On the Analysis of Extensive Cosmic-Ray Shower Data

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Formulae are derived, on the basis of the primary electron hypothesis, which describe the behavior of counters exposed to the large air showers. In particular, expressions for the variation of counting rate with area of the counters, number of counters, and altitude are given, and seem to agree fairly well with experiment, except for the latter at very high altitudes.

I. INTRODUCTION

THERE are at present at least two separate mechanisms which have been suggested for the development of the large air showers. The first of these regards the showers as generated by the ordinary cascade process from extremely energetic primary electrons (tacitly assuming that the ordinary formulae of electrodynamics are valid at such high energies). Here one has a definite picture whose consequences can be worked out in detail. One usually assumes that primary electrons impinge upon the top of the atmosphere with an energy spectrum proportional to $E_0^{-\lambda}dE_0$; λ is adjusted to fit the experiments and usually chosen to be about 2.8. The second mechanism describes the showers as produced by the high energy tail of a primary proton spectrum, and one can choose one of a number of different means of converting a fraction of the primary energy into the soft component (either through the decay of heavy mesons into lighter ones with emission of soft radiation, or other possibilities). Clearly a great deal of flexibility in detail is possible here, so that it does not seem profitable at present to work out the details of any particular version until one knows somewhat more precisely whether there are any real difficulties with the primary electron hypothesis, and, if so, where they are. One might mention, however, one feature which has to do with the penetration of the primaries that is common to all theories that ascribe the air showers to primary protons. Experiments on the penetrating showers, which are probably characteristic of the primary events, both at sea level and at high altitudes have suggested that the primaries have a mean free path of about 125 g/cm^2 in air. Since this distance is roughly equal to three shower units, one might expect that this comparatively high penetrating power would manifest itself in the altitude variation of the properties of a shower deep in the atmosphere. In particular, the asymptotic altitude variation should be characteristic of this distance rather than the distance of one shower unit which describes the decay of individual cascade showers after they have passed their maxima. For these reasons we have decided to make a study of the behavior of the showers near sea level, using the primary electron hypothesis and the cascade theory, to see whether any obvious difficulties are yet apparent and to indicate in which directions experiments might profitably be performed.

In making this study, we have had in mind in particular counter experiments, in which one studies the variation of the counting rate with either the altitude, the number of counters in coincidence, their area, or possibly the zenith angle of the shower axis. We will not consider variations in the geometry of the counter arrangement, since we will always assume that all the counters involved are concentrated at a point. We will discuss this condition in more detail later. For example, experiments have been performed by Hilberry,[1] and by Kraybill and Ovrebo,[2] in which the altitude was varied, and by Cocconi, Loverdo, and Tongiorgi,[3] in which both number of counters and area were varied. In the latter case the information was combined to calculate a density spectrum.

II. GENERAL ASSUMPTIONS

We will use the radial distribution of the shower electrons given by Moliere,[4] and will

[1] N. Hilberry, Phys. Rev. **60**, 1 (1941).
[2] H. Kraybill and P. Ovrebo, Phys. Rev. **72**, 351 (1947).
[3] G. Cocconi, A. Loverdo, and V. Tongiorgi, Phys. Rev. **70**, 846 (1946).
[4] Edited by W. Heisenberg, *Vorträge über Kosmische Strahlung* (Berlin, Germany, 1943).

assume, with him, that the radial distribution does not vary appreciably in the vicinity of the shower maximum. Then the scattering in a shower is measured by a distance r_1 which is equal to R/h where R is the value of r_1 at sea level, about 60 meters, and h the fraction of the atmosphere above the point of observation (assuming an isothermal atmosphere). If the distance from the shower axis measured in units of this characteristic distance is called ρ, then the particle density d is given by $(n/2\pi r_1^2)(w(\rho)/\rho)$, with n the number of particles in the shower, and $w(\rho)$ a function which is plotted by Moliere, and which goes to 3.25 as ρ goes to zero. It is more useful for us to express ρ in terms of d and this has been done empirically, yielding, as an approximate expression good to within about 10 percent for $\rho\leqslant 1$, or $(\beta n/2\pi r_1^2 d)\leqslant 10$

$$\rho\approx(\beta n/2\pi r_1^2 d)\exp[-1.1(\beta n/2\pi r_1^2 d)^{0.35}], \quad (1)$$

where $\beta=w(0)=3.25$.

