Refraction of Waves from a Point Source into a Medium of Higher Velocity*

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A point source is placed in one medium and the fields in a second medium, separated from the first by an infinite plane boundary, are calculated for the case that the wave velocity in the second medium exceeds that in the first medium. Absorption is neglected and the problem is solved both by ray methods and by evaluating, by the method of steepest descents, of an exact solution of the wave equation. The agreement between the solutions by different methods permits a ready and expected interpretation of the wave solution. Rays incident on the boundary at angles exceeding the critical angle are totally reflected; however, directly transmitted energy penetrates to all points in the second medium. At points in the second medium outside the critical angle, near the surface and far from the source, the directly transmitted fields are much smaller than the fields, exponentially decaying from the surface, which result from totally reflected rays. This suggests an experiment to measure quantitatively the penetration of the fields into the second medium in total reflection, easily performable for the case of sound waves penetrating from air into water.

I. INTRODUCTION

IN Fig. 1 a point harmonic source of sound of circular frequency ω is located at P, a height h above an infinite flat water surface. The origin of coordinates O is on the surface directly beneath P, whose rectangular coordinates are $(0, 0, h)$. If the subscripts 1 and 2 refer to air and water, respectively, the sound pressure p in either medium can be found from the relations

$$p_1 = -i\omega\rho_1\varphi_1 e^{-i\omega t},\quad p_2 = -i\omega\rho_2\varphi_2 e^{-i\omega t}, \tag{1}$$

where φ_1 and φ_2 satisfy the wave equation in their respective media,

$$\Delta\varphi_1 + k_1^2\varphi_1 = 0,\quad z>0;\qquad \Delta\varphi_2 + k_2^2\varphi_2 = 0,\quad z<0. \tag{2}$$

In these equations ρ denotes density, c sound velocity, and $k=\omega/c$. The boundary conditions determining the solution are: (a) φ_1 and φ_2 each represent outgoing waves of vanishing amplitude at infinity; (b) the solutions are everywhere finite except in the neighborhood of P where, except for an arbitrary multiplicative factor (which we shall ignore),

$$\varphi_1 = e^{ik_1R}/R, \tag{3}$$

R being the distance from the point P; (c) on the bounding plane $z=0$

$$\rho_1\varphi_1 = \rho_2\varphi_2,\quad \partial\varphi_1/\partial z = \partial\varphi_2/\partial z.$$

Formulated in this way this acoustic problem is formally identical with the electromagnetic problem of determining the radiation from a dipole antenna a height h above a plane earth. We have chosen to describe the problem in terms of a point source of sound radiating from air into water merely because we are interested here primarily in the case where the wave velocity in medium 2 is greater than that in medium 1. Sound incident from air to water is, in fact, totally reflected at angles of incidence greater than 13°. The electromagnetic problem was first solved by Weyl,[1] who extended a solution by Sommerfeld[2] of the case $h=0$. The radiation from an antenna located above the earth has since been treated by a number of writers,[3] principally interested, however, in the fields above the earth. Consequently, an evaluation of the fields in the second medium has only been given very recently by Kruger.[4] Kruger solves the electromagnetic problem for the general case that the second medium has a complex index of refraction. His analysis which is long and difficult cannot be applied to the

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[1] H. Weyl, Ann. d. Physik u. Chemie **60**, 481 (1919).
[2] A. Sommerfeld, Ann. d. Physik **28**, 665 (1909).
[3] Cf. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 573 ff.
[4] M. Kruger, Zeits. f. Physik **121**, 377 (1943).

problem under consideration for two reasons. Kruger's solution is immediately applicable only when the wave velocity in the second medium is less than the wave velocity in the first medium. Furthermore, Kruger makes the permeability of the second medium unity *ab initio*. With this simplification the analogy to the acoustic problem is no longer exact.[5]

In the following sections expressions are obtained for the sound pressure in the second medium, for the case that the index of refraction is real and less than unity. The solutions are valid at large distances from the source and are derived in two ways, first by solving the wave equation subject to the boundary conditions, next by ray acoustics.

The first solution is accomplished with considerably less effort than Kruger's, the simplification resulting from the fact that absorption is neglected. The solution by ray acoustics is even easier to obtain, and contains the essential features of the wave solution. Energy reaches a point C (Fig. 1), close to the surface and far from the source, in two ways. The ray PD, lying inside the critical angle, OPE, is refracted to C. In addition, the total reflection of a ray PB results in an exponential decrease of sound pressure with increasing depth.

