cipitate was washed with 1 percent oxalic acid solution, dried, and ignited at 800–1000°C. The product weighed 74 mg and consisted of CaO and practically the entire quantity (2 mg) of Sc_2O_3. The calcium was added as a carrier to assure complete recovery of the radioactive scandium.

By this procedure it was intended to obtain scandium free of radioactive phosphorus, calcium, and iron.

APPENDIX II: CHEMICAL PROCEDURE FOR TITANIUM

Chemically pure TiO_2 was irradiated by neutrons in the pile. The titanium dioxide was fused in platinum with ten times its weight of sodium carbonate. The melt was then boiled with water and filtered. The insoluble residue of titania was well washed with boiling water, and the filtrate and washings were rejected. This procedure should have removed phosphorus and part of any scandium impurity. The partly purified titania was then fused in platinum with potassium acid sulfate, and the melt was dissolved in cold $6N-H_2SO_4$. Orthotitanic acid was precipitated, at room temperature, with dilute ammonium hydroxide. The precipitate was dissolved in concentrated HCl and the solution diluted to $2N$. One volume of saturated oxalic acid solution was added to three volumes of the warm solution. After standing four hours, a trace of precipitate, which may have been calcium or scandium oxalate, was filtered out.

The filtrate was again made ammoniacal, and the orthotitanic acid precipitate was filtered out, washed, and dissolved in HCl. After dilution to $2N$, sufficient tartaric acid was added to prevent precipitation of titanium. The solution was neutralized with ammonia, made about $0.3N$ in H_2SO_4, and saturated with H_2S. Ammonia was added in decided excess, and the solution was again saturated with H_2S. The precipitate, probably iron and platinum sulfides, was filtered out. The filtrate, made strongly acid with H_2SO_4, was boiled to expel H_2S, diluted, and cooled. The titanium was precipitated with a 4 percent solution of cupferron added in excess, and the precipitate was ignited to TiO_2. The procedure outlined gave a sample of TiO_2 free of phosphorus, scandium, calcium, and iron.

PHYSICAL REVIEW VOLUME 73, NUMBER 2 JANUARY 15, 1948

The Propagation of a Pulse in the Atmosphere. Part II

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(Received September 5, 1947)

The previous investigation of the dispersion of long waves in the atmosphere has been extended to shorter periods of the order of one minute. Both the phase velocity and group velocity have been determined. The results are applied to the interpretation of the pressure wave produced by the Great Siberian Meteor and to the pressure oscillations recorded by microbarographs in England.

1. INTRODUCTION AND SUMMARY

FOR the purpose of interpreting the observed features of the pressure wave produced by the explosion of the Great Siberian Meteor in 1908,[1] the previous investigation by the author[2] of the dispersion of long waves in the atmosphere (Krakatoa wave, period of one hour) has been extended to shorter periods of the order of a minute. Two model atmospheres have been treated, one (a) having a finite height in which the temperature gradient is constant and equal to 7/11 of the adiabatic, and another (b) in which the same temperature gradient prevails in the troposphere, but a constant temperature is assumed above 10.3 km. The variation of phase

* This work was carried out while the writer served as consultant to project N6-ori-139, task order #1, sponsored by the O.N.R.

[1] F. J. W. Whipple, Q. J. Roy. Meteor. Soc. **56**, 287 (1930).

[2] C. L. Pekeris, Proc. Roy. Soc. **A171**, 434 (1939); this paper will be referred to as I.

velocity with period in model a was studied by Solberg,[3] while G. I. Taylor[4] first gave the limiting value of the phase velocity at long waves for the more realistic model b. In this study both the phase velocity and group velocity have been determined. It is found (see Fig. 1) that in model a there exists, in addition to the oscillation studied by Solberg (I), a second type of oscillation (II) which is limited to periods less than the free period of the atmosphere for purely vertical oscillation (3.5 minutes). In this type II of oscillation the group velocity passes through a maximum, equal to 0.64 of the value of the sound velocity at the ground, at a period of about 2.5 minutes. No type II oscillation was found in model b. However, it turns out that in b the free oscillation of type I exists only for periods greater than 2 minutes. Shorter periods are not propagated freely without horizontal attenuation, essentially because they leak out vertically to space.

A steady-state solution is obtained for the oscillation of atmosphere b when excited by a point source situated at the ground, in which the vertical velocity w varies with time like $e^{i\sigma t}$. It is found that at large distances from the source the resulting pressure oscillation at the ground has a relative maximum at a period of about 4 minutes, tending to zero both at very long periods and at the cut-off period (see curve A, Fig. 2). When this steady-state solution is used to determine the oscillation excited by an *impulsive* point source, in which the spectrum of w is uniform, the higher dispersion at the shorter periods results in an excitation which starts from zero at the cut-off period and increases in a monotone fashion towards the longer periods.

On the other hand, if the pulse is such that the *pressure variation* at the source has a uniform spectrum, the excitation is a maximum at a period equal to Brunt's period for the vertical oscillation of a particle near the ground, i.e., 9.5 minutes. These results are applied to the interpretation of the pressure wave produced by the Great Siberian Meteor and to pressure oscillations recorded by microbarographs in England.

[3] H. Solberg, Astrophys. Norvegica 2, 123 (1936).
[4] G. I. Taylor, Proc. Roy. Soc. A126, 728 (1929); H. Lamb, *Hydrodynamics* (Cambridge University Press, Teddington, England, 1932), p. 541.

