PHYSICAL REVIEW VOLUME 73, NUMBER 2 JANUARY 15, 1948

On the Reactive Terms in Quantum Electrodynamics

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November 24, 1947

I. INTRODUCTION

THE recent discovery by Lamp and Retherford[1] of the displacement of the $2S_{\frac{1}{2}}$ level in the hydrogen spectrum and its subsequent interpretation by Bethe[2] as a reactive* effect of radiation have suggested an investigation of the role of the inertial and reactive terms in quantum electrodynamics. Such studies have up to now been largely frustrated by the appearance of divergences, but Bethe's results in the line shift problem indicate that, at least non-relativistically, it is possible to separate the inertial and reactive effects and to identify the latter as the level shift, in good agreement with experiment. Lamb[3] has shown that this can also be done relativistically, and that all the divergent effects are indeed associated with the electromagnetic mass. The procedure that has been used is to calculate consistently the electromagnetic self-energy of the electron in the hydrogen atom and to identify those terms which are associated with the increase in apparent mass of the electron through its interaction with the radiation field. Here one assumes that the electromagnetic mass of the electron is a small effect and that its apparent divergence arises from a failure of present day quantum electrodynamics above certain frequencies. It is impossible at present to say just what these frequencies are, but one might suppose them to be of the order of 137 $mc^2/\hbar$, or of nuclear dimensions, so that there is a region in which relativistic effects can manifest themselves and in which our present electrodynamics might be supposed to have a sort of validity. As long as the major contributions to the line shift come from considerably lower frequencies, it is possible to schematize this cut-off by using the phenomenological mass of the electron, which must include these effects. Under these conditions, it is unlikely that the exact nature of the failure of our electrodynamics can appreciably affect the results.

This viewpoint suggests that one re-examine some other areas in which the electrodynamics has failed, to see whether these considerations affect the conclusions that have been drawn. In particular, a problem closely connected to the line shift is that of the radiative effects on the scattering of electrons from an electrostatic field, which is just the dynamic analog to the line shift, both involving the simultaneous interaction of an electron with the electrostatic field of a scattering center, and with the electromagnetic field. Indeed, if one considers the line shift non-relativistically for a highly excited continuum level and expands the wave function in Born approximation, one obtains for the line shift just a sum of radiative scattering corrections of the form (2) below, corrected for the fact that the scatterings involved are virtual rather than real. We would like, therefore, to study the radiative effects on electron scattering in order to see whether the electromagnetic mass of the electron plays any part in the problem.

II. REACTIVE EFFECTS ON ELECTRON SCATTERING

This problem has been studied in a non-relativistic approximation by Bloch and Nordsieck,[4] Braunbeck and Weinmann,[5] and Pauli and Fierz,[6] relativistically by Dancoff,[7] and then discussed in detail by Bethe and Oppenheimer.[8]

If one calculates non-relativistically in perturbation theory, the cross section for the elastic

[1] W. E. Lamb, Jr. and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[2] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
* We use the term reactive to apply to the radiative effects which survive after the electromagnetic inertial effects are properly taken into account.
[3] W. E. Lamb, Jr., unpublished.
[4] F. Bloch and A. Nordsieck, Phys. Rev. **52**, 54 (1937).
[5] W. Braunbeck and E. Weinmann, Zeits. f. Physik **110**, 360 (1938).
[6] W. Pauli and M. Fierz, Nuovo Cimento **15**, 167 (1938).
[7] S. M. Dancoff, Phys. Rev. **55**, 959 (1939).
[8] H. A. Bethe and J. R. Oppenheimer, Phys. Rev. **70**, 451 (1946).

scattering of an electron of momentum $\mathbf{k}$ into the solid angle $d\omega$, with final momentum $\mathbf{p}$, one obtains

$$d\sigma=(mV_{kp}/2\pi\hbar^2)^2d\omega, \tag{1}$$

where m is the mass of the electron, and V_{kp} the matrix element of the electrostatic scattering potential V between the initial and final states. If one now includes the effects of interaction with the radiation field, in the usual way, one finds that the cross section is changed by an amount

$$\delta d\sigma=-\frac{2\alpha}{3\pi}\frac{(\mathbf{p}-\mathbf{k})^2}{m^2c^2}\cdot d\sigma\cdot\int_T^\infty\frac{dq}{q}, \tag{2}$$