These qualitative results are well known. The quantitative analysis developed below does, however, add to our understanding of the problem. The elementary analysis by ray acoustics gives a surprisingly accurate picture of the transmission.* The method of steepest descents used to evaluate the integrals obtained in the wave solution shows very beautifully that, although according to Huygens' principle wavelets must reach C (Fig. 1) from all points on the surface, actually these waves interfere so that only wavelets from the immediate vicinity of the point D remain to transmit any important amount of energy to C. A further result is that at points C near the surface, sufficiently far from the source and outside the critical angle, the directly transmitted pressure (along the path

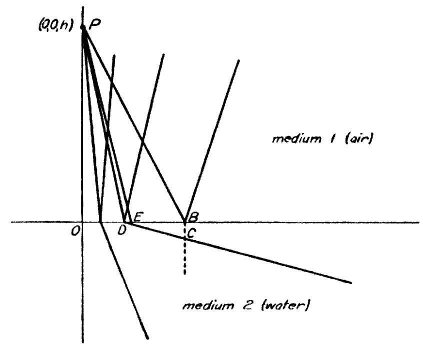

FIG. 1. Ray diagram.

PDC) is proportional to the depth and to the inverse square of the distance from the source, because of the increased divergence of the refracted rays as the incident angle approaches the critical angle. This means that at points C (Fig. 1), within a fraction of a wave-length of the surface, the sound pressure is produced primarily by the exponentially decreasing term, since the pressure at B on the surface varies as the inverse first power of the distance from the source, and the exponential decrease from the pressure at the surface is negligible for depths a fraction of the wave-length.

This result suggests the possibility of a quantitative verification of the theoretical prediction that in total reflection the wave amplitude does penetrate into the second medium. While this penetration is scarcely in doubt, nonetheless it is noteworthy that past optical experiments to test this prediction are all essentially qualitative. That is, they show the presence of some energy in the second medium, but do not quantitatively measure the amount of energy present.[6] The long wave-lengths of sound waves at sonic frequencies make it relatively simple to vary the depth of a hydrophone at points near the surface and outside the critical angle, thereby directly measuring the variation of sound pressure with depth and permitting comparison with the predicted exponential decay. In such an experiment a hydrophone sensitive to pressure rather than to velocity must be used, since the velocity at an

[5] G. Joos and J. Teltow, Physik. Zeits. **40**, 289 (1939).

* A general discussion of the approximations involved in the ray analysis has been given by C. Eckart in a paper "The Approximate Solution of One-Dimensional Wave Equations," soon to appear in the Reviews of Modern Physics.

[6] Cf., for example, R. W. Wood, *Physical Optics* (The Macmillan Company, New York, 1934), pp. 418 ff. A quantitative experiment by C. Schaefer and G. Gross, Ann. d. Physik **32**, 648 (1910), did not directly measure the fields in the second medium.

air-water surface is very small owing to the very great density of water compared to air. It should perhaps be noted that there is no contradiction between the fact that a hydrophone can respond to the exponentially decreasing pressure and the very meaning of total reflection, namely, that all the energy in the incident ray goes into the reflected ray. The presence of a hydrophone in the water introduces a new boundary with new boundary conditions. The result is to permit energy to enter the water from the otherwise totally reflected incident ray in just sufficient amount to energize the hydrophone. Although this assertion is difficult to prove in detail, it must be true because it can be shown directly from the fundamental equations that energy is always conserved.

II. SOLUTION OF THE WAVE EQUATION

If (x, y, z) are the coordinates of a point in medium 1, then[3]

$$\frac{e^{ik_1R}}{R}=\frac{ik_1}{2\pi}\int_0^{2\pi}d\psi\int_0^{(\pi/2)-i\infty}d\theta\,\sin\theta \times e^{ik_1[x\sin\theta\cos\psi+y\sin\theta\sin\psi+(h-z)\cos\theta]}, \quad (5)$$

when $0\leqslant z\leqslant h$.

Write

$$\varphi_1=\frac{e^{ik_1R}}{R}+\frac{ik_1}{2\pi}\int_0^{2\pi}d\psi\int_0^{(\pi/2)-i\infty}d\theta\,\sin\theta f_r \times e^{ik_1[x\sin\theta\cos\psi+y\sin\theta\sin\psi+(z+h)\cos\theta]}, \quad (6)$$

$$\varphi_2=\frac{ik_1}{2\pi}\int_0^{2\pi}d\psi\int_0^{(\pi/2)-i\infty}d\theta\,\sin\theta f_t \times e^{ik_2(x\sin\theta'\cos\psi+y\sin\theta'\sin\psi-z\cos\theta')}, \quad (7)$$

where f_r, f_t, and θ' are as yet undetermined functions of θ, ψ. The boundary conditions at $z=0$ are satisfied when

$$k_1\sin\theta=k_2\sin\theta', \quad (8)$$

$$f_t=\frac{2\rho_1k_1\cos\theta e^{ik_1h\cos\theta}}{\rho_2k_1\cos\theta+\rho_1k_2(1-k_1^2k_2^{-2}\sin^2\theta)^{\frac{1}{2}}}, \quad (9)$$

$$f_r=\frac{\rho_2k_1\cos\theta-\rho_1k_2(1-k_1^2k_2^{-2}\sin^2\theta)^{\frac{1}{2}}}{\rho_2k_1\cos\theta+\rho_1k_2(1-k_1^2k_2^{-2}\sin^2\theta)^{\frac{1}{2}}}. \quad (10)$$