2. THEORY OF ATMOSPHERIC OSCILLATIONS

The theory of atmospheric oscillations dates back to Laplace,[5] but Lamb[6] was the first to develop it without making restrictive assumptions about the physical processes involved. Since we shall be interested in periods ranging from a minute to about an hour, we may neglect the stabilizing effects arising from the earth's rotation which become significant only at periods of the order of a day.[7] We shall, therefore, confine the discussion to the oscillations of a horizontally stratified atmosphere resting on a flat ground when disturbed from static equilibrium. If the positive direction of the z axis be taken downwards, the pressure distribution in the equilibrium state is given by

$$(dp_0/dz)=g\rho_0, \quad p_0=R\rho_0T, \tag{1}$$

$$\begin{aligned} p_0(z)&=p_0(z_0)\exp\left[-(g/R)\int_z^{z_0}(dz/T)\right],\\ \rho_0(z)&=\rho_0(z_0)(T_0/T)\exp\left[-(g/R)\int_z^{z_0}(dz/T)\right]. \end{aligned} \tag{2}$$

Upon this equilibrium state we superimpose small oscillations which are governed by the equations of motion and continuity:

$$\rho_0\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}, \quad \rho_0\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial y}, \quad \rho_0\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+g\rho, \tag{3}$$

$$\frac{\partial \rho}{\partial t}+w\frac{\partial \rho_0}{\partial z}=-\rho_0\chi, \quad \chi\equiv\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \tag{4}$$

p and ρ denoting perturbations from equilibrium values. The adiabatic energy equation,

$$\frac{Dp}{Dt}=c^2\frac{D\rho}{Dt}, \quad c^2=\gamma p_0/\rho_0=\gamma RT_0, \tag{5}$$

yields the fifth equation for the determination of the five variables (u, v, w, p, ρ):

$$\frac{\partial p}{\partial t}+g\rho_0w=-\gamma p_0\chi. \tag{6}$$

[5] Laplace, Mécanique Céleste, Livre 4, Chap. 5 (1845).
[6] H. Lamb, *Hydrodynamics* (Cambridge University Press, Teddington, England, 1932), p. 550; Proc. Roy. Soc. A84, 552 (1910); Proc. Lond. Math. Soc. 7, 122 (1909).
[7] In the case of long period oscillations G. I. Taylor has shown that the effect of the earth's rotation can be taken into account in the manner used in the theory of oceanic tides. Proc. Roy. Soc. A156, 378 (1936).

Eliminating p and ρ, we get

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}(c^2\chi+gw),\quad \frac{\partial^2 v}{\partial t^2}=\frac{\partial}{\partial y}(c^2\chi+gw), \tag{7}$$

$$\frac{\partial^2 w}{\partial t^2}=\frac{\partial}{\partial z}(c^2\chi+gw)+\left[(\gamma-1)g-\frac{dc^2}{dz}\right]\chi. \tag{8}$$

Let the dependence on time and on $r(=(x^2+y^2)^{\frac{1}{2}})$ be given by a factor $e^{i\sigma t}J_0(kr)$, then the above equations yield

$$c^2\ddot{\chi}+(\dot{c}^2+\gamma g)\dot{\chi}+[\sigma^2-k^2(Q^2/\sigma^2)]\chi=0,\quad Q^2\equiv g\dot{c}^2+\sigma^2c^2-(\gamma-1)g^2, \tag{9}$$

$$\delta w=\sigma^2c^2\dot{\chi}+(g\gamma\sigma^2-gk^2c^2)\chi,\quad \delta\equiv g^2k^2-\sigma^4, \tag{10}$$

$$\delta p=i\sigma\rho_0[gc^2\dot{\chi}+(\gamma g^2-\sigma^2c^2)\chi],\quad \text{or}\quad p=(i\rho_0/\sigma)(gw+c^2\chi), \tag{11}$$

where the dots denote differentiation with respect to z.

In order to bring out the contrast with the laws of propagation of acoustic vibrations ($\sigma\rightarrow\infty$), where the motion is irrotational and is derivable from a velocity potential φ,

$$\mathbf{v}=-\nabla\varphi,\quad p=\rho_0(\partial\varphi/\partial t),\quad \nabla^2\varphi=(1/c^2)\partial^2\varphi/\partial t^2, \tag{12}$$

we write in (7) and (8)

$$gw+c^2\chi=-\partial^2\varphi/\partial t^2,\quad u=-\partial\varphi/\partial x,\quad v=-\partial\varphi/\partial y,\quad p=\rho_0\partial p/\partial t, \tag{13}$$

and find that

$$w=-(\partial\varphi/\partial z)+\Delta,\quad \Delta=[\dot{c}^2-(\gamma-1)g](\sigma^2\varphi+g\dot{\varphi})/Q, \tag{14}$$

$$Q^2c^2\sigma^2\ddot{\varphi}+(\gamma\sigma^2gQ^2-\sigma^2c^2\dot{Q}^2)\dot{\varphi}+[-Q^4k^2g^2+Q^4\sigma^4+\gamma\sigma^4g^2Q^2-\sigma^6c^2Q^2-\sigma^4c^2\dot{Q}^2g^2]\varphi/g^2=0. \tag{15}$$

The motion is now rotational, as evidenced by the Δ term in (14). This arises from the fact that, whereas when an air particle is displaced horizontally it is restrained only by elastic forces, when it is displaced vertically an additional force stemming from the vertical stable stratification comes into play. When a particle is displaced adiabatically upwards, its temperature drops in dry air at the rate of about 10° per km., the so-called dry-adiabatic lapse rate. On the other hand, the normal temperature gradient in the atmosphere is only about 0.6 of the adiabatic. Hence the particle is cooler than the surrounding air at the new elevation, and is therefore pulled down to its original position by a force which is initially proportional to the displacement. The period of free vertical oscillation of an individual air particle was shown by Brunt[8] to be determined from

$$\sigma^2c^2=(\gamma-1)g^2-g\dot{c}^2;\quad Q=0. \tag{16}$$

Only when the actual temperature gradient is equal to the adiabatic is the right-hand side of (16) equal to zero, and only then do the atmospheric oscillations become strictly isotropic. In the acoustic limit $(\partial\varphi/\partial z)\simeq(2\pi/\lambda)\varphi$, whereas

$$\Delta\simeq\varphi[(\dot{c}^2/c^2)-g(\gamma-1)/c^2]\simeq\varphi(1/T)(dT/dz)\simeq\varphi\,10^{-7}\text{ cm}^{-1}.$$

Δ is then of the order of 10^{-4} smaller than $(\partial\varphi/\partial z)$. To the same degree of approximation Eqs. (9) and (15) then reduce to the acoustic wave Eq. (12). On the other hand, at periods of several minutes, in which we shall be interested in this study, the motion deviates radically from acoustic isotropy, especially near Brunt's resonant period, where Q in (9) and (15) passes through zero.

