effects: the exact calculation will, in general, give greater phase shifts than the Born approximation for an attractive potential, but the cross section is proportional to $\sin^2\delta_l$ in the exact theory instead of δ_l^2 in the Born approximation. In the case of a repulsive potential, the exact δ_l is smaller in absolute magnitude than the Born approximation value, and the replacement of δ_l^2 by $\sin^2\delta_l$ causes a further reduction in the cross section. For the symmetric force, the calculations in the Born approximation were not carried out explicitly, but it was found that the result is nearly the same as for exchange forces. The same is seen to be true in the exact theory.

Since the nuclear forces appear to be of the exchange or symmetric type, the total cross section at 80 Mev, and probably still at higher energies, is considerably smaller than the value derived from the Born approximation. This tends to bring about better agreement with the experimental value of the cross section of about 0.08 barns[4] at 90 Mev. Shorter range of the forces would also help to lower the cross section and thereby improve the agreement with experiment. But it must again be remembered that these results may be modified by tensor forces, and that also relativistic corrections may easily have an effect of about 10 percent. The sign of these corrections is unknown.

At lower energies, agreement between Born approximation and exact calculation is not to be expected. Considering the magnitude of the nuclear potential, the actual agreement at the lower energies is surprisingly good. It might be expected that the Born approximation gives too large a cross section at lower energies—this, at least, is its behavior in most problems of *atomic* collisions. In contrast to this expectation, the Born approximation cross section is smaller than the exact one for 20 Mev and $a=2.0\times10^{-13}$ cm. This can be understood from a calculation at zero energy; here the Born approximation would give 0.6 barns as against the correct value of 20 barns. (With $a=2.8\times10^{-13}$, the Born approximation gives 1.8 barns.) The high value from the exact calculation is known to be caused by a resonance effect which, of course, cannot be simulated by the Born approximation. Since the latter gives a cross section which increases montonically with decreasing energy, it must fall below the actual cross section already at some finite energy.

APPENDIX

The radial wave function for a given l inside the potential well is given by

$$v=r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(\kappa r), \tag{2}$$

and outside the potential well by:

$$u=Ar^{-\frac{1}{2}}Z_{l+\frac{1}{2}}(kr), \tag{3}$$

where Z is the generalized Bessel function,

$$Z_{l+\frac{1}{2}}(kr)=J_{l+\frac{1}{2}}(kr)\cos\delta_l-N_{l+\frac{1}{2}}(kr)\sin\delta_l, \tag{4}$$

where J is the Bessel and N the Neumann function, and

$$k^2=ME/\hbar^2,\quad \kappa^2=M(E+V_0)/\hbar^2. \tag{5}$$

E is the energy in the center-of-mass system and V_0 the depth of the potential. The two functions u and v must join with continuous derivative at $r=a$. Setting the logarithmic derivatives equal, we obtain the condition:

$$k\frac{Z_{l+\frac{1}{2}}'(ka)}{Z_{l+\frac{1}{2}}(ka)}=\kappa\frac{J_{l+\frac{1}{2}}'(\kappa a)}{J_{l+\frac{1}{2}}(\kappa a)}, \tag{6}$$

where the prime denotes differentiation with respect to the argument. Equations (6) and (4) serve to determine the phase shift δ.

Equation (6) may be simplified by using the following relations between the generalized Bessel function:

$$\begin{aligned} Z_p'(x)&=Z_{p-1}(x)-(p/x)Z_p(x),\\ Z_{p+1}(x)&=(2p/x)Z_p(x)-Z_{p-1}(x). \end{aligned} \tag{7}$$

Therefore, we have

$$\begin{aligned} \frac{Z_{l+\frac{1}{2}}'(x)}{Z_{l+\frac{1}{2}}(x)}&=-\frac{l+\frac{1}{2}}{x}+\frac{Z_{l-\frac{1}{2}}(x)}{Z_{l+\frac{1}{2}}(x)}\\ &=-\frac{l+\frac{1}{2}}{x}+\cfrac{1}{\cfrac{2l-1}{x}-\cfrac{Z_{l-3/2}}{Z_{l-\frac{1}{2}}}} \end{aligned} \tag{8}$$

$$=-\frac{l+\frac{1}{2}}{x}+\cfrac{1}{\cfrac{2l-1}{x}-\cfrac{1}{\cfrac{2l-3}{x}-\cfrac{Z_{l-5/2}}{Z_{l-3/2}}}}. \tag{9}$$

Continuing to express the Bessel functions by those of lower order, we finally arrive at the ratio,

$$\frac{Z_{-\frac{1}{2}}(ka)}{Z_{\frac{1}{2}}(ka)}=\cot(ka+\delta). \tag{10}$$

Equating both sides of Eq. (6), the term $-(l+\frac{1}{2})/x$ cancels and we obtain

$$\cfrac{k}{\cfrac{2l-1}{ka}-\cfrac{1}{\cfrac{2l-3}{ka}-\cfrac{1}{\cfrac{\cdots}{\cfrac{1}{ka}-\cot(ka+\delta)}}}} =\cfrac{\kappa}{\cfrac{2l-1}{\kappa a}-\cfrac{1}{\cfrac{2l-3}{\kappa a}-\cfrac{1}{\cfrac{\cdots}{\cfrac{1}{\kappa a}-\cot\kappa a}}}}. \tag{11}$$

This can be solved and gives the formula,

$$\cot(ka+\delta)=\frac{1}{ka}-\cfrac{1}{\cfrac{3}{ka}-\cfrac{1}{\cfrac{5}{ka}-\cfrac{1}{\cfrac{\cdots}{\cfrac{2l-1}{ka}-\cfrac{ka}{\kappa a}\left(\cfrac{2l-1}{\kappa a}-\cfrac{1}{\cfrac{2l-3}{\kappa a}-\cfrac{1}{\cfrac{\cdots}{\cfrac{1}{\kappa a}-\cot\kappa a}}}\right)}}}}. \tag{12}$$

The use of this formula obviates the necessity of calculating the explicit formulae for the Bessel functions of high order. It also reduces the numerical work for higher l partly to that already done for lower l.

In some cases, the potential is repulsive and greater than the energy of the system. In this case κ comes out imaginary, and if we use $\kappa'=|\kappa|$ instead, we must replace Eq. (12) by

$$\cot(ka+\delta)=\frac{1}{ka}-\cfrac{1}{\cfrac{3}{ka}-\cfrac{1}{\cfrac{5}{ka}-\cfrac{1}{\cfrac{\cdots}{\cfrac{2l-1}{ka}+\cfrac{ka}{\kappa' a}\left(\cfrac{2l-1}{\kappa' a}+\cfrac{1}{\cfrac{2l-3}{\kappa' a}+\cfrac{1}{\cfrac{\cdots}{\cfrac{1}{\kappa' a}-\coth\kappa' a}}}\right)}}}}. \tag{13}$$

Note the change of sign and the change from cot to coth.

It is worth noting that in actual calculations with Eq. (13) and occasionally with (12), especially for large l, the two terms which have to be added in each denominator, tend to cancel each other so that considerable numerical accuracy is needed.

For the higher values of l it is convenient to use the recursion formula derived from perturbation theory:

$$\delta_l=\delta_{l-1}\frac{(ka)^2}{(2l+1)(2l+3)}. \tag{14}$$

This formula was found to be quite accurate if the phase was less than 1°.