Choose a system of spherical coordinates with the polar axis directed along the positive z axis. The contour in the complex θ plane for the above integrals lies along the real axis from $\theta=0$ to $\theta=\pi/2$, and then in the fourth quadrant to $\theta=\pi/2-i\infty$. For real θ, therefore, the integral (5) represents a system of plane waves moving downward, i.e., outward from the source. The polar angle of the direction of advance of the plane wave defined by ψ, θ in (5) is $\pi-\theta$. Thus θ is the angle of incidence of the wave on the surface. The corresponding wave in (6) is moving upward in the direction whose polar angle is θ, and in (7) is moving in a direction whose polar angle is $\pi-\theta'$. Since (7) must represent an outgoing wave at infinity, the appropriate solution of (8) for real θ' is $\theta'<\pi/2$. Hence θ' is the angle of refraction of a plane wave striking the surface with incident angle θ, and f_r and f_t are, respectively, the reflection and transmission coefficients of the wave amplitude. For real θ', $\cos\theta'$, the square root in (9) and (10), is positive. Along the contour, $\sin\theta$ increases monotonically. Consequently, in order that (7) be finite for negative z,

$$(1-k_1^2k_2^{-2}\sin^2\theta)^{\frac{1}{2}}=i(k_1^2k_2^{-2}\sin^2\theta-1)^{\frac{1}{2}}, \quad (11)$$

when $k_2<k_1\sin\theta$.

If z is now taken as a *positive* depth beneath the surface, $\alpha=c_1/c_2$, $\beta=\rho_2/\rho_1$, (7) can be rewritten as

$$\varphi_2=\frac{ik_1}{\pi}\int_0^{2\pi}d\psi\int_0^{(\pi/2)-i\infty}d\theta\,\sin\theta \times\frac{e^{ik_1(x\sin\theta\cos\psi+y\sin\theta\sin\psi+h\cos\theta+z(\alpha^2-\sin^2\theta)^{\frac{1}{2}})}}{\beta\cos\theta+(\alpha^2-\sin^2\theta)^{\frac{1}{2}}}. \quad (12)$$

For transmission from air to water, $\beta=850$ and $\alpha=0.23$.

Introducing $r^2=x^2+y^2$, letting $\sin\theta=u$, and using Eq. (15) below, it is easily shown that[3]

$$\varphi_2=2ik_1\int_0^{\infty}du\,uJ_0(k_1ru)(e^{ik_1Q}/P). \quad (13)$$

For sufficiently large k_1ru, J_0 in (13) can be replaced by its asymptotic expansion. Suppose the asymptotic expansion is valid for $|u|>a_0$, where a_0 will be determined later. Then introducing the asymptotic expansion for J_0, with a

a complex number whose magnitude is a_0,

$$\varphi_2 = 2ik_1 \int_0^\alpha du u J_0(k_1 r u) \frac{e^{ik_1 Q}}{P} + ie^{-(i\pi/4)} \left(\frac{2k_1}{\pi r}\right)^{\frac{1}{2}} \int_\alpha^\infty du u^{\frac{1}{2}} \frac{e^{ik_1 A}}{P} + ie^{i\pi/4} \left(\frac{2k_1}{\pi r}\right)^{\frac{1}{2}} \int_\alpha^\infty du u^{\frac{1}{2}} \frac{e^{ik_1 B}}{P}. \quad (14)$$

In (13) and (14)

$$\begin{aligned} P(u) &= \beta(1-u^2)^{\frac{1}{2}} + (\alpha^2-u^2)^{\frac{1}{2}}, \\ Q(u) &= h(1-u^2)^{\frac{1}{2}} + z(\alpha^2-u^2)^{\frac{1}{2}}, \\ A(u) &= h(1-u^2)^{\frac{1}{2}} + z(\alpha^2-u^2)^{\frac{1}{2}} + ru, \\ B(u) &= h(1-u^2)^{\frac{1}{2}} + z(\alpha^2-u^2)^{\frac{1}{2}} - ru. \end{aligned} \quad (15)$$

The phase of u lies between $-\pi$ and π. Figure 2 shows the complex plane with cuts starting at 1, -1, α, and $-\alpha$, in such directions as to fulfill the required conditions that, on the positive real axis, the expressions $(1-u^2)^{\frac{1}{2}}$ and $(\alpha^2-u^2)^{\frac{1}{2}}$ must be positive real when u is small and positive imaginary when u is large.

