times the experimental value of $\Gamma(t)$, which is certainly very plausible. At these short times, the calculated values depend very sensitively on the values of the disintegration energies and lifetimes of the primary fission products. The agreement with experiment at these times thus lends some support to the assumptions which governed the choice of the initial energies and lifetimes which were (1) that the parabolic mass formula holds for nuclei quite far removed from the region of stability and (2) that the chance of finding a given charge on the primary fission product is given by Eq. (5).

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Dirac's One-Electron Problem in Momentum Representation

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In the present note the eigenfunctions of the problem have been transformed into a form suitable for numerical calculations.

I. THE INTEGRAL EQUATION FOR THE PROBLEM

THE non-relativistic one-electron problem for a Coulomb field has been treated in the momentum representation very exhaustively and from several points of view.[1] As far as I know the corresponding relativistic problem for a Dirac electron has, however, not yet been discussed. In the present note we will, therefore, deal with this problem.

In this section we deduce the integral equation for our problem. For this purpose we use the operator[2]

$$T\cdots=\frac{1}{h^{\frac{3}{2}}}\iiint_{-\infty}^{+\infty} dxdydz e^{-(i/\hbar)(\xi x+\eta y+\zeta z)}\cdots \quad (1)$$

(ξ, η, ζ = momentum components)

which transforms the space coordinate representation

$$u(x, y, z)=(u_1, u_2, u_3, u_4)$$

of the Dirac wave function into the momentum representation

$$Tu(x, y, z)=v(\xi, \eta, \zeta)=(v_1, v_2, v_3, v_4).$$

Making use of the well-known formula[3]

$$\frac{1}{r}=\frac{1}{2\pi^2}\iiint_{-\infty}^{+\infty}\frac{e^{-i(\lambda x+\mu y+\nu z)}}{\lambda^2+\mu^2+\nu^2}d\lambda d\mu d\nu$$
$$=\frac{e^{(i/\hbar)(\xi x+\eta y+\zeta z)}}{\pi h}$$
$$\times\iiint_{-\infty}^{+\infty}\frac{e^{-(i/\hbar)(\xi' x+\eta' y+\zeta' z)}}{(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2}d\xi' d\eta' d\zeta',$$

we get

$$T\frac{u}{r}=\frac{1}{\pi h}\iiint_{-\infty}^{+\infty}\frac{v(\xi', \eta', \zeta')}{(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2}\times d\xi' d\eta' d\zeta'.$$

[1] H. Weyl, Zeits. f. Physik **48**, 1 (1928); E. Podolsky and L. Pauling, Phys. Rev. **34**, 109 (1929); E. A. Hylleraas, Zeits. f. Physik **74**, 216 (1932); W. Elsasser, Zeits. f. Physik **81**, 332 (1931); V. Fock, Zeits. f. Physik **98**, 145 (1936).

[2] Cf. W. Pauli, *Handbuch der Physik* **XXIV**, 227 (1933), second edition.

[3] We can get it using the Fourier integral theorem and the integral relation given by Weyl, see reference 1, p. 41.

Since we also have

$$T[-i\hbar(\partial/\partial x)u]=\xi v,$$

the application of (1) to the Dirac differential equation

$$Hu=\left[E+\frac{Ze^2}{r}+\beta E_0 -ic\hbar\left(\alpha_1\frac{\partial}{\partial x}+\alpha_2\frac{\partial}{\partial y}+\alpha_3\frac{\partial}{\partial z}\right)\right]u=0$$

yields the integral equation

$$[E+\beta E_0+c(\alpha_1\xi+\alpha_2\eta+\alpha_3\zeta)]v(\xi,\eta,\zeta)+\frac{Ze^2}{\pi h}\iiint_{-\infty}^{+\infty}\frac{v(\xi',\eta',\zeta')}{(\xi'-\xi)^2+(\eta'-\eta)^2+(\zeta'-\zeta)^2}\times d\xi'd\eta'd\zeta'=0. \quad (2)$$

This is a system of four linear integral equations for the four functions v_1, v_2, v_3, v_4.

