Resonance Reactions Involving Dirac-Type Incident Particles

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An investigation has been made to determine whether the results of Wigner and Eisenbud can be extended to include the scattering of a particle which obeys a Dirac-type wave equation. It is found that no additional difficulties arise as a result of the use of Dirac particles, and that the treatment of Wigner and Eisenbud can be carried through step by step with relatively minor changes.

In order to carry out this program, however, it was necessary to determine the appropriate relativistic boundary conditions to be satisfied by the external and internal wave functions at the boundary of the internal region. It is hoped that a comparison of the relativistic and non-relativistic boundary conditions will lead to a clearer understanding of the boundary conditions in the non-relativistic problem.

THE present discussion attempts to show how the treatment of resonance reactions given by Wigner and Eisenbud[1] may be extended to include the scattering cross section (elastic and inelastic) of a neutron obeying a Dirac-type wave function incident on a nucleus of zero spin. The development and notation used in this discussion will closely parallel that of reference 1, which will be referred to as A.

As in A, functions $F_s(r_s, \Omega_s)\psi_s(i_s)$ are defined such that in the external region the most general solutions of the quantum-mechanical equations with a definite energy are linear combinations of these functions. The F_s is a solution of a Dirac-type wave equation where[2] M is the relative mass of the neutron and nucleus.

$$\{c\boldsymbol{\alpha}\cdot\mathbf{p}+\beta Mc^2+E\}F_s(r_s, \Omega_s)=0, \qquad (1)$$
$$\mathbf{p}=-i\hbar\nabla,$$

$\mathfrak{D}$ and $\mathfrak{V}$ type wave functions similar to A (9) but satisfying (1) and the boundary conditions

$$\int_{r=a_s}\mathfrak{D}^*_{sljm}\mathfrak{D}_{sljm}dS=1, \qquad (2a)$$

$$[i\alpha_r\mathfrak{D}_{sljm}]_{r=a_s}=-[\mathfrak{V}_{sljm}]_{r=a_s}, \qquad (2b)$$

may be defined.

The Eqs. (2) do not completely specify the four-component wave functions $\mathfrak{V}$ and $\mathfrak{D}$. This may be done by using the additional condition

$$[(1+\beta)/2](1-i\delta\alpha_r)\mathfrak{V}_{sljm}|_{r=a_s}=0, \qquad (3)$$

where δ is a parameter at our disposal. δ may depend on a_s, l, and j but not on the energy, and α_r is the radial component of $\boldsymbol{\alpha}$.[3]

The R matrix is defined in a manner similar to that of A: (compare A (13)).

$$\varphi_{sljm}=\mathfrak{D}_{sljm}+\sum_{s'l'}R^{(j)}_{sl,s'l'}\mathfrak{V}_{s'l'jm}. \qquad (3)$$

R is a real matrix. Now one considers those solutions $X_{\lambda jm}$ of the quantum-mechanical equations in the internal region which satisfy boundary conditions similar to the $\mathfrak{V}_{sljm}$. That is,

$$\int_{r=a_s}dsP_l^{m*}[(1+\beta)/2](1-i\delta\alpha_r)X_{\lambda jm}=0. \qquad (4)$$

In A the next step is to apply Green's theorem to the $X_{\lambda\mu}$ and $\varphi_{sl\mu}$. For the present work the following relation is equivalent to Green's

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[1] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).