For large real u the complex numbers $1-u^2$ and α^2-u^2 have phase π. The contour for each of the infinite integrals in (14) runs from somewhere near the origin to $+\infty$ without crossing any of the cuts. In the various regions of the plane the signs of the real and imaginary parts of $(1-u^2)^{\frac{1}{2}}$ and $(\alpha^2-u^2)^{\frac{1}{2}}$ are negative in the following regions.

$$\begin{aligned} Re(1-u^2)^{\frac{1}{2}} < 0&: \; III,\ V \\ Re(\alpha^2-u^2)^{\frac{1}{2}} < 0&: \; II,\ III,\ V,\ VI \\ I(1-u^2)^{\frac{1}{2}} < 0&: \; I,\ II,\ V,\ VIII \\ I(\alpha^2-u^2)^{\frac{1}{2}} < 0&: \; I,\ V,\ VI,\ VIII. \end{aligned} \quad (16)$$

Otherwise they are positive. The real and imaginary parts of the radicals change sign discontinuously at the cuts but become zero on the real and imaginary axes if they change sign crossing these axes.

Assume, specifically, $\alpha < 1$, $\beta > 1$. Then the imaginary parts of both radicals have the same sign at $u = \pm(\beta^2-\alpha^2)^{\frac{1}{2}}/(\beta^2-1)^{\frac{1}{2}}$, the only points where $P(u)$ can possibly be zero. Thus $P(u)$ is never zero and the contours can be deformed without fear of passing through a pole.[7] Asymptotic expressions for the infinite integrals in (14) can be obtained by the method of steepest descents.[8] The saddle points are found from

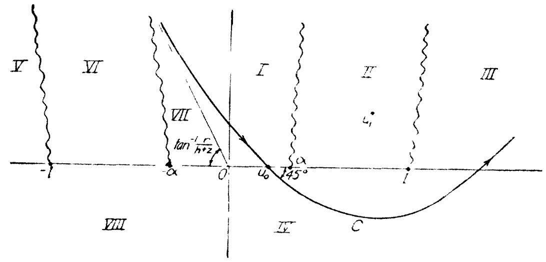

FIG. 2. Regions of the complex plane and contour along which $ReA(u) = A(u_0)$, when $r^2/(r^2+h^2) < \alpha^2$.

$$A'(u) = [-uh/(1-u^2)^{\frac{1}{2}}] - [uz/(\alpha^2-u^2)^{\frac{1}{2}}] + r = 0, \quad (17)$$

$$B'(u) = [-uh/(1-u^2)^{\frac{1}{2}}] - [uz/(\alpha^2-u^2)^{\frac{1}{2}}] - r = 0. \quad (18)$$

Equation (17) has a root u_0 somewhere on the real axis between 0 and α, and may have a root u_1 somewhere in II (Fig. 2). There are no other roots in the right half of the plane nor are there any roots in VII. It is necessary to find the contour passing through u_0 along which

$$ReA(u) = ReA(u_0),$$

and along which $IA(u)$ increases as u approaches infinity. Along this contour the phase of e^{ik_1A} in (14) is constant, while the real part of e^{ik_1A} decreases exponentially as infinity is approached. If this contour leads from u_0 to a point at infinity from which the integral to $+\infty$ is zero, the integral involving $A(u)$ in (14) can at once be evaluated. If the contour does not lead to such a point at infinity, it will be necessary to return to $+\infty$ along another path, possibly passing through u_1.

In the vicinity of u_0

$$A(u) = A(u_0) + \tfrac{1}{2}A''(u_0)(u-u_0)^2. \quad (19)$$

$A''(u_0)$ is real and negative. Consequently, the appropriate contour cuts the real axis at u_0 at an angle of $-45°$, as in Fig. 2. Furthermore, $A(u)$ is real and finite on the real axis between $-\alpha$ and α, and has but one maximum in that interval, at u_0. It follows that the extension to

[7] A somewhat lengthier argument shows that P is never zero whatever the value of β.

[8] G. N. Watson, *Treatise on the Theory of Bessel Functions* (The Macmillan Company, New York, 1944), p. 235.

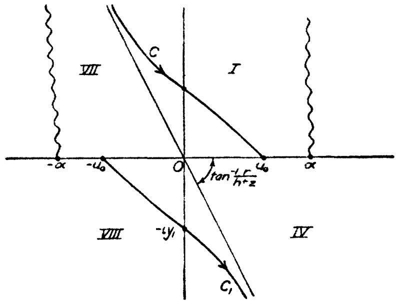

FIG. 3. Contours C and C_1.

the left of the contour $ReA(u)=A(u_0)$ never crosses the real axis again but continues into regions I or VII. In these regions, for large u,

$$(1-u^2)^{\frac{1}{2}}=(\alpha^2-u^2)^{\frac{1}{2}}=-iu. \tag{20}$$

Substituting (20) in (15) with $u=\rho e^{i\varphi}$, ρ approaching infinity, it is seen that $ReA(u)$ can be finite for large u only if φ asymptotically approaches the value $\tan^{-1}-r/(h+z)$. Thus the contour $ReA(u)=A(u_0)$ extends to the left into region VII, and in this region $IA(u)$ approaches $+\infty$ as u becomes infinite. The appropriate contour $ReA(u)=A(u_0)$ has therefore been determined to the left of u_0, as shown in Fig. 2.