II. THE EIGENFUNCTIONS IN MOMENTUM REPRESENTATION

We obtain a solution of (2) applying the operator T, Eq. (1), to the corresponding eigenfunctions in space coordinate representation. For $j=l+\frac{1}{2}$ they are given by

$$\begin{aligned} u_1&=\left(\frac{l-m+1}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,m}(\Theta,\Phi)iR(r),\\ u_2&=\left(\frac{l+m+2}{2l+3}\right)^{\frac{1}{2}}Y_{l+1,m+1}(\Theta,\Phi)iR(r),\\ u_3&=\left(\frac{l+m+1}{2l+1}\right)^{\frac{1}{2}}Y_{l,m}(\Theta,\Phi)S(r),\\ u_4&=-\left(\frac{l-m}{2l+1}\right)^{\frac{1}{2}}Y_{l,m+1}(\Theta,\Phi)S(r).\end{aligned} \quad (3)$$

Putting

$$\rho=\frac{2p_0}{h}r,\quad \gamma=((l+1)^2-\alpha^2Z^2)^{\frac{1}{2}},$$
$$N=((n'+\gamma)^2+\alpha^2Z^2)^{\frac{1}{2}}$$

(n' = radial quantum number) the functions R and S are expressible in the form

$$R=A(1-\epsilon)^{\frac{1}{2}}e^{-\rho/2}\rho^{\gamma-1}\{n'F(-(n'-1),2\gamma+1,\rho)+(N+l+1)F(-n',2\gamma+1,\rho)\},$$
$$S=A(1+\epsilon)^{\frac{1}{2}}e^{-\rho/2}\rho^{\gamma-1}\{-n'F(-(n'-1),2\gamma+1,\rho)+(N+l+1)F(-n',2\gamma+1,\rho)\},$$

where

$$A=-\frac{(\Gamma(2\gamma+n'+1))^{\frac{1}{2}}}{\Gamma(2\gamma+1)(n'!)^{\frac{1}{2}}}\frac{1}{(4N(N+l+1))^{\frac{1}{2}}}\times\left(\frac{2p_0}{h}\right)^{\frac{3}{2}},\quad \epsilon=\frac{E}{E_0}=\frac{n'+\gamma}{N},$$

and $F(\alpha,\beta,\rho)$ means the confluent hypergeometric function.[4]

To calculate $v(\xi,\eta,\zeta)$ we introduce into T, instead of the Cartesian space and momentum coordinates x, y, z and ξ, η, ζ, the polar coordinates r, Θ, Φ and p, ϑ, φ, respectively, and replace r by $r=(\hbar/2p_0)\rho$ and p by $p=p_0\sigma$, where $p_0=(E_0/c)(1-\epsilon^2)^{\frac{1}{2}}$. Applying the procedure used by Podolsky and Pauling[1] for a non-relativistic hydrogen atom we obtain

$$Th(\rho)Y_{l,m}(\Theta,\Phi)=Y_{l,m}(\vartheta,\varphi)\frac{i^l h^{\frac{3}{2}}}{4\sqrt{2}p_0^3}\frac{1}{(\sigma)^{\frac{1}{2}}}\int_0^\infty h(\rho)y_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right)\rho^{\frac{3}{2}}d\rho.$$

The application of T to a function of the form

$$h(\rho)Y_{l,m}(\Theta,\Phi)$$

replaces only the variables Θ, Φ of the spherical harmonic by ϑ, φ, while the radial function $h(\rho)$ goes over into a function $H(\sigma)$ of σ

$$H(\sigma)=i^l\frac{h^{\frac{3}{2}}}{4\sqrt{2}p_0^3}\frac{1}{(\sigma)^{\frac{1}{2}}}\int_0^\infty h(\rho)J_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right)\rho^{\frac{3}{2}}d\rho.$$

The factor $i^l(h^{\frac{3}{2}}/4\sqrt{2}p_0^3)$ causes that eigenfunctions normalized in coordinate representation go over in eigenfunctions normalized in momentum representation.

The application of the operator T to (3) gives us, therefore, the eigenfunctions in momentum representation in the form[5]

[4] Cp. H. Bethe, *Handbuch der Physik* **XXIV**, 316, second edition.