For future reference we shall point out here that Eq. (9) can be reduced by the substitution

$$u=\int_z^{z_0}\left(\frac{c_0^2}{c^2}\right)dz,\quad H_0=\frac{RT_0}{g},\quad q\equiv\left[\frac{c^4}{c_0^4}-\frac{1}{\gamma H_0\sigma^2}\frac{dc^2}{du}-\frac{(\gamma-1)c^2}{\gamma^2H_0^2\sigma^2}\right], \tag{17}$$

to

$$\chi=\exp(u/2H_0)\psi(u),\quad \frac{d^2\psi}{du^2}+\left[-\frac{1}{4H_0^2}+\frac{\sigma^2c^2}{c_0^4}-k^2q\right]=0. \tag{18}$$

3. FREE OSCILLATIONS OF AN ATMOSPHERE WITH CONSTANT TEMPERATURE

In addition to the anisotropic effects mentioned above, arising from the vertical stability, the atmosphere has free modes of oscillation that are analogous to Rayleigh waves. The nature of these can best be illustrated in the simple case of an atmosphere in which the temperature is uniform at all heights. In the case of a liquid

[8] D. Brunt, Q. J. Roy. Meteor. Soc. **53**, 30 (1927).

half-space of constant sound velocity, say an ocean of infinite depth, the field produced by a point source situated at some depth can be represented by two spherical waves emanating from the source and from its image (negative) in the surface. When the half space is rigid, Rayleigh[9] has shown that the source excites, in addition, a free surface wave whose energy decreases exponentially with depth, and which spreads out cylindrically. Two types of surface waves of this nature can be propagated in the atmosphere. These, of course, have not the same physical origin as Rayleigh's waves, since the rigidity of the air is zero.

It will be convenient in this section to take the positive z axis upwards. Equations (9) and (10) reduce in our case to

$$c^2\ddot{\chi}-\gamma g\dot{\chi}+[\sigma^2-k^2c^2+k^2(\gamma-1)g^2/\sigma^2]\chi=0, \quad (19)$$

$$(g^2k^2-\sigma^4)w=\sigma^2c^2\dot{\chi}+(gk^2c^2-g\gamma\sigma^2)\chi=0. \quad (20)$$

We also have for the distribution of normal density with height

$$\rho_0(z)=\rho_0(0)\exp(-z/H_0),\quad H_0=(RT_0/g)=(c_0^2/\gamma g). \quad (21)$$

Assuming

$$\chi=A\cdot\exp(\lambda z+i\sigma t-ikx), \quad (22)$$

we get from (19)

$$\lambda=\frac{\gamma g}{2c^2}-\mu,$$

$$\mu=\left[\frac{\gamma^2g^2}{4c^4}-\frac{\sigma^2}{c^2}+k^2-\frac{g^2k^2(\gamma-1)}{\sigma^2c^2}\right]^{\frac{1}{2}}, \quad (23)$$

where μ is to be taken positive. The choice of the negative sign in front of μ was made in order to assure that the kinetic energy of wave motion integrated over a column of the atmosphere is finite. The kinetic energy density, which is proportional to $\rho_0(z)\chi^2$, varies then like $\exp(-2\mu z)$. In the other solution with the positive sign in front of μ, this quantity is proportional to $\exp(2\mu z)$. Now a free oscillation is possible if w vanishes at the ground, or from (20)

$$\lambda=\frac{\gamma g}{c^2}-\frac{gk^2}{\sigma^2}. \quad (24)$$

Substituting in (23), we obtain a relation between σ and k which determines the speed of propagation $V(=\sigma/k)$ of the free waves and the group velocity $U(=d\sigma/dk)$ as functions of the frequency σ. This relation has two roots corresponding to two possible types of free oscillation:

$$\text{I.}\quad k=\sigma/c,\quad V=U=c,\quad \mu=g(2-\gamma)/2c^2,\quad \lambda=(\gamma-1)g/c^2, \quad (25)$$

$$\text{II.}\quad k=\sigma^2/g,\quad V=(g/\sigma),\quad U=\tfrac{1}{2}V,\quad \mu=\frac{\sigma^2}{g}-\frac{\gamma g}{2c^2},\quad \lambda=\frac{\gamma g}{2c^2}-\frac{\sigma^2}{g}. \quad (26)$$

While oscillation I can exist at all frequencies, in oscillation II only frequencies above a critical value can be propagated:

$$\text{II.}\quad \sigma^2>\sigma_c^2=\gamma g^2/2c^2,\quad k>1/2H_0,\quad U<c/(2\gamma)^{\frac{1}{2}}. \quad (27)$$

When Eq. (27) is violated, no root of (23) and (24) belonging to class II exists which yields an integrable wave energy in an atmospheric column. As $\sigma\rightarrow\sigma_c$ the energy content per atmospheric column grows indefinitely, so that it becomes increasingly difficult to excite the oscillation near σ_c.

In the next section, when dealing with an atmosphere having a constant temperature in the stratosphere, we shall also find a cut-off frequency conditioned by the vanishing of μ. Then, however, frequencies less than the cut-off frequency are disallowed.

It should be noted that because the dependence on z is represented by a single exponential term, the condition of the vanishing of w at the ground makes it vanish at all heights. In an atmosphere of constant temperature the free oscillations manifest themselves only by horizontal motion u and by a pressure oscillation p,

$$u=(ikc^2/\sigma^2)\chi,\quad p=(i\rho_0c^2/\sigma^2)\chi,\quad w=0. \quad (28)$$

Had we assumed a factor

$$H_0^{(2)}(kr)\rightarrow(2/\pi kr)^{\frac{1}{2}}\exp(-ikr+i\pi/4), \quad (29)$$

instead of $\exp(-ikx)$ in (22), the displacement would have been in the r-direction, and the amplitude would have decreased like $r^{-\frac{1}{2}}$.