The rules (16) and Eqs. (15) show that on the real axis for $-\alpha<u<\alpha$, $B(-u)=A(u)$. Hence $B(u)$ has a saddle point at $-u_0$. Also $A(u)$, u in IV, equals $B(-u)$, $-u$ in VII. $A(u)$, u in I, equals $B(-u)$, $-u$ in $VIII$. The contour C_1, therefore, along which $ReB(u)=B(-u_0)$ and $IB(u)$ increases as $u\rightarrow\infty$, leaves $-u_0$ at an angle of $-45°$, as in Fig. 3, and in IV mirrors the contour C in VII, C the contour along which $ReA(u)=A(u_0)$. $P(u)$, u in IV, equals $P(-u)$, u in VII. $IB(u)\rightarrow+\infty$ as u becomes infinite anywhere in region IV. Let $-iy_1$ be the point at which C_1 crosses the negative y axis, and choose $a=-ia_0$ in (14). Then in (14)

$$\int_{-ia_0}^{\infty} duu^{\frac{1}{2}}\frac{e^{ik_1B}}{P}=\int_{-ia_0}^{-iy_1} duu^{\frac{1}{2}}\frac{e^{ik_1B}}{P}+\int_{\substack{-iy_1\\ C_1}} duu^{\frac{1}{2}}\frac{e^{ik_1B}}{P}. \tag{21}$$

In (21) the integral along C_1 proceeds from $-iy_1$ along the direction shown by the arrow (Fig. 3). Furthermore,

$$\int_{\substack{-iy_1\\ C_1}} duu^{\frac{1}{2}}\frac{e^{ik_1B}}{P}=e^{-i(\pi/2)}\int_{C}^{iy_1} duu^{\frac{1}{2}}\frac{e^{ik_1A}}{P}. \tag{22}$$

In (22) the integral along C proceeds to iy_1 along the direction shown by the arrow (Fig. 3). Substituting Eqs. (21) and (22) in (14) gives, therefore,

$$\varphi_2=2ik_1\int_0^{-ia_0} duuJ_0(k_1ru)\frac{e^{ik_1Q}}{P}+ie^{-(i\pi/4)}\left(\frac{2k_1}{\pi r}\right)^{\frac{1}{2}}\int_{-ia_0}^{ia_0} duu^{\frac{1}{2}}\frac{e^{ik_1A}}{P}+ie^{-(i\pi/4)}\left(\frac{2k_1}{\pi r}\right)^{\frac{1}{2}}\int_C^{\infty} duu^{\frac{1}{2}}\frac{e^{ik_1A}}{P}. \tag{23}$$

In (23) the integral along C proceeds from infinity in region VII through u_0 and along the appropriate extension of $ReA(u)=A(u_0)$ in the part of the plane to the right of u_0, arriving finally at infinity on the positive real axis.

On the real axis, between α and 1, $ReA(u)=h(1-u^2)^{\frac{1}{2}}+ru$. Let $r^2/(r^2+h^2)=\gamma^2$. If $\gamma^2<\alpha^2$, $ReA(u)$ decreases monotonically as u varies from α to 1. Therefore, since $ReA(u)$ also decreases as u varies from u_0 to α, if $\gamma^2<\alpha^2$ the contour $ReA(u)=A(u_0)$ cannot cross the real axis between α and 1. Since $ReA(u)$ is constant on the contour, the contour can become parallel to the imaginary axis only when $\partial ReA(u)/\partial y=0$, $u=x+iy$. This means that dA/du is pure real whenever the contour is parallel to the y axis. It is readily shown that in IV the imaginary parts of $u/(1-u^2)^{\frac{1}{2}}$ and $u/(\alpha^2-u^2)^{\frac{1}{2}}$ have the same sign. Thus, from (17), the contour never becomes parallel to the y axis in IV. In III and IV for large u, $(1-u^2)^{\frac{1}{2}}=(\alpha^2-u^2)^{\frac{1}{2}}=iu$. Then, as before, using $u=\rho e^{i\varphi}$, $ReA(u)=A(u_0)$ must approach infinity in III with $\varphi=\tan^{-1}r/(h+z)$. In III $IA(u)\rightarrow\infty$ as $u\rightarrow\infty$. This completes the determination of the contour C in (23) for the case $r^2/(r^2+h^2)<\alpha^2$ (Fig. 2).

When $\gamma^2>\alpha^2$, the contour may be more complicated. In this case, $ReA(u)$ has a maximum on the real axis at $u=\gamma$, at which point $ReA(u)=(r^2+h^2)^{\frac{1}{2}}$.