[5] The fact that $v(\xi,\eta,\zeta)$ has the same form as $u(x,y,z)$ Eq. (3), can be proved also directly. The form (3) of $u(x,y,z)$ follows from the supposition that u is an eigenfunction of the operator corresponding to the z component of the total angular momentum and the operator $\beta((\mathbf{m}\boldsymbol{\sigma})+h)$, where $\mathbf{m}$ and $\boldsymbol{\sigma}$ are the orbital and the spin angular momenta. But both these operators are of the same form in both our representations, because this statement is true for the operator corresponding to the orbital angular momentum $\mathbf{m}$. Therefore, $v(\xi,\eta,\zeta)$, being an eigenfunction of these operators, has the form of $u(x,y,z)$.

$$v_1=\left(\frac{l-m+1}{2l+3}\right)^{\frac{1}{2}} Y_{l+1,m}(\vartheta,\varphi)M(\sigma),$$

$$v_2=\left(\frac{l+m+2}{2l+3}\right)^{\frac{1}{2}} Y_{l+1,m+1}(\vartheta,\varphi)M(\sigma),$$

$$v_3=\left(\frac{l+m+1}{2l+1}\right)^{\frac{1}{2}} Y_{l,m}(\vartheta,\varphi)N(\sigma),$$

$$v_4=-\left(\frac{l-m}{2l+1}\right)^{\frac{1}{2}} Y_{l,m+1}(\vartheta,\varphi)N(\sigma).$$

Using

$$G(\nu,\beta,\delta,l,\sigma)=\frac{1}{(\sigma)^{\frac{1}{2}}}\int_0^\infty e^{-\rho/2}\rho^{\delta+\frac{1}{2}} \times F(-\nu,\beta,\rho)J_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right)d\rho, \quad (4)$$

the functions $M(\sigma)$ and $N(\sigma)$ are given by

$$M(\sigma)=+B(1-\epsilon)^{\frac{1}{2}} \times\{n'G(n'-1,2\gamma+1,\gamma-1,l+1,\sigma) +(N+l+1)G(n',2\gamma+1,\gamma-1,l+1,\sigma)\},$$

$$N(\sigma)=-B(1+\epsilon)^{\frac{1}{2}} \times\{-n'G(n'-1,2\gamma+1,\gamma-1,l,\sigma) +(N+l+1)G(n',2\gamma+1,\gamma-1,l,\sigma)\},$$

where

$$B=i^l\frac{(\Gamma(2\gamma+n'+1))^{\frac{1}{2}}}{\Gamma(2\gamma+1)(n'!)^{\frac{1}{2}}}\frac{1}{(N(N+l+1))^{\frac{1}{2}}}\frac{1}{4p_0^{\frac{1}{2}}}.$$

Our last task is to give the function G, (4), a form suitable for numerical calculations. To get an expression for $J_{l+\frac{1}{2}}$ applicable for all σ we use the Hankel functions $H^{(1)}_{l+\frac{1}{2}}$ and $H^{(2)}_{l+\frac{1}{2}}$

$$J_{l+\frac{1}{2}}(z)=\tfrac{1}{2}(H^{(1)}_{l+\frac{1}{2}}(z)+H^{(2)}_{l+\frac{1}{2}}(z)),$$

where

$$H^{(1)}_{l+\frac{1}{2}}(z)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} i^{-(l+1)}e^{iz}\sum_{m=0}^{l}\frac{1}{m!}\frac{(l+m)!}{(l-m)!}\left(\frac{i}{2}\right)^m\frac{1}{z^{m+\frac{1}{2}}}$$

and where for real z the function $H^{(2)}_{l+\frac{1}{2}}(z)$ is equal to the conjugate complex value of $H^{(1)}_{l+\frac{1}{2}}(z)$.

Denoting by G_k the function G in which $J_{l+\frac{1}{2}}$ has been replaced by $H^{(k)}_{l+\frac{1}{2}}$, we get

$$G=\tfrac{1}{2}(G_1+G_2),$$

where, for instance,

$$G_1(\nu,\beta,\delta,l,\sigma)=\frac{1}{(\sigma)^{\frac{1}{2}}}\int_0^\infty e^{-\rho/2}\rho^{\delta+\frac{1}{2}} \times F(-\nu,\beta,\rho)H^{(1)}_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right)d\rho.$$

Since

$$F(-\nu,\beta,\rho) =\nu!\Gamma(\beta)\sum_{n=0}^{\nu}\frac{(-\rho)^n}{n!\Gamma(\nu-n+1)\Gamma(\beta+n)} \quad (5)$$

and

$$\int_0^\infty e^{-\alpha\rho}\rho^{s-1}d\rho=\frac{\Gamma(s)}{\alpha^s} \text{ for } R(\alpha)>0 \text{ and } R(s)>0,$$

we get

$$G_1(\nu,\beta,\delta,l,\sigma)=i^{-(l+1)}\frac{2}{(\pi)^{\frac{1}{2}}}\nu!\Gamma(\beta)\sum_{n=0}^{\nu} \sum_{m=0}^{l}\frac{\Gamma(n-m+\delta+2)(l+m)!}{n!\Gamma(\nu-n+1)\Gamma(\beta+n)m!(m-l)!}(-1)^n i^m\frac{1}{\sigma^{n+1}}\left(\frac{2}{1-i\sigma}\right)^{n-m+\delta+2} \quad (6)$$