[9] Lord Rayleigh, Sci. Pap. 2, 441 (1900); H. Lamb, Phil. Trans. Roy. Soc. **A203**, 1 (1904); C. L. Pekeris, Proc. Nat. Acad. Sci. **26**, 433 (1940).

4. FREE OSCILLATIONS OF AN ATMOSPHERE HAVING A CONSTANT TEMPERATURE GRADIENT IN THE TROPOSPHERE AND A CONSTANT TEMPERATURE IN THE STRATOSPHERE (MODEL *b*)

With the z axis pointing downwards, let in the troposphere

$$T=\beta z,\quad m\equiv -1+(g/R\beta),\quad c^2=\gamma gz/(m+1). \tag{30}$$

Write

$$\tau=\sigma^2/gk,\quad x=2kz,$$

$$2\alpha=\left[-2-m+\frac{(m+1)\tau}{\gamma}-\frac{1}{\tau}+\frac{(\gamma-1)(m+1)}{\gamma\tau}\right], \tag{31}$$

then Eq. (9) reduces to

$$x\frac{d^2\chi}{dx^2}+(m+2)\frac{d\chi}{dx}+\left(-\frac{x}{4}+\alpha+1+\frac{m}{2}\right)\chi=0, \tag{32}$$

whose solution is

$$\chi=e^{-x/2}[AF_1(x)+Bx^{-1-m}F_2(x)],$$
$$F_1=F(-\alpha,\,m+2,\,x);$$
$$F_2=F(-\alpha-1-m,\,-m,\,x), \tag{33}$$

provided m is not an integer. The condition of the vanishing of w at the ground ($x=x_0$) yields

$$\frac{B}{A}=\frac{x_0^{1+m}[\dot F_1(x_0)-NF_1(x_0)]}{\{[N+(1+m)/x_0]F_2(x_0)-\dot F_2(x_0)\}},$$

$$N=\frac{1}{2}\left(1+\frac{1}{\tau}\right)-\frac{(m+1)}{x_0}. \tag{34}$$

In the stratosphere ($x<x_1$)

$$c^2=\gamma gx_1/2k(m+1),\quad \chi=De^{\lambda z}, \tag{35}$$

$$\frac{\lambda}{2k}=-\frac{(m+1)}{2x_1}\left\{1-\left[1+\frac{x_1^2}{(m+1)^2}-\frac{2x_1\tau}{\gamma(m+1)}-\frac{2(\gamma-1)x_1}{\gamma\tau(m+1)}\right]^{\frac{1}{2}}\right\}. \tag{36}$$

At the tropopause ($x=x_1$) we must have continuity of the perturbation pressure p and of w. From (10) and (11) it follows that both $\dot\chi$ and χ must be continuous, and therefore also $\dot\chi/\chi$. Equating $\dot\chi/\chi$ obtained from (33) and (35), we arrive at a relation which determines $\sigma(k)$:

$$g(x)=\frac{\dot F_1(x_1)+(B/A)x_1^{-1-m}[\dot F_2(x_1)-(1+m)F_2(x_1)/x_1]}{[F_1(x_1)+(B/A)x_1^{-1-m}F_2(x_1)]}-\frac{1}{2}+\frac{(m+1)}{2x_1}\left\{1-\left[1+\frac{x_1^2}{(m+1)^2}-\frac{2x_1\tau}{\gamma(m+1)}-\frac{2(\gamma-1)x_1}{\gamma\tau(m+1)}\right]^{\frac{1}{2}}\right\}=0. \tag{37}$$

Here (B/A) is determined from (34) and

$$(x_1/x_0)=(T_s/T_0), \tag{38}$$

where T_s denotes the temperature in the stratosphere, and T_0 the temperature at the ground. For a given value of α one computes τ from (31) and then finds such a value of x_0 (and with it of x_1) that (37) is satisfied. Having obtained x_0 as a function of α, the frequency σ and associated phase velocity V and group velocity U can be determined from

$$\sigma=\left[\frac{g\tau x_0}{2H_0(m+1)}\right]^{\frac{1}{2}},\quad \frac{V}{c_0}=\left(\frac{2\tau(m+1)}{\gamma x_0}\right)^{\frac{1}{2}},$$

$$\frac{U}{c_0}=\left(\frac{2m+2}{\gamma}\right)^{\frac{1}{2}}\frac{d(\tau x_0)^{\frac{1}{2}}}{dx_0}. \tag{39}$$

It should be noted that for a given α, Eq. (31) yields two values of τ:

$$\tau_1=\frac{\gamma(2+2\alpha+m)}{2(m+1)}+\left[\frac{\gamma^2(2+2\alpha+m)^2}{4(m+1)^2}+\frac{\gamma}{(m+1)}+1-\gamma\right]^{\frac{1}{2}}, \tag{40}$$

$$\tau_2=\frac{\gamma(2+2\alpha+m)}{2(m+1)}-\left[\frac{\gamma^2(2+2\alpha+m)^2}{4(m+1)^2}+\frac{\gamma}{(m+1)}+1-\gamma\right]^{\frac{1}{2}}, \tag{41}$$

both of which are relevant to the dispersion relation. The limiting values for long wave-lengths