If $A(u_0)<(r^2+h^2)^{\frac{1}{2}}$ the contour may cross the real axis between α and 1 and go to infinity in II. If $A(u_0)>(r^2+h^2)^{\frac{1}{2}}$, the contour must go to infinity in III, as in Fig. 2. When $\gamma^2>\alpha^2$, u_0 approaches α as z approaches zero. In fact, for small z, (17) is solved by

$$u_0=\alpha-z^2/2\alpha b^2, \tag{24}$$

where

$$b=r/\alpha-h/(1-\alpha^2)^{\frac{1}{2}} \tag{25}$$

is positive when $\gamma^2>\alpha^2$. The equation of the contour $ReA(u)=A(u_0)$ becomes in this case

$$ReA(u)=\alpha r+h(1-\alpha^2)^{\frac{1}{2}}+z^2/2b. \tag{26}$$

In the vicinity of $u=\alpha$, for small z, the equation of the contour can be found by letting $u=\alpha+w$, w small, and solving for w. The only question is the choice of sign for $(\alpha^2-w^2)^{\frac{1}{2}}=(-2\alpha w-w^2)^{\frac{1}{2}}$, which is solved by noting from Fig. 2, that the cut through α is so drawn that at $u=u_0$ the phase of w is $-\pi$. Thus $(\alpha^2-u^2)^{\frac{1}{2}}$ is positive real when $u=u_0$ if, for small w, $(\alpha^2-u^2)^{\frac{1}{2}}=i(2\alpha w)^{\frac{1}{2}}$. The solution to (26) is then found to be

$$w=\frac{z^2e^{i\varphi}}{2b^2\alpha(1-\sin\varphi)}. \tag{27}$$

Equation (27) shows that as φ varies from $-\pi$ to 0, u varies continuously from u_0 to a point on the real axis to the right of α. In other words, for small z the contour does cross the real axis into region II (Fig. 2), and, in fact, it follows from (27) that it crosses the axis to the right of α at an angle of 45°. For small w

$$A'(u)=\frac{-h\alpha}{(1-\alpha^2)^{\frac{1}{2}}}-\frac{h\alpha^2w}{(1-\alpha^2)^{\frac{3}{2}}}+i\alpha z(2\alpha)^{\frac{1}{2}}w^{-\frac{1}{2}}+r. \tag{28}$$

Or with $w=\rho e^{i\varphi}$

$$\begin{aligned}A'(u)=\alpha b&-h\alpha^2(1-\alpha^2)^{-\frac{3}{2}}\rho\cos\varphi\\&+\alpha z(2\alpha)^{\frac{1}{2}}\rho^{-\frac{1}{2}}\sin\tfrac{1}{2}\varphi\\&+i[-h\alpha^2(1-\alpha^2)^{-\frac{3}{2}}\rho\sin\varphi\\&\qquad+\alpha z(2\alpha)^{\frac{1}{2}}\rho^{-\frac{1}{2}}\cos\tfrac{1}{2}\varphi].\end{aligned} \tag{29}$$

Equation (29) shows that for small ρ and $0<\varphi<\pi$, i.e., for points in II near α, $ReA'(u)$ is not zero. On the other hand, $IA'(u)=0$ on a contour which for sufficiently small z lies arbitrarily close to the real axis. This means that for sufficiently small z the contour $ReA(u)=A(u_0)$ does not become parallel to the x axis but necessarily becomes parallel to the y axis in the immediate vicinity of $u=\alpha$. In other words, for small z, the contour $ReA(u)=A(u_0)$ curves around the point $u=\alpha$ and, in II, proceeds toward the positive imaginary axis. For small z, at points not near α,

$$A'(u)=-hu(1-u^2)^{-\frac{1}{2}}+r, \tag{30}$$

and it can be shown that the real part of (30) is not zero anywhere along the contour $ReA(u)=A(u_0)$ in II.

This completes the proof that for small z and $r^2/(r^2+h^2)>\alpha^2$ the contour C in (23) after passing through u_0 goes to infinity in the extension of region II into the second quadrant. To get to $+\infty$ on the real axis, the contour has to return via the saddle point u_1 (Fig. 2). By methods similar to those used above it can be shown that the returning contour goes through u_1 at an angle of $-45°$, for small z, and does in fact proceed to $+\infty$ in region III (Fig. 2).

$$\begin{aligned}(d/dz)A(u_0)&=(\alpha^2-u_0^2)^{\frac{1}{2}}+A'(u_0)(du_0/dz)\\&=(\alpha^2-u_0^2)^{\frac{1}{2}}.\end{aligned} \tag{31}$$

Equation (31), which follows from (15) and the definition of u_0, shows that as z increases, for any fixed values of h and r, $A(u_0)$ increases, since $u_0<\alpha$. In fact, for sufficiently large z, (17) shows u_0 approaches zero, and $A(u_0)$ therefore approaches infinity with increasing z. Since the maximum value of $ReA(u)$ on the real axis between α and 1 is $(r^2+h^2)^{\frac{1}{2}}$, the preceding statements mean that as z is increased from zero, with $\gamma^2>\alpha^2$, the contour C in (23), which for small z passes through both u_0 and u_1, somehow deforms with increasing z so that for sufficiently large z the contour after leaving u_0 goes to infinity in III without first going to infinity in II. The complexity of the algebra makes it difficult to determine explicitly the value of z, as a function of r and h, at which the contour changes abruptly from one which goes through both u_0 and u_1 to a contour which goes only through u_0. However, examination of the contours indicates the transition point probably occurs when $A(u_0)=ReA(u_1)$, and that for values of z such that $A(u_0)$ exceeds $ReA(u_1)$ the contour does not pass through u_1.