where α is the fine structure constant, T the kinetic energy of the electron, q the frequency of the photon whose emission and re-absorption produces this effect, and, for consistency, m in both (1) and (2) must be specified to be the mechanical mass of the electron. This result is in itself somewhat surprising since, as is well known, the electromagnetic mass of the electron diverges linearly in non-relativistic approximation, so that the expression (2) seems to have no effects of a mass change represented. The reason for this omission may be understood in the following way: In the use of continuum perturbation theory, one is accustomed to assuming that the first- and second-order energy shifts caused by the perturbation are zero, which is quite correct for a scattering potential, but not for the interaction with radiation. The factor of m^2 in (1) comes from the density of states and from the flux corresponding to a given momentum, hence from a factor $p^2(dp/d\epsilon)^2$. If one carried out the perturbation theory consistently, one would insert the energy-momentum relationship that is given by the Hamiltonian to first order in α, therefore involving the linear divergence of the electromagnetic mass. However, one has made a sort of inadvertent separation of effects of order α, and has used here the empirical mass of the electron. This procedure is quite analogous to Bethe's approach to the line shift calculation. However, one must be quite careful in this, since in discarding some terms of order α, and keeping others, one must be sure that the ones that are kept are the correct ones. In this non-relativistic treatment the form of the perturbation calculation makes the separation between inertial and reactive effects of radiation automatic and correct, but, as we shall see, an unambiguous separation in a relativistic calculation, while also possible, is not automatically performed by the method of calculation. It is just this difference that led to the failure of Dancoff's relativistic attack on the problem, as we shall see later.

Indeed, one can predict immediately the effect of the mass in a relativistic calculation, by considering the effect of a small variation in mass in the relativistic equivalent to (1),**

$$d\sigma_{\mathrm{rel}}=(\epsilon V_{kp}/2\pi\hbar^2c^2)^2d\omega\left[\frac{1+(\mathbf{p}+\mathbf{k}/2mc)^2}{1+(\mathbf{p}/mc)^2}\right], \tag{1'}$$

where, as before, $|k|=|p|$. The terms are grouped in this way since in the hybrid perturbation theory ordinarily used the explicit effects of the electromagnetic mass would only appear in the term in square brackets. The other two factors of ϵ arise exactly as the m^2 term above, and one would ordinarily use the empirical mass of the electron in them. If one now increases the mass in the square brackets by a small amount μ, the change in cross section, to order v^2/c^2, is given by

$$\frac{\delta d\sigma_{\mathrm{rel}}}{d\sigma_{\mathrm{rel}}}=\frac{\mu}{2m}\frac{(\mathbf{p}-\mathbf{k})^2}{m^2c^2}, \tag{3}$$

and if one inserts for μ the electromagnetic mass of the electron, calculated in hole theory, $\mu=(3\alpha m/2\pi)\int^\infty(dq/q)$, one obtains an expression which differs only in the multiplicative constant from the divergent terms that Dancoff found, suggesting quite strongly that these are identifiable as manifestations of the electromagnetic mass of the electron. In fact, the numerical difference arises from Dancoff's omission of certain electrostatic transitions, which are, of course, essential to the covariance of the scheme, and, if included, make the agreement exact.

We will discuss this situation in somewhat greater detail in the next section.

** This argument is due Dr. S. Epstein, to whom I am indebted for permission to quote his results.

III. RELATIVISTIC EFFECTS

As motivation to a relativistic attack on the problem, Dancoff argued that, for energetic virtual photons, the relativistic mass increase of the recoil electrons cuts down the current matrix elements, and thereby would induce convergence into the expression (2) at the upper limit, if one were to consistently treat the recoil particles relativistically in hole theory. He performed a relativistic calculation of the radiative effects, and did indeed find that this was the case, but the relativistic treatment introduced a new set of terms with no non-relativistic analogs, which brought in a new divergence. In this calculation, he used the same hybrid perturbation theory that was discussed in the preceding section, thereby leaving out some divergent effects of the electromagnetic mass. It will be pertinent to our discussion to analyze in some detail the nature and origin of the new divergent terms, and we will show that they are all attributable to the (divergent) mass increase of the electron, as was suggested in the preceding section. What remains is just the expression (2) with an upper limit at kmc^2, where k is a number of order unity that can be calculated directly from the theory. Thus, the change of cross section caused by interaction with radiation is just

$$\delta d\sigma = -\frac{2\alpha}{3\pi}\frac{(\mathbf{p}-\mathbf{k})^2}{m^2c^2}\cdot d\sigma\cdot\ln\frac{kmc^2}{T}. \tag{4}$$

All of Dancoff's divergent terms (his $4d$, $4d'$, $4e$, $4e'$) involve transitions in which a negative energy electron excites itself to a positive energy state of the same momentum and spin through its interaction with the radiation field, or vice versa. Thus, an electron with momentum $\mathbf{p}$, and energy $-|\epsilon|$, could emit a photon with momentum $\mathbf{q}$, recoiling to the positive energy state $\mathbf{p}-\mathbf{q}$, and then re-absorb the photon, going to the state $\mathbf{p},|\epsilon|$. The net effect of the transition is just that the electron has changed the sign of its energy. There are also electrostatic terms of exactly this same nature, which Dancoff has omitted. The matrix element for the process above is