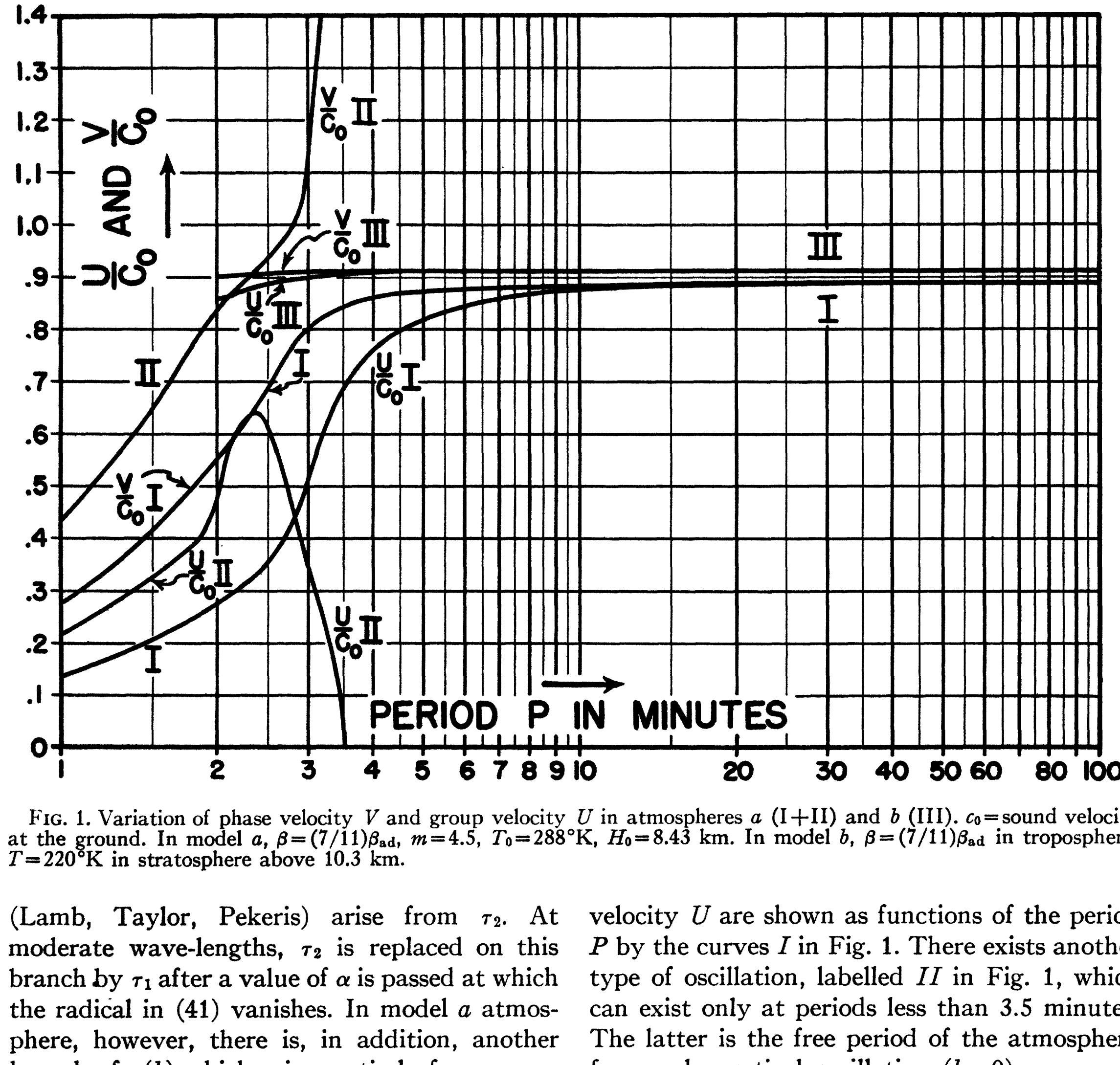

FIG. 1. Variation of phase velocity V and group velocity U in atmospheres a (I+II) and b (III). c_0 = sound velocity at the ground. In model a, $\beta=(7/11)\beta_{ad}$, $m=4.5$, $T_0=288°K$, $H_0=8.43$ km. In model b, $\beta=(7/11)\beta_{ad}$ in troposphere, $T=220°K$ in stratosphere above 10.3 km.

(Lamb, Taylor, Pekeris) arise from τ_2. At moderate wave-lengths, τ_2 is replaced on this branch by τ_1 after a value of α is passed at which the radical in (41) vanishes. In model a atmosphere, however, there is, in addition, another branch of $\sigma(k)$ which arises entirely from τ_1.

In the model a atmosphere in which the tropospheric temperature gradient is assumed to extend also into the stratosphere ($x_1=0$), the solution (33) for χ cannot contain the B term. Hence it follows from (34) that the frequency relation is given by

$$F_1'(x_0)-\left[\frac{1}{2}\left(1+\frac{1}{\tau}\right)-\frac{(m+1)}{x_0}\right]F_1(x_0)=0. \quad (42)$$

One branch of the solution of (42) was studied by Solberg[8] for several integral values of m (temperature gradients). We have evaluated this branch for the case $m=4.5$, $\beta=(7/11)\beta_{ad}$, and the resulting phase velocity V and group velocity U are shown as functions of the period P by the curves I in Fig. 1. There exists another type of oscillation, labelled II in Fig. 1, which can exist only at periods less than 3.5 minutes. The latter is the free period of the atmosphere for purely vertical oscillation ($k=0$), a general formula for which was given by Lamb[6] for a model a atmosphere. In the limit of long wavelengths ($k\rightarrow 0$), the motion in oscillation I is mainly horizontal ($w\simeq ku$), while in II the oscillation of individual particles is mainly in the vertical direction as in stellar pulsation. In the case of an atmosphere of constant temperature, discussed in the previous section, (V/c_0) and (U/c_0) of branch I are both equal to 1 for all periods, while in branch II they increase proportionately to the period from zero at $P=0$, and then terminate at the critical period.

The dispersion curves for model b, as obtained from (37), are shown by curves III in Fig. 1. These terminate at a period of two minutes. For shorter periods the radical in (36) and (37) becomes imaginary and free waves do not exist. We note that at $P=2$, V is about 2 percent above c_1, the sound velocity in the stratosphere, while U is about 2 percent less than c_1. The limiting value of U and V for long periods is 4 percent above c_1 and 8.5 percent below c_0. It is clear that the cut-off period arises from the fact that in contrast to atmosphere *a*, the temperature in the stratosphere of atmosphere *b* is too high to allow the propagation (trapping) of slow-speed, short-period waves. It is therefore probable that other model atmospheres having a finite temperature minimum in the stratosphere would also exhibit a short period cut-off.