[2] The matrices $\boldsymbol{\alpha}$ and β are defined as follows:

$$\alpha_x=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix},\quad \alpha_y=\begin{bmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{bmatrix},$$

$$\alpha_z=\begin{bmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{bmatrix},\quad \beta=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}.$$

[3] Explicitly

$$\alpha_r=\begin{bmatrix}0&0&\cos\vartheta&\sin\vartheta e^{-i\varphi}\\0&0&\sin\vartheta e^{i\varphi}&-\cos\vartheta\\\cos\vartheta&\sin\vartheta e^{-i\varphi}&0&0\\\sin\vartheta e^{i\varphi}&-\cos\vartheta&0&0\end{bmatrix},$$

where ϑ and φ are, respectively, the polar and azimuthal angles.

theorem:[4]

$$\int_V dV[\Psi^* H\varphi-(H\Psi)^*\varphi]=i\int_S dS\Psi^* ch\alpha_n\varphi, \quad (5)$$

where S is the bounding surface of the volume V and H is the operator $ich\boldsymbol{\alpha}\cdot\nabla$ added to any function of the space and spin coordinates. α_n is the component of $\boldsymbol{\alpha}$ along the outward normal to S. An application of (5) to φ_{sljm} and $X_{\lambda jm}$ yields, since the $X_{\lambda jm}$ and $\mathfrak{V}_{sljm}$ on the surface $r=a_s$ are orthogonal to the $\mathfrak{D}_{sljm}$,

$$-ch\int_{r=a_s} dSX_{\lambda jm}{}^*\mathfrak{V}_{sljm}$$
$$=(E-E_\lambda)\int dIX_{\lambda jm}{}^*\varphi_{sljm}=-(ch)^{\frac{1}{2}}\gamma_{\lambda slj}. \quad (6)$$

The last relation defines the energy-independent real quantities γ. One may also write for the internal region

$$\varphi_{sljm}=\sum\nolimits_\lambda A_{slj\lambda}X_{\lambda jm}, \quad (7)$$

whence (6) gives

$$A_{slj\lambda}=(ch)^{\frac{1}{2}}[\gamma_{\lambda slj}/(E_\lambda-E)]. \quad (8)$$

Since φ_{sljm} must be continuous at the surface $r=a_s$ one has further, on S,

$$(ch)^{\frac{1}{2}}\sum\nolimits_\lambda[\gamma_{\lambda slj}/(E_\lambda-E)]X_{\lambda jm}$$
$$=\mathfrak{D}_{sljm}+\sum_{s''l''}R^{(j)}_{sl,s''l''}\mathfrak{V}_{s''l''jm}. \quad (9)$$

If both sides are multiplied by $\mathfrak{V}_{s'l'jm}$ and integrated over the surface S one obtains

$$R^{(j)}_{sl,s'l'}=+\sum\nolimits_\lambda[\gamma_{\lambda slj}\gamma_{\lambda s'l'j}/(E_\lambda-E)]. \quad (10)$$

This is the present result for the R matrix and is identical in form, despite the different wave equation used, with A (23).

The relation between the R matrix and the collision matrix may now be determined. To do this it is necessary to specify the form of the wave functions in the external region. Define

$$\chi_{j-\frac{1}{2},j,m}=\begin{pmatrix}[(j+m)/2j]^{\frac{1}{2}} & Y_{j-\frac{1}{2}}^{m-\frac{1}{2}}\\ [(j-m)/2j]^{\frac{1}{2}} & Y_{j-\frac{1}{2}}^{m+\frac{1}{2}}\end{pmatrix}, \quad (11a)$$

$$\chi_{j+\frac{1}{2},j,m}=\begin{pmatrix}[(j+1-m)/(2j+2)]^{\frac{1}{2}} & Y_{j+\frac{1}{2}}^{m-\frac{1}{2}}\\ -[(j+1+m)/(2j+2)]^{\frac{1}{2}} & Y_{j+\frac{1}{2}}^{m+\frac{1}{2}}\end{pmatrix}. \quad (11b)$$

It is convenient to use the σ_r matrix

$$\sigma_r=\begin{pmatrix}\cos\vartheta & \sin\vartheta e^{-i\varphi}\\ \sin\vartheta e^{i\varphi} & -\cos\vartheta\end{pmatrix}, \quad (12)$$

and note that

$$\sigma_r\chi_{j\pm\frac{1}{2},j,m}=\chi_{\mp j\frac{1}{2},j,m}.$$

Y_l^m is the spherical harmonic normalized to unity.