We will also need an expression for the number of particles in a shower as a function of the initiating energy and the distance in shower units from the point of origin of the shower. We take, as a basis for this, Heisenberg's[4] approximate expression and simplify it slightly, giving

$$n=(1/4l^{\frac{1}{2}})\exp[2(yl)^{\frac{1}{2}}-l], \quad (2)$$

where l is the distance in shower units from the point of origin of the shower, and $y=ln(E_0/\epsilon)$, where ϵ, the critical energy for air, is about 10^8 ev. In terms of y, the primary spectrum will be proportional to $e^{-y(\lambda-1)}dy$, and we will make no attempt to conserve constant factors in this work, since we are primarily interested in density spectra and altitude variations.

We will also assume everywhere that all the counters involved are at the same point, and will discuss the range of validity of this assumption later.

III. CALCULATIONS

We are now ready to calculate first the density spectra of the showers at a point of observation l_0 shower units below the top of the atmosphere. The number of primary electrons striking the atmosphere with an "energy" between y and $y+dy$, and an angle with the vertical between ϑ and $\vartheta+d\vartheta$ is proportional to $e^{-y(\lambda-1)}dy\sin\vartheta d\vartheta$, and each of these has to travel through a distance $l=l_0/\cos\vartheta$ to reach the point of observation. Thus the number with l between l and $l+dl$ is $e^{-y(\lambda-1)}dyl_0dl/l^2$. These showers have a number of particles given by Eq. (2) above, and distance from the shower axis within which the density is larger than Δ is given by

$$\rho\leqslant(\beta nh^2/2\pi R^2\Delta)\exp[-(\alpha\beta nh^2/2\pi R^2\Delta)^{1/\gamma}], \quad (3)$$

where $\alpha=(1.1)^{2.85}=1.32$, and $1/\gamma=0.35$. Thus the actual area involved is $\pi(\rho R/h)^2$ which is proportional to

$$f(n,l_0,\Delta)=(n^2h^2/\Delta^2)\times\exp[-2(\alpha\beta nh^2/2\pi R^2\Delta)^{1/\gamma}], \quad (4)$$

and our problem is simply to integrate this over all energies and zenith angles. This is most easily done if one first makes a Mellin transformation on f, with respect to n, obtaining

$$g(\sigma)=\int_0^\infty f(n)n^{\sigma-3}dn \propto (h^2/\Delta^2)(2\pi R^2\Delta/2^\gamma\alpha\beta h^2)^\sigma\Gamma(\gamma\sigma), \quad (5)$$

with Γ the usual Γ-function. As is well known, this can be inverted by means of the contour integration

$$f(n,l_0,\Delta)=(1/2\pi i)\int_C g(\sigma)n^{2-\sigma}d\sigma, \quad (6)$$

where the contour C is taken parallel to the imaginary axis to the right of the singularity of the Γ-function at $\sigma=0$. We will not invert this until later. We must now evaluate the following expression for the integral density spectrum:

$$N(l_0,\Delta)\propto(1/2\pi i)(h^3/\Delta^2)\int_C d\sigma(8\pi R^2\Delta/2^\gamma\alpha\beta h^2)^\sigma \times\Gamma(\gamma\sigma)\int_{l_0}^\infty(dl/l^{3-(\sigma/2)})\exp[(\sigma-2)l]\int_0^\infty dy \times\exp[-y(\lambda-1)+2(2-\sigma)(yl)^{\frac{1}{2}}], \quad (7)$$

where we have used Heisenberg's expression for $n(y,l)$ and the fact that $h\approx l_0/25$, since there are about 25 shower units in the atmosphere. The last integral can be performed either by setting $y=x^2$, and then setting the lower limit for x equal to $-\infty$, or by the saddle point method; it is assumed in either case that $\Re(2-\sigma)>0$, which restricts the contour to the strip $0<\Re(\sigma)<2$. The result is proportional to

$$(2-\sigma)l^{\frac{1}{2}}\exp[(2-\sigma)^2 l/(\lambda-1)],$$

where it is assumed that $l\gg 1$. We must now integrate

$$\int_{l_0}^{\infty}(dl/l^{(5-\sigma)/2})\exp\left[l\left[\frac{(2-\sigma)^2}{\lambda-1}-(2-\sigma)\right]\right]$$
$$=\int_{l_0}^{\infty}dl l^{(\sigma-5)/2}\exp\left[\frac{(2-\sigma)(3-\lambda-\sigma)l}{\lambda-1}\right],$$

which is convergent for $3-\lambda\leqslant\Re(\sigma)\leqslant 2$. In this region both factors in the integrand are decreasing, and one has contributions only from values of l near l_0. This is equivalent to the statement that most of the showers recorded will be vertical, and follows from the fact that we are dealing with counters that are close together deep in the atmosphere. In this region, again using the method of steepest descents, the integral is approximately given by

$$l_0^{(\sigma-3)/2}\exp\left[\frac{(2-\sigma)(3-\lambda-\sigma)}{(\lambda-1)}l_0\right]$$
$$\frac{1}{[(5-\sigma)/2]+[(2-\sigma)(\lambda+\sigma-3)/(\lambda-1)]l_0}.$$