5. EXCITATION OF THE FREE OSCILLATION OF ATMOSPHERE *b* BY A POINT SOURCE SITUATED AT THE GROUND IN WHICH *w* HAS A UNIFORM SPECTRUM

The fact that the cut-off period of model *b* occurs when the radical in (37) vanishes, shows that at the cut-off the wave energy in the free oscillation per column of the atmosphere is infinite. This suggests a vanishing normalization factor, i.e., a difficulty in exciting periods near the cut-off. In order to investigate this point quantitatively, with a view of application to the pressure wave produced by the North Siberian Meteor, we shall determine first the relative amplitudes of the pressure oscillation excited by a point source situated at the ground, in which w varies with time like $e^{i\sigma t}$. Such an investigation was carried out by the writer for long periods in connection with the interpretation of the Krakatoa wave.[2] There, a w-point source suggested itself by the nature of the volcanic eruption, and the problem was to determine the relative excitation of two modes at the same frequency. In this case of the question of relative excitation of various periods in the same mode, the nature of the point source is important. One obtains a different answer if the comparison is made on the basis of a uniform spectrum of the pressure oscillation at the point source, i.e., a p-source.

Referring to Eqs. (9) and (10), we seek a solution of the form

$$\chi(z,r,t)=e^{i\sigma t}\int_0^\infty A(k,\sigma)K(z,\sigma,k)\times J_0(kr)kdk, \quad (43)$$

where, by (9),

$$c^2\ddot{K}+(\dot{c}^2+\gamma g)\dot{K}+[\sigma^2-k^2(Q/\sigma^2)]K=0. \quad (44)$$

The function $A(k,\sigma)$ has to be chosen so as to satisfy the boundary conditions at the ground and at the point source. Since the point source is at the ground, w must be zero except at the source where we shall assume it to become integrably infinite. Such a discontinuous function can be represented by the discontinuous integral

$$w(0,r,t)=e^{i\sigma t}\int_0^\infty J_0(kr)kdk. \quad (45)$$

With the aid of (10) one obtains

$$\chi(z,r,t)=e^{i\sigma t}\int_0^\infty J_0(kr)K(z,\sigma,k)kdk/W(k,\sigma), \quad (46)$$

$$W(k,\sigma)=[1/(g^2k^2-\sigma^4)]\cdot[\sigma^2c^2\dot{\chi}+(g\gamma\sigma^2-gk^2c^2)\chi]_{z=0}. \quad (47)$$

At points outside of the source, where w is zero, we get from (11)

$$p(0,r,t)=(i\rho_0c_0^2/\sigma)e^{i\sigma t}\int_0^\infty J_0(kr)\times K(0,\sigma,k)kdk/W(k,\sigma). \quad (48)$$

The integral in (48) can be evaluated by the residue method at the zeros of W, which yield the various modes of free oscillation. The contribution from the first mode can be shown to be given at large ranges r by[10]

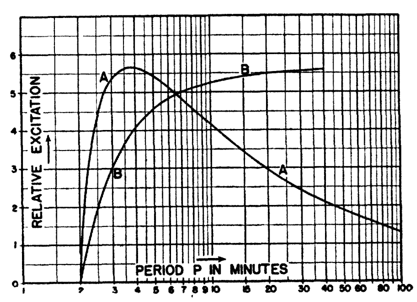

FIG. 2. Relative excitation of atmpsphere *b* by a *w*-point source situated at the ground. A = steady-state solution; B = solution for an impulsive point source (including effect of dispersion).

[10] H. Lamb, Phil. Trans. Roy. Soc. **A203**, 1 (1904); C. L. Pekeris, J. Acous. Soc. Am. **18**, 295 (1946); see also reference 2.

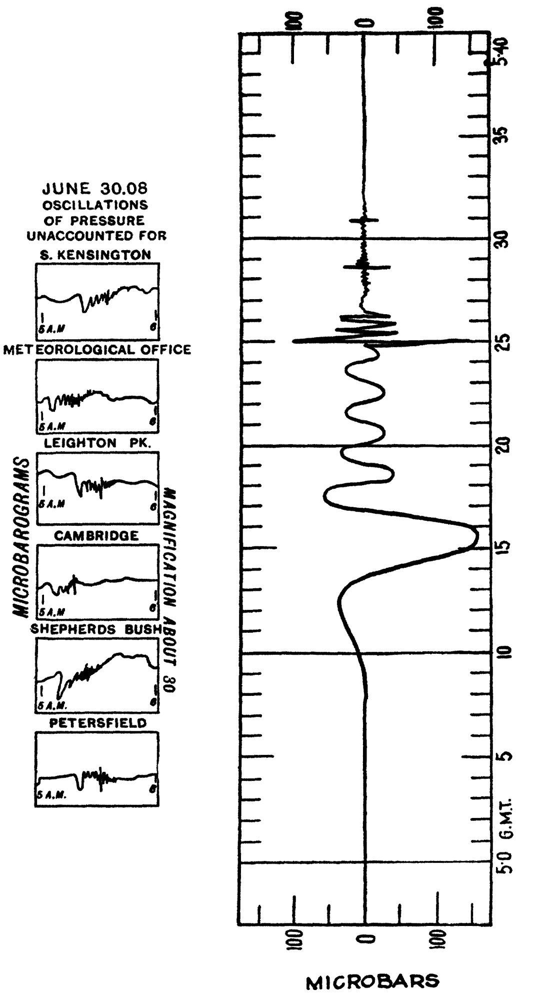

FIG. 3. Microbarograms of the pressure wave produced by the Great Siberian Meteor, recorded at English stations. On the right is a composite drawing made by F. J. W. Whipple.[1]

$$p(0,r,t)\to\frac{\rho_0c_0^2}{2\sigma}\left(\frac{2k_1}{\pi r}\right)^{\frac{1}{2}}\exp\left(i\sigma t-ik_1r+i\frac{3\pi}{4}\right)\times\frac{K(0,k_1,\sigma)}{(\partial W/\partial k)}. \quad (49)$$