Incident and emergent waves are then defined as

$$I_{sljm}=\Psi_s(i_s)\frac{1}{(2\epsilon_s)^{\frac{1}{2}}}\frac{e^{-ik_sr}}{r}\times\begin{pmatrix}(\epsilon_s/c)\sigma_r & \chi_{ljm}\\ & \chi_{ljm}\end{pmatrix}, \quad (13a)$$

$$E_{sljm}=\Psi_s(i_s)\frac{1}{(2\epsilon_s)^{\frac{1}{2}}}\frac{e^{ik_sr}}{r}\times\begin{pmatrix}-(\epsilon_s/c)\sigma_r & \chi_{ljm}\\ & \chi_{ljm}\end{pmatrix}, \quad (13b)$$

where

$$k_s=p_s/h, \quad \epsilon_s=p_sc^2/(E_s+Mc^2).$$

One then may further define the quantities A, B, and ω by specifying the asymptotic behavior of the $\mathfrak{D}$'s and $\mathfrak{V}$'s in terms of the I's and E's.

$$\mathfrak{D}_{sljm}\simeq i(c/2)^{\frac{1}{2}}[A_{slj}I_{sljm}-A_{slj}{}^*E_{sljm}], \quad (14a)$$

$$\mathfrak{V}_{sljm}\simeq(c/2)^{\frac{1}{2}}[B_{slj}\omega_{slj}{}^*I_{sljm}+B_{slj}\omega_{slj}E_{sljm}]. \quad (14b)$$

Since $\mathfrak{D}_{sljm}$ and $\mathfrak{V}_{sljm}$ both satisfy (1),

$$\int_s dS\mathfrak{V}_{slj\mu}{}^*i\alpha_n\mathfrak{D}_{slj\mu}$$

is a constant for any surface S including the origin, and this constant is seen to be equal to -1 for the surface $r=a_s$ (from (2)). Hence,

$$B\omega A+B\omega^*A^*=2, \quad (15)$$

so that one may also write

$$B\omega A=1-iC, \quad (16)$$

[4] A derivation of (5) is given in the Appendix.

TABLE I.

l	$B\omega$	C
0	$-\left(\frac{\epsilon_s}{c}\right)^{\frac{1}{2}}\left(\frac{1}{1+\delta^2}\right)^{\frac{1}{2}}\left(1+i\left[\frac{1}{k_sa_s}+\frac{c}{\epsilon_s}\delta\right]\right)e^{-ik_sa_s}$	$-\frac{\epsilon_s}{c}\frac{1}{1+\delta^2}\left[\left(\frac{\delta}{k_sa_s}-\frac{c}{\epsilon_s}\right)\left(\frac{1}{k_sa_s}+\frac{c}{\epsilon_s}\delta\right)+\delta\right]$
1	$\left(\frac{\epsilon_s}{c}\right)^{\frac{1}{2}}\left(\frac{1}{1+\delta'^2}\right)^{\frac{1}{2}}\left(\left[\delta'\frac{c}{\epsilon_s}\frac{1}{k_sa_s}-1\right]-i\frac{c}{\epsilon_s}\delta'\right)e^{-ik_sa_s}$	$\frac{\epsilon_s}{c}\frac{1}{1+\delta'^2}\left[\left(\delta'+\frac{c}{\epsilon_s}\frac{1}{k_sa_s}\right)\left(\delta'\frac{c}{\epsilon_s}\frac{1}{k_sa_s}-1\right)-\delta'\frac{c^2}{\epsilon_s^2}\right]$

where C is a real diagonal matrix. Now

$$\varphi_{sljm}=\mathfrak{D}_{sljm}+\sum_{s'l'}R^{(j)}_{sl,\,s'l'}\mathfrak{V}_{sljm}$$
$$=(c/2)^{\frac{1}{2}}\sum_{s'l'}\{(iA+RB\omega^*)_{sl's'l'}I_{s'l'jm}+(-iA+RB\omega)_{sl,\,s'l'}E_{s'l'jm}\}.$$