We have, then, finally

$$N(l_0,\Delta)\propto(1/2\pi i)(l_0^{\frac{1}{2}}/\Delta^2)\int_C d\sigma(490R^2\Delta)^\sigma\Gamma(\gamma\sigma)$$
$$\cdot\left[l_0^{-(3\sigma/2)}\cdot\exp\left[\frac{(2-\sigma)(3-\lambda-\sigma)}{\lambda-1}l_0\right]\right]\Big/$$
$$\left[(\lambda+\sigma-3)l_0+\left(\frac{\lambda-1}{2}\right)\left(\frac{5-\sigma}{2-\sigma}\right)\right],\quad(8)$$

where we have used again $h\approx l_0/25$.

We may now consider an experiment in which an N-fold set of coincidence counters, each of area A, is used to record shower counts at a depth l_0. Since the probability of recording a count is $(1-e^{-A\Delta})^N$, we must integrate this over the differential density spectrum at the point of observation. In the integral spectrum (8) Δ enters only as $\Delta^{\sigma-2}$, so that the counting rate is given by

$$C(l_0)\propto\frac{1}{2\pi i}l_0^{\frac{1}{2}}\int_C\frac{(2-\sigma)(490R^2)^\sigma\Gamma(\gamma\sigma)l_0^{-(3\sigma/2)}\exp\left[\frac{(2-\sigma)(3-\lambda-\sigma)l_0}{\lambda-1}\right]}{(\lambda+\sigma-3)l_0+\left(\frac{\lambda-1}{2}\right)\left(\frac{5-\sigma}{2-\sigma}\right)}d\sigma\int_0^\infty d\Delta\cdot\Delta^{\sigma-3}(1-e^{-A\Delta})^N,\quad(9)$$

where the latter integral is proportional to

$$A^{2-\sigma}\int_0^\infty d\delta\cdot\delta^{\sigma-3}(1-e^{-\delta})^N,$$

which is convergent for $N\geqslant 2$. Now, since $\Re(\sigma)<2$ always, and usually <1, the contributions to this integral come from near the origin, and it is sufficient to replace $(1-e^{-\delta})$ by $\delta e^{-\delta/2}[1+(\delta^2/12)+(\delta^3/48)\cdots]$ giving us for the integral

$$(2/N)^N(NA/2)^{2-\sigma}\Gamma(N-2+\sigma)$$
$$\times[1+[(N-1+\sigma)(N-2+\sigma)/3N]$$
$$+[(N-1+\sigma)(N+\sigma)(N-2+\sigma)/6N^3]\cdots]$$
$$=(2/N)^N(NA/2)^{2-\sigma}\Gamma(N-2+\sigma)\cdot S(N,\sigma),$$

yielding, finally,

$$C(l_0)\propto\frac{1}{2\pi i}\left(\frac{2}{N}\right)^N l_0^{\frac{1}{2}}\int_C\frac{(2-\sigma)l_0^{-(3\sigma/2)}d\sigma}{(\lambda+\sigma-3)l_0+[(\lambda-1)/2][(5-\sigma)/(2-\sigma)]}\left(\frac{NA}{980R^2}\right)^{2-\sigma}\Gamma(\gamma\sigma)\Gamma(N-2+\sigma)$$
$$\times\exp[(2-\sigma)(3-\lambda-\sigma)l_0/\lambda-1]S(N,\sigma).\quad(10)$$

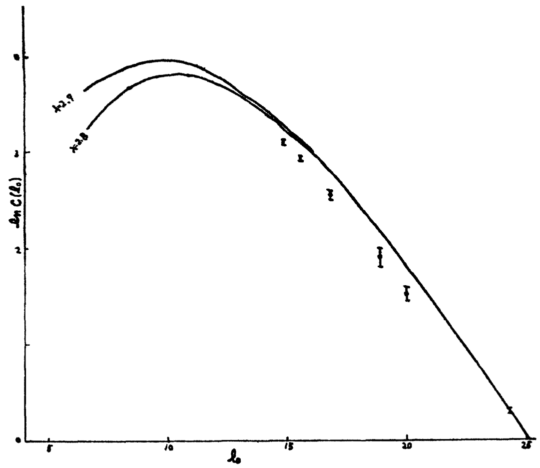

FIG. 1. Variation of counting rate with altitude for $A=200$ cm², and $\lambda=2.8$ and 2.9. Experimental points are those of Hilberry and are normalized so that the sea level point lies on the curve for $\lambda=2.8$.