Using the notation of (30) and (31) and putting

$$R=\frac{gW}{c_0^2\cdot K}=\frac{2\tau}{(1-\tau^2)}\left[\frac{1}{\chi}\frac{d\chi}{dx}+\frac{(m+1)}{x_0}-\frac{1}{2\tau}\right]_{x=x_0}, \quad (50)$$

we find that

$$p(0,r,t)=Ar^{-\frac{1}{2}}\exp(i\sigma t-ik_1r+i3\pi/4)\times[(\tau)^{\frac{1}{2}}\partial R/\partial x]^{-1}. \quad (51)$$

The quantity $[(\tau)^{\frac{1}{2}}\partial R/\partial x]^{-1}$, giving the relative excitation of the various frequencies in the pressure oscillation by a w-point source, is shown for atmosphere b by curve A of Fig. 2. As expected, it vanishes at the cut-off frequency. This curve applies only to the steady-state solution. For an impulsive excitation, such as the explosion of the Great Siberian Meteor was, one must take into account the additional weakening of the amplitude with increasing range because of the stretching of the wave by dispersion. This dispersive stretching introduces an additional factor $[r(d^2k/d\sigma^2)]^{-\frac{1}{2}}$ into (51). It arises from the evaluation of a σ-integral over (49) at the point of stationary phase of the exponential term. This factor, which is inversely proportional to the square root of the slope of the (U/c_0) curve *III* in Fig. 1, favors the short periods over the long ones. As a result, a relative excitation of atmosphere b by an impulsive w-point source, as shown by curve B in Fig. 2, is obtained. While this curve would fit the Krakatoa wave, for which the w-source is, moreover, a plausible excitation, the records of the Siberian Meteor wave shown in Fig. 3 suggest a preference for a period of from 5 to 8 minutes.

6. EXCITATION OF THE FREE OSCILLATION OF ATMOSPHERE *b* BY A POINT SOURCE SITUATED AT THE GROUND IN WHICH THE PRESSURE VARIATION HAS A UNIFORM SPECTRUM

In the impulsive excitation curve B of Fig. 2 the comparison was made on the basis of a uniform spectrum in w. It was, moreover, assumed that at the source the spacial distribution of w was represented by the discontinuous integral in (45). While it is of interest to analyze the field produced by a point source of well-defined characteristics, and w in (45) is a function which meets the required boundary condition at the ground, we must examine more closely the nature of the field in the immediate vicinity of a point source which radiates in a medium governed by the rather complex Eqs. (9) and (10), or (15). Because of the anisotropy resulting from the vertical stability of the atmosphere, we should not expect that the emitted wave will have even initially a spherical wave front. The deviation from initial spherical symmetry should be small at acoustic frequencies, but could become appreciable near Brunt's resonant frequency. Now the integral in (45) has the following origin. In the solution of the acoustic wave equation

$$\nabla^2\varphi=(1/c^2)(\partial^2\varphi/\partial t^2), \qquad (52)$$

where c^2 may be a function of z, in the form

$$\varphi(z, r, t)=e^{i\sigma t}\int_0^\infty F(z, \sigma, k)J_0(kr)dk, \qquad (53)$$

$$\ddot{F}+[(\sigma^2/c^2)-k^2]F=0, \qquad (54)$$

the conditions at a point source situated at $z=z_1$ are determined from the behavior of F at large wave numbers k. The asymptotic behavior of F for large k and small values of $(z-z_1)$ is $\exp[-k|z-z_1|]$, so that near the source

$$\varphi\rightarrow e^{i\sigma t}\int_0^\infty \exp[-k|z-z_1|]J_0(kr)dk = e^{i\sigma t}/[(z-z_1)^2+r^2]^{\frac{1}{2}}. \qquad (55)$$

It follows from (55) that at $z=z_1$

$$\frac{\partial\varphi^-}{\partial z}-\frac{\partial\varphi^+}{\partial z}=w^+-w^-=e^{i\sigma t}\int_0^\infty J_0(kr)kdk. \qquad (56)$$

where + and − refer to $z>z_1$ and $z<z_1$, respectively. This function is equal to the limit assumed by $2e^{i\sigma t}(z-z_1)/R$, as $z\rightarrow z_1$, namely, zero everywhere except at $r=0$, where it becomes infinite like $2e^{i\sigma t}/(z-z_1)$. The specification of a point-source solution of (52) by (56), which is attributable to Lamb,[10] gives the right result in the case of constant c and in the case of $c=az$, for which an explicit solution of the form (55) has been derived.[11]

In the free oscillations of the atmosphere, the asymptotic solution of (9), or of (15), for large k and small $(z-z_1)$ is

$$A\exp\left[-(k/\sigma)\int_{z_1}^{z} Qdz\right],\ z>z_1;$$
$$A\exp\left[-(k/\sigma)\int_{z}^{z_1} Qdz\right],\quad z<z_1. \qquad (57)$$

Hence, if we are considering a point source situated at $z=z_1$ in which the *pressure p varies like $e^{i\sigma t}$*, we have for the boundary condition at the source

$$\frac{\partial p^-}{\partial z}-\frac{\partial p^+}{\partial z}=2(Q/\sigma)\int_0^\infty J_0(kr)kdk,\quad z=z_1. \qquad (58)$$