Hence, a solution of the quantum-mechanical equation in which only one incident beam occurs may be written

$$(c/2)^{\frac{1}{2}}I_{sljm}+(c/2)^{\frac{1}{2}}\sum_{s'l'}[(RB\omega^*+iA)^{-1}\times(RB\omega-iA^*)]_{sl,\,s'l'}E_{s'l'jm},$$

so that the collision matrix becomes

$$U_{sl,\,s'l'}=-[(RB\omega^*+iA)^{-1}(RB\omega-iA^*)]_{s,\,l,\,s'l'}$$
$$=[\omega(1-iBRB-iC)^{-1}\times(1+iBRB+iC)\omega]_{sl,\,s'l'}. \quad (17)$$

The wave function for a collision experiment is obtained in the usual manner in terms of the U. In fact, a plane wave representing a particle of type s moving along the z axis, with $j_z=\frac{1}{2}$, is given by

$$\Psi_{s,\,\frac{1}{2}}=(\sqrt{\pi}/k_s)\sum_{l=0}^{\infty}[\{(l+1)^{\frac{1}{2}}E_{s,\,l,\,l+\frac{1}{2},\,\frac{1}{2}}+\sqrt{l}\,E_{s,\,l,\,l-\frac{1}{2},\,\frac{1}{2}}-(-)^l\{(l+1)^{\frac{1}{2}}I_{s,\,l,\,l+\frac{1}{2},\,\frac{1}{2}}+\sqrt{l}\,I_{s,\,l,\,l-\frac{1}{2},\,\frac{1}{2}}\}], \quad (18)$$

so that a continuable solution with the same incident waves as (18) is given by

$$\Psi_{s,\,\frac{1}{2}}+(\sqrt{\pi}/k_s)\sum_{l,\,s',\,l'}\{(l+1)^{\frac{1}{2}}[(-)^lU^{(l+\frac{1}{2})}_{sl,\,s'l'}-\delta_{sl,\,s'l'}]E_{s',\,l',\,l+\frac{1}{2},\,\frac{1}{2}}+\sqrt{l}[(-)^lU^{(l-\frac{1}{2})}_{sl,\,s'l'}-\delta_{sl,\,s'l'}]E_{s',\,l',\,l-\frac{1}{2},\,\frac{1}{2}}\}. \quad (19)$$

Expressions for the various differential cross sections may be obtained from (19).

The values of B, ω, C as calculated for $j=\frac{1}{2}$ are given in Table I. Note that they contain the energy independent, but otherwise arbitrary, parameters δ and δ'.

The results as given in Table I appear relatively complicated. Nonetheless, for $j=\frac{1}{2}$, $l=1$, the choice $\delta'=0$ gives results no more complex than the non-relativistic formalism. For $j=\frac{1}{2}$, $l=0$, the complication is not so easily removed. However, one must, of course, be able to reduce the present result to the non-relativistic for low energy particles.

The low energy form of the present results is discussed below. For $j=\frac{1}{2}$, $l=1$, $\delta'=0$

$$B_{s,\,1,\,\frac{1}{2}}=-(\epsilon_s/c)^{\frac{1}{2}},$$
$$\omega_{s,\,1,\,\frac{1}{2}}=e^{-ik_sa_s},$$
$$C_{s,\,1,\,\frac{1}{2}}=-(1/k_sa_s).$$

These results agree with those found in reference 1 for $l=1$, except for the B. However, the sign of B is immaterial and in the low energy limit

$$B_{s,\,\frac{1}{2}}\simeq-[\tfrac{1}{2}(\hbar/Mc)]^{\frac{1}{2}}k_s^{\frac{1}{2}},$$

which, except for a difference in normalization, agrees also with reference 1.