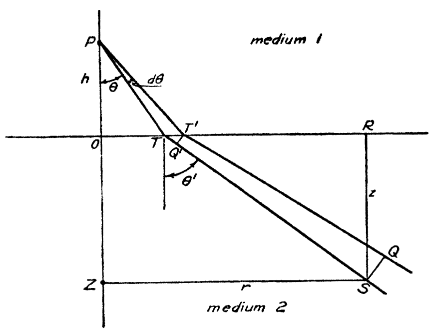

FIG. 4. Divergence of two infinitesimally separated rays.

An asymptotic expansion of the contour integral in (23) can now be obtained[8] by reverting the relation between u and A. The dominant terms of the asymptotic expansion are most simply found by approximating $A(u)$ by (19) in the vicinity of its saddle points, and replacing the contour integral by the integral of a Gaussian function from $-\infty$ to $+\infty$. With this approximation the last term in (23) becomes

$$\frac{2u_0^{\frac{1}{2}}r^{-\frac{1}{2}}e^{ik_1[h(1-u_0^2)^{-\frac{1}{2}}+z\alpha^2(\alpha^2-u_0^2)^{-\frac{1}{2}}]}}{P(u_0)[h(1-u_0^2)^{-\frac{3}{2}}+\alpha^2 z(\alpha^2-u_0^2)^{-\frac{3}{2}}]^{\frac{1}{2}}} \qquad (32)$$

when the contour does not pass through u_1. When the contour passes through u_1, its contribution must be added to (32). For small z this contribution is

$$\frac{2h(r^2+h^2)^{-1}e^{[ik_1(r^2+h^2)^{\frac{1}{2}}-k_1 z(\gamma^2-\alpha^2)^{\frac{1}{2}}]}}{[\beta h(r^2+h^2)^{-\frac{1}{2}}+i(\gamma^2-\alpha^2)^{\frac{1}{2}}]}. \qquad (33)$$

The other integrals in (23) can be shown to be negligible compared to (32) and (33) for sufficiently large k_1r. Suppose the asymptotic expansion of $J_0(z)$ represents $J_0(z)$ with sufficient accuracy for $|z|>n$, n some number. Then a_0 in (23) equals n/k_1r. With the substitution $u=w/k_1r$, the first two integrals in (23) are readily seen to be of order $1/k_1r$ compared to (32) and (33). This completes the wave solution of the problem.

III. SOLUTION BY RAY ACOUSTICS

In Fig. 4, PT and PT' are two rays, lying in the same plane with OP, whose angles of incidence on the surface are θ and $\theta+d\theta$. The angles of refraction for these rays are θ' and $\theta'+d\theta'$. θ is assumed less than the critical angle. The ray PT is refracted to a point S, at depth $OZ=z$, and distance $OR=r$ from the axis OP. The amplitude of the pressure at S will now be calculated.

Consider the surfaces formed by rotating Fig. 4 about the line POZ. Energy leaves P in a volume between the cones formed by rotating PT and PT'. Some of this energy is reflected and some refracted. According to ray acoustics, the refracted energy always lies between the cones formed by rotating TS and $T'Q$. The pressure at S can therefore be determined by finding the relation between SQ, which is perpendicular to TS, and $T'Q'$, which is perpendicular to TS at Q'. $T'Q'$ measures the energy density at the point where the energy first enters the water. The relation between the pressure in medium 2 at T, and the pressure in medium 1 at T is obtained from the coefficient f_t, Eq. (9), assuming that the curvature of the incident wave front at T is sufficiently small that the incident wave at T can be regarded as plane. Referring to Fig. 4,

$$\begin{aligned} OT &= h\tan\theta, \\ TT' &= h\sec.^2\theta d\theta, \\ T'Q' &= TT'\cos\theta' = h\sec.^2\theta\cos\theta' d\theta. \end{aligned} \qquad (34)$$

The distance SQ to first order in differentials is the perpendicular distance $T'Q'$ at the surface plus the additional distance resulting from the rotation $d\theta'$.

$$SQ = T'Q' + TSd\theta' = T'Q' + z\sec.\theta' d\theta'. \qquad (35)$$

From (8) and the definition of α,

$$\sin\theta = \alpha\sin\theta', \quad \cos\theta d\theta = \alpha\cos\theta' d\theta'. \qquad (36)$$