We shall now obtain the complete solution for a p-point source situated at $z=z_1$, and, after allowing $z_1\rightarrow z_0$ (the ground), shall compute the relative excitation function for various periods. For the region above the source, let

$$\chi=\chi_1=CN,\quad \beta p_1=C(gc^2\dot{N}+\omega N),\quad z<z_1, \qquad (59)$$

$$\beta\equiv(k^2g^2-\sigma^4)/(i\rho_0\sigma),\quad \omega\equiv(\gamma g^2-\sigma^2c^2), \qquad (60)$$

where N is the solution of (9) which gives an integrable wave energy per atmospheric column. In the region between the source and the ground let

$$\chi=\chi_2=AM+BN,$$
$$\beta p_2=A(gc^2\dot{M}+\omega M)+B(gc^2\dot{N}+\omega N),$$
$$z_0>z>z_1. \qquad (61)$$

From the boundary condition of the vanishing of w at the ground we get by (10) and (61),

$$(B/A)=-\{[\sigma^2c^2\dot{M}+g(\gamma\sigma^2-k^2c^2)M]/[\sigma^2c^2\dot{N}+g(\gamma\sigma^2-k^2c^2)N]\}_{z=z_0}. \qquad (62)$$

At the source we must have continuity of p, and

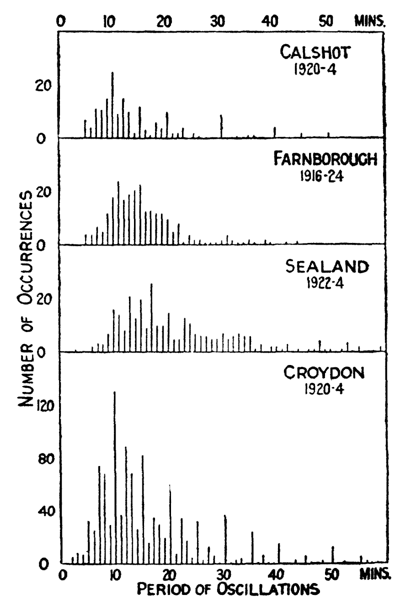

FIG. 4. Frequency of occurrence of pressure oscillations of well developed periods on microbarograms at English stations (after N. K. Johnson, reference 13).

[11] C. L. Pekeris, see reference 10.

we must satisfy (58):

$$p_1=p_2,\quad \dot{p}_1-\dot{p}_2=2(Q/\sigma)k, \tag{63}$$

which determine A, B, and C. After letting $z_1\rightarrow z_0$ in these relations, we arrive at the desired solution for a p-point source situated on the ground:

$$p(z_0,r,t)=\frac{-2\sigma c^2e^{i\sigma t}}{Q}\times\int_0^\infty \frac{J_0(kr)kdk(gc^2\dot{N}+\omega N)}{[\sigma^2c^2\dot{N}+g(\gamma\sigma^2-k^2c^2)N]}, \tag{64}$$

where N and N are evaluated at z_0. This integral can now be evaluated in terms of normal modes by the method of residues, and the result is a steady-state excitation function of the form

$$p=Ar^{-\frac{1}{2}}\exp(i\sigma t-ik_1r+i3\pi/4)\times[(\tau)^{\frac{1}{2}}Q(\partial R/\partial x)]^{-1}, \tag{65}$$

which differs from (51) by the extra factor $(1/Q)$. The same factor attaches also to the impulsive excitation function B of Fig. 2. Since Q vanishes at Brunt's period (9.5 minutes in atmosphere b), it follows that the impulsive excitation function for a p-point source vanishes at the cut-off period of 2 m and is peaked at 9.5 m. The integral of the excitation function over a finite spectral band width, including the Brunt period, is of course finite.

7. APPLICATIONS TO THE PRESSURE WAVE PRODUCED BY THE GREAT SIBERIAN METEOR AND TO PRESSURE OSCILLATIONS RECORDED ON MICROBAROGRAPHS IN ENGLAND

Figure 3 shows some original pressure records of the Great Siberian Meteor wave of 1908 obtained on microbarographs at several stations in England, and, on the right, a composite drawing of the principal features of these records made by F. J. W. Whipple.[1] A first pressure rise, lasting about 3 minutes, is followed by a rapid drop in the next two minutes, and then by a succession of four damped oscillations of about 2 minutes' period. The ratio of the velocities of the first arrival (323 m/sec.) to that of the last of the 2-min. waves (308 m/sec.) agrees with the ratio of the corresponding group velocities in atmosphere b shown in Fig. 1. With an assumed surface temperature of 288°K, $c_0=341$ m/sec., and the group velocity at a period of 2 min. in atmosphere b is 293 m/sec. This is 5 percent less than the observed value of 308 m/sec. The difference could be due either to a lower value of the surface temperature or to an actual stratospheric temperature of about 240°K, as against the assumed value of 220°K. The fact that no periods shorter than 2 min. were recorded, suggests the cut-off period shown in Fig. 1. The shocks recorded at 5^h25^m and later are probably due to the various rays (or normal modes) which travel through the sound channel associated with a minimum sound velocity in the stratosphere, in the manner observed by Ewing and co-workers[12] in the oceanic sound channel.

Figure 4 gives an analysis made by Johnson[13] of the frequency of occurrence of oscillations of well developed periods on microbarograms at English stations. He interprets the peaks as being associated with Brunt's resonant period. This view is supported by the excitation function for a pressure point source given in (65). The lack of periods less than 2 minutes is also manifest. In this connection the following quotation from Johnson's paper is of interest:

"In the second place we may notice that oscillations with a period of two minutes (or even less) are extremely obvious and very easily detected in the records of any properly adjusted microbarograph. The small number of oscillations of short period (say less than seven minutes) shown in Fig. 1 cannot therefore be attributed to difficulty of detection."

There are a number of problems raised by this investigation to which the writer hopes to return.

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[12] M. Ewing *et al.*, Bull. Geol. Soc. Am. **5**, 930 (1946); B. Gutenberg, Bull. Seis. Soc. Am. **36**, 327 (1946); J. Meteor. **3**, 27 (1946).

[13] N. K. Johnson, Q. J. Roy. Meteor. Soc. **55**, 20 (1929).