For $j=\frac{1}{2}$, $l=0$, one writes

$$-\delta=\tfrac{1}{2}(\hbar/Mc)/a_s$$

and neglects $[\frac{1}{2}(\hbar/Mc)k_s]^2$ with respect to unity, to find, with $\chi_s=k_sa_s$,

$$B_{s,\,0,\,\frac{1}{2}}=-[\tfrac{1}{2}(\hbar/Mc)]^{\frac{1}{2}}k_s^{\frac{1}{2}}[(1+\delta^4\chi_s^2)/(1+\delta^2)]^{\frac{1}{2}},$$
$$\omega_{s,\,0,\,\frac{1}{2}}=\exp-i(\chi_s-\text{arc}\tan\delta^2\chi_s),$$
$$C_{s,\,0,\,\frac{1}{2}}=\delta^4\chi_s.$$

In addition, one will usually have $k_sa_s<1$, $\delta^2\ll1$, so that

$$B_{s,\,0,\,\frac{1}{2}}=-[\tfrac{1}{2}(\hbar/Mc)]^{\frac{1}{2}}k_s^{\frac{1}{2}},$$
$$\omega_{s,\,0,\,\frac{1}{2}}=\exp-i\chi_s,$$
$$C_{s,\,0,\,\frac{1}{2}}=\delta^4\chi_s,$$

which differs from reference 1 primarily in the non-zero C.

The formal discussion given by Wigner and Eisenbud[1] for the generalized one-level formula may be taken over in toto. We list here for convenience, in the limit where $[\frac{1}{2}(\hbar/Mc)k_s]^2$ is neglected in respect to unity, the partial widths and energy shifts.

$$\Gamma_{\lambda,s,0,\frac{1}{2}}=\frac{\hbar}{Mc}\frac{k_s}{1+\delta^8\chi_s^2}\frac{1+\delta^4\chi_s^2}{1+\delta^2}\gamma^2_{\lambda,s,0,\frac{1}{2}},$$

$$\Delta_{\lambda,s,0,\frac{1}{2}}=\tfrac{1}{2}\delta^4\chi_s\Gamma_{\lambda,s,0,\frac{1}{2}},$$

$$\Gamma_{\lambda,s,1,\frac{1}{2}}=\frac{\hbar}{Mc}\frac{k_s\chi_s^2\gamma^2_{\lambda,s,1,\frac{1}{2}}}{1+\chi_s^2}$$

$$\Delta_{\lambda,s,1,\frac{1}{2}}=-(1/\chi_s)\Gamma_{\lambda,s,1,\frac{1}{2}}$$

where $\chi_s=k_sa_s$

$$-\delta=\tfrac{1}{2}(\hbar/Mc)/a_s.$$

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APPENDIX: DERIVATION OF EQ. (5)

If $H=ich\boldsymbol{\alpha}\cdot\nabla+Q$ and Ψ and φ are four-component wave functions regular on the interior and surface of a volume V, and if Q is an operator such that $\Psi^*(Q\varphi)=(Q\Psi)^*\varphi$, then

$$\begin{aligned}\int_V d\tau(\Psi^*H\varphi-(H\Psi)^*\varphi)\\ &=\int_V d\tau(\Psi^*ich\boldsymbol{\alpha}\cdot\nabla\varphi-(ich\boldsymbol{\alpha}\cdot\nabla\Psi)^*\varphi)\\ &=\int_V d\tau ich[\Psi^*\boldsymbol{\alpha}\cdot\nabla\varphi+(\alpha\cdot\nabla\Psi)^*\varphi]\\ &=ich\int_V d\tau\nabla\cdot(\Psi^*\boldsymbol{\alpha}\varphi)\\ &=ich\int_s d\sigma(\Psi^*\alpha_n\varphi),\end{aligned}$$

where α_n is the component of $\boldsymbol{\alpha}$ along the outward normal to s.