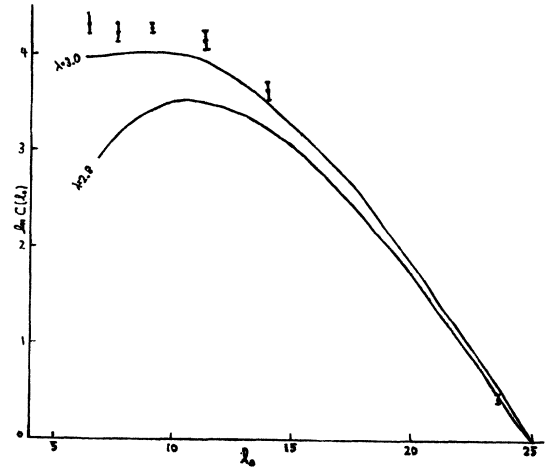

FIG. 2. Variation of counting rate with altitude for $A=100$ cm², and $\lambda=2.8$ and 3.0. Experimental points are those of Kraybill and Ovrebo and are normalized so that the sea level point lies on the curve for $\lambda=2.8$.

We will see that there will, in general, be a saddle on the real axis at $\sigma=\sigma_0(N, l_0, A, \lambda)$. The width of the saddle will be principally determined by the two Γ-functions, and by the term in $\exp[\sigma^2 l_0(\lambda-1)]$, hence proportional to

$$(l_0+[(\lambda-1/2][\gamma^2\psi'(\gamma\sigma_0)+\psi'(N-2+\sigma_0)])^{-\frac{1}{2}}$$

so that we have finally

$$C(l_0)\propto(2/N)^N\frac{(2-\sigma_0)l_0^{\frac{1}{2}(1-\sigma_0)}\Gamma(\gamma\sigma_0)\Gamma(N-2+\sigma_0)S(N,\ \sigma_0)\ \exp[-(2-\sigma_0)(\lambda+\sigma_0-3)l_0/(\lambda-1)]}{\left[(\lambda+\sigma_0-3)l_0+\frac{\lambda-1}{2}\left(\frac{5-\sigma_0}{2-\sigma_0}\right)\right]\left\{l_0+\frac{\lambda-1}{2}[\gamma^2\psi'(\gamma\sigma_0)+\psi'(N-2+\sigma_0)]\right\}^{\frac{1}{2}}}\times(NA/980R^2)^{2-\sigma_0},\quad(11)$$

where σ_0 is obtained from

$$\gamma\psi(\gamma\sigma_0)+\psi(N-2+\sigma_0)+[2\sigma_0 l_0/(\lambda-1)]-[(5-\lambda)/(\lambda-1)]l_0+\ln(980R^2/NAl_0^{\frac{1}{2}})=0,\quad(12)$$

with the aid of a table of ψ-functions, such as are given by Jahnke and Emde.[5] Here $\psi(z)$ is the logarithmic derivative of the Γ-function, and $\psi'(z)$ is its derivative, and the $\psi(N-2+\sigma_0)$ can usually be neglected in (12).

IV. RESULTS AND COMPARISON WITH EXPERIMENTS

There are now two ways in which the formulae above can be used, of which the more accurate is, of course, the more laborious. This is to solve Eq. (12) for σ_0 for every experimental situation, and then substitute into (11). One has to do this for each point on the curve that is required. This has been done, for example, for the altitude variation of a three counter coincidence arrangement, for counter areas of 100 cm² and 200 cm², respectively, and for two values of λ in each case. The results are shown in Figs. 1 and 2. In the former case the experimental points are those of Kraybill and Ovrebo and in the latter case, Hilberry. It is seen that the agreement leaves something to be desired, especially at high altitudes. The assumption of point geometry breaks down there, but in the wrong direction, as will be discussed later. At sea level the counting rate $\sim e^{-0.3l_0}$, so that one cannot draw any conclusions on the point mentioned in Section I.