Let the pressure in the water at T be $p(T)$ and the pressure at S be $p(S)$. The energy which leaves P between the cones formed by PT and PT', and which enters the water, must pass through a surface of area $2\pi(OT')(T'Q')$ at T, and through a surface of area $2\pi(ZS)(SQ)$ at S. The energy density is proportional to the square of the pressure. Therefore, the ratio of the absolute magnitudes of $p(S)$ and $p(T)$ is given by

$$|(p(S)/p(T))| = \{[(OT')(T'Q')/(ZS)(SQ)]\}^{\frac{1}{2}}. \qquad (37)$$

The magnitude of the pressure in the water at T is, from (1), (5), and (7),

$$|p(T)| = \rho_2 p_0 |f_t| / \rho_1(PT), \qquad (38)$$

where p_0 is the magnitude of the outgoing pressure from P at unit distance from P. The phase at S is simply given by the path length from P to S. In other words, from (37) and (38)

$$p(S) = \frac{\rho_2 p_0 |f_t|}{\rho_1(PT)} \left[\frac{(OT')(T'Q')}{(ZS)(SQ)} \right]^{\frac{1}{2}} e^{ik_1(PT)+ik_2(TS)}. \qquad (39)$$

With the aid of Eqs. (9) and (34) to (36), and some elementary trigonometry, Eq. (39) yields

$$p(S) = \frac{2\beta p_0 r^{-\frac{1}{2}} (\sin\theta)^{\frac{1}{2}} e^{ik_1[h(1-\sin^2\theta)^{-\frac{1}{2}} + \alpha^2 z(\alpha^2 - \sin^2\theta)^{-\frac{1}{2}}]}}{[\beta \cos\theta + (\alpha^2 - \sin^2\theta)^{\frac{1}{2}}][h(1-\sin^2\theta)^{-\frac{3}{2}} + z\alpha^2(\alpha^2-\sin^2\theta)^{-\frac{3}{2}}]^{\frac{1}{2}}}. \qquad (40)$$

The angle θ of the incident ray which reaches S (Fig. 4) is determined by the equation

$$h \tan\theta + z \tan\theta' = r,$$

or

$$[h \sin\theta/(1-\sin^2\theta)^{\frac{1}{2}}] + [z \sin\theta/(\alpha^2 - \sin^2\theta)^{\frac{1}{2}}] = r. \qquad (41)$$

Equation (41) in $\sin\theta$ is identical with Eq. (17) which defines u_0. Furthermore, when (33) is of no account the pressure in the water is $\omega\rho_2$ times the expression (32), while the magnitude of the outgoing pressure at unit distance from P is $\omega\rho_1$, according to (1) and (3). Thus the expression (40), derived by ray acoustics, agrees exactly with Eq. (32), obtained by solving the wave equation. Remembering that Eq. (13) was obtained with the substitution $\sin\theta = u$, it appears that the saddle point u_0 simply determines the real angle of incidence contributing primarily to the pressure in the second medium, and this angle is just the angle determined by Snell's law and ray acoustics. The expression (32) thus represents the directly transmitted energy to any point in the second medium, and the approximations made in deriving (32) are apparently equivalent to assuming the curvature of the wave front negligible, the approximation made in deriving (40).

The expression (33) does not contribute to the pressure if $\gamma^2 < \alpha^2$. Evidently, at $z=0$, $\gamma^2 < \alpha^2$ means a point on the surface within the critical angle. Thus any point in the second medium lying directly beneath a point on the surface which is within the critical angle receives only directly transmitted energy. A point near the surface for which $\gamma^2 > \alpha^2$, i.e., a point near the surface outside the critical angle, also is reached by the exponentially decaying term (33). However, for sufficiently large z, for any point such that $\gamma^2 > \alpha^2$, the term (33) no longer contributes to the received pressure. This latter result is a little difficult to interpret. It may be a spurious result of the approximations made, or it may actually be a real effect. If it is real, it may be experimentally verifiable at points in the immediate vicinity of $\gamma^2 = \alpha^2$.

For the case of sound incident from air to water, $\beta \gg 1$, the magnitude of (32) is for small z, using (24)

$$2z\beta^{-1}(\alpha r)^{-\frac{1}{2}}(1-\alpha^2)^{-\frac{1}{2}} b^{-\frac{3}{2}}. \qquad (41)$$

Thus, for small z the directly transmitted pressure is smaller than the exponentially damped term (33) whose magnitude is, for $\beta \gg 1$,

$$2\beta^{-1}(r^2+h^2)^{-\frac{1}{2}} \exp[-k_1 z(\gamma^2 - \alpha^2)^{\frac{1}{2}}]. \qquad (42)$$

For very large r, $r \gg z$ and $r \gg h$, the magnitude of (32) becomes

$$2\beta^{-1}\alpha z(1-\alpha^2)^{-\frac{1}{2}}(r^2+z^2)^{-1}. \qquad (43)$$

The expression (43) shows that at sufficiently large distances from the source, at any given depth, the increased divergence of the rays causes the pressure to vary as the inverse square of the distance from the source.

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