$$-\sum_{\mathbf{q}\lambda}\frac{2\pi e^2}{q(q+\epsilon_{\mathbf{p}}+\epsilon_{\mathbf{p}-\mathbf{q}})} \times\bar{X}_f(\boldsymbol{\alpha}\cdot\mathbf{1}_\lambda)(\Lambda_{\mathbf{p}-\mathbf{q}}^+-\Lambda_{\mathbf{p}-\mathbf{q}}^-)(\boldsymbol{\alpha}\cdot\mathbf{1}_\lambda)X_i, \tag{5}$$

where λ is the photon polarization, $\mathbf{1}_\lambda$ a unit vector in the direction of polarization, the X normalized spinors, and the Λ projection operators as indicated. The corresponding electrostatic term is

$$\sum_{\mathbf{q}}\frac{2\pi e^2}{q^2}\bar{X}_f(\Lambda_{\mathbf{p}-\mathbf{q}}^+-\Lambda_{\mathbf{p}-\mathbf{q}}^-)X_i, \tag{5'}$$

and the sum of (5) and (5′), since $\bar{X}_fX_i=0$, is

$$U_{\mathbf{p}-\mathbf{p}+}=\sum_{\mathbf{q}}\frac{2\pi e^2}{q^2\epsilon_{\mathbf{p}-\mathbf{q}}}\bar{X}_f\left\{H_{\mathbf{p}-\mathbf{q}}-\frac{q}{q+\epsilon_p+\epsilon_{p-q}}\sum_\lambda(\alpha\cdot 1_\lambda)H_{p-q}(\alpha\cdot 1_\lambda)\right\}X_i, \tag{6}$$

where H is the Dirac Hamiltonian. For large q this gives a divergent term

$$U_{p-p+}=-\sum_{\mathbf{q}}\frac{3\pi e^2}{q^3}\bar{X}_f\boldsymbol{\alpha}\cdot\mathbf{p}X_i=+\sum_{\mathbf{q}}\frac{3\pi e^2}{q^3}\bar{X}_f\beta mX_i$$
$$=\frac{3e^2m}{2\pi}\int^\infty\frac{dq}{q}\cdot\bar{X}_f\beta X_i, \tag{7}$$

so that these transitions behave as if they were caused by a perturbation equal to $\beta\cdot(3\alpha mc^2/2\pi)\times\int^\infty(dq/q)$. However, the coefficient of β here is just the divergent part of the electromagnetic mass of the electron, so that these transitions result from the electromagnetic mass effects of the radiation field and, if the empirical mass is used in the Hamiltonian $\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m$, one must omit transitions of this kind. Thus Dancoff's divergent terms arise entirely from the fact that the mass term is not diagonal in the Dirac Hamiltonian, so that a small (but divergent) increment in mass mixes positive and negative energy states, and makes new transition schemes possible.

In the case of the Pauli-Weisskopf theory, the situation is exactly analogous. The terms that give the highest (quadratic) divergence, are the terms in the square of the vector potential, whose matrix element for creating a pair in the sense discussed above is $(e^2/\pi\epsilon_p)\int_0^\infty qdq$, while the electromagnetic mass due to these terms gives exactly the same result when added to the zero-order Hamiltonian.

It should be noted that we have only shown that the divergent part of these matrix elements corresponds to a mass effect and, in fact, there are finite parts that do not, so that transitions of this type should properly be included to a finite extent. However, one is faced, in doing this, with the problem of separating a finite part of a formally infinite term, so that a specification of the way in which the infinite integral is performed is required for an unambiguous result.*** Schwinger[9] has given a procedure for separation of mass terms, by means of a canonical transformation on the Hamiltonian, which is equivalent to ours for this problem and which leads to finite terms of this character.

We conclude, therefore, that one can adopt a fairly clear procedure for eliminating the divergent mass effects from these transition problems. In the first place, one must use the empirical mass of the electron wherever the energy-momentum relationship is required, and second, must omit from the transition schemes transitions that are caused by mass increments.

*** An estimate of these finite terms indicates that they produce effects which are of order unity compared with the logarithm in (4), so that they are not important for low velocities.

[9] J. Schwinger, unpublished.

IV. CONCLUSIONS

We have seen that the use of the perturbation method in certain transition problems provides an automatic separation between the inertial and reactive effects of the radiation field. In a relativistic calculation, however, the non-diagonal character of βm in the Dirac Hamiltonian makes the separation of inertial effects a slightly more subtle procedure, and it must be performed explicitly. However, an unambiguous identification of divergent terms can indeed be made, and the reactive terms in the scattering of an electron by an electrostatic field are found to converge. The suggested procedure is easily extended to other types of fields, and one can see no clear obstacle to the calculation of the reactive effects on other problems involving electrons.

In conclusion, I would like to express my sincere appreciation to Professor J. R. Oppenheimer, for having suggested this problem, and for many helpful conversations concerning it.