If one does not vary the parameters over too large a range, one can obtain a good estimate without much trouble. We notice that the results

[5] E. Jahnke and F. Emde, *Funktionentafeln* (Leipzig, Germany, 1933). In this book $\psi(z)$ is what we call $\psi(1+z)$.

are, to a large extent, characterized by the value of σ_0. For example, if one differentiates Eq. (10) with respect to l_0, the result is multiplied by $(2-\sigma)(3-\lambda-\sigma)/(\lambda-1)$ plus some other terms that are not large if one isn't near the top of the atmosphere. This is a slowly varying factor and will not change the position of the saddle, so that one has, to good approximation

$$dC(l_0)/dl_0 = -[(2-\sigma_0)(\lambda+\sigma_0-3)/(\lambda-1)]C(l_0)$$

or

$$C(l_0) \propto \exp[-[(2-\sigma_0)(\lambda+\sigma_0-3)/(\lambda-1)]l_0] \quad (13)$$

over not too large a range. This fits the curves of Figs. 1 and 2 fairly well near sea level.

By a similar argument, one can find that the variation with A is given by $A^{2-\sigma_0}$. At $l_0=23.5$, around $A=100$ cm^2 and $\lambda=2.8$, one finds that $\sigma_0=0.54$, so that the variation with area should be $A^{1.46}$ in good agreement with the experimental results of Cocconi, Loverdo, and Tongiorgi. One finds similar agreement with their work at $l_0=18.5$, after their correction for effects of the roof. They have performed these analyses by graphical integration and find the same results, combining the effects of changes of A and N.

For the variation with N one finds, in the same way,

$$C(N_1)/C(N_2)=2^{N_1-N_2}(N_2^{N_2}/N_1^{N_1})(N_1/N_2)^{2-\sigma_0} \times[\Gamma(N_1-2+\sigma_0)/\Gamma(N_2-2+\sigma_0)] \times[S(N_1,\sigma_0)/S(N_2,\sigma_0)], \quad (14)$$

or, in the special case when $N_1=3$, $N_2=4$,

$$C(3)/C(4)=[8/3(1+\sigma_0)](4/3)^{\sigma_0} \times[S(3,\sigma_0)/S(4,\sigma_0)], \quad (14')$$

which again seems to be in fair but not fully satisfactory agreement with the experiments of Cocconi, Loverdo, and Tongiorgi as shown in Table I if one chooses $\lambda=2.8$. Small variations of λ do not essentially change this.

TABLE I.

A	$[C(3)/C(4)]_{\text{theor.}}$, $\lambda=2.8$, $l_0=23.5$	$[C(3)/C(4)]_{\text{exp}}$
27 cm^2	1.66	2.23 ± 0.18
54		2.45 ± 0.25
129		1.65 ± 0.05
258	1.63	1.48 ± 0.06
516		1.70 ± 0.04
774		1.63 ± 0.03
1032	1.61	1.66 ± 0.03

V. SOME LIMITS OF VALIDITY

We wish now to estimate the effective radii of the showers we are considering to find the limitations of the assumption that the counters are at one point. Since we will only be interested in distances of the order of a few meters, it will be sufficient to consider the radial distribution of shower electrons to be following a $1/r$ law. Thus $d\sim\beta nh^2/2\pi R^2\rho=\beta nh/2\pi Rr$. Since the densities we are counting are of order $1/A$, the important distances are $r\sim\beta nhA/2\pi R$, where $n\sim\frac{1}{10}e^{[(5-\lambda-2\sigma_0)/(\lambda-1)]l_0}$, so

$$r\sim(hA/20R)e^{[(5-\lambda-2\sigma_0)/(\lambda-1)]l_0}.$$

For areas of the order of 100 cm^2 and $\lambda\sim2.8$, the coefficient of l_0 in the exponent is close to 1 at high altitudes, so $r\sim(hA/20R)e^{l_0}$. Now for $A=200$ cm^2, $h\sim\frac{1}{2}$, $r\sim e^{l_0}/10^5$ meters, which becomes of order 1 meter at $l_0\sim11$. At sea level one finds $r\sim20$ meters. Thus, below $l_0\sim11$ one can expect our formulae to be valid, and at higher altitudes to be a sort of upper limit to the counting rate (limiting curve as counter separation approaches zero). It is therefore of particular interest that the experimental points in Fig. 2 lie above the theoretical curves.