But using

$$F\left(-l,l+1,-(n+\delta+1),\frac{i+\sigma}{2\sigma}\right) =\frac{1}{\Gamma(n+\delta+2)} \times\sum_{m=0}^{l}\frac{(l+m)!\Gamma(n-m+\delta+2)}{m!(l-m)!}\left(\frac{i+\sigma}{2\sigma}\right)^m$$

or

$$F\left(-\nu,\delta+2-m,\beta,\frac{2}{1-i\sigma}\right)=\frac{\Gamma(\beta)\Gamma(\nu+1)}{\Gamma(\delta+2-m)} \sum_{n=0}^{\nu}\frac{\Gamma(n-m+\delta+2)}{n!\Gamma(\nu-n+1)\Gamma(\beta+n)}\left(-\frac{2}{1-i\sigma}\right)^n,$$

we can write G_1 in the form

$$G_1(\nu,\beta,\delta,l,\sigma)=i^{-(l+1)}\frac{2}{(\pi)^{\frac{1}{2}}}\nu!\Gamma(\beta)\frac{1}{\sigma}$$
$$\times\sum_{n=0}^{\nu}\frac{\Gamma(n+\delta+2)}{n!\Gamma(\nu-n+1)\Gamma(\beta+n)}(-1)^n\left(\frac{2}{1-i\sigma}\right)^{n+\delta+2}$$
$$\times F\left(-l,l+1,-(n+\delta+1),\frac{i+\sigma}{2\sigma}\right)$$
$$\equiv i^{-(l+1)}\frac{2}{(\pi)^{\frac{1}{2}}}\sum_{m=0}^{l}\frac{(l+m)!\Gamma(\delta-m+2)}{m!(l-m)!}\frac{i^m}{\sigma^{m+1}}$$
$$\times\left(\frac{2}{1-i\sigma}\right)^{\delta+2-m}F\left(-\nu,\delta+2-m,\beta,\frac{2}{1-i\sigma}\right).\quad(7)$$

G_2 is given by the conjugate complex value of G_1.

For the calculation of G_1 we can use (6) or (7) for all values of σ. But the practical use of this formula is very tedious, especially for small values of σ, because we must use complex numbers. For $|\sigma|<1$, however, we can obtain for G a real expression. Using

$$\int_0^\infty e^{-\rho/2}\rho^{n+\delta+\frac{1}{2}}J_{l+\frac{1}{2}}\left(\frac{\rho\sigma}{2}\right)d\rho$$
$$=\frac{\Gamma(n+l+\delta+3)}{\Gamma(l+\frac{3}{2})}2^{n+\delta-l+2}\sigma^{l+\frac{1}{2}}$$
$$\times F\left(\frac{n+\delta+l+3}{2},\frac{n+\delta+l+4}{2},l+\tfrac{3}{2},-\sigma^2\right),$$

we get with regard to (5) the function G in the form

$$G(\nu,\beta,\delta,l,\sigma)=\frac{\nu!\Gamma(\beta)}{\Gamma(l+\frac{3}{2})}2^{\delta-l+2}\sigma$$
$$\times\sum_{n=0}^{\nu}\frac{\Gamma(n+l+\delta+3)}{n!\Gamma(\nu-n+1)\Gamma(\beta+n)}(-2)^n$$
$$\times F\left(\frac{n+l+\delta+3}{2},\frac{n+l+\delta+4}{2},l+\tfrac{3}{2},-\sigma^2\right)$$

we wished to obtain.

But this formula can also be further transformed into a form which contains formally only one summation. With the help of well-known relations we get

$$F\left(\frac{n+l+\delta+3}{2},\frac{n+l+\delta+4}{2},l+\tfrac{3}{2},-\sigma^2\right)$$
$$=F\left(\frac{n+\delta-l+1}{2}+l+1,\frac{n+\delta-l+2}{2}+l+1,\tfrac{1}{2}+l+1,-\sigma^2\right)$$
$$=\frac{\Gamma\left(\frac{n+\delta-l+1}{2}\right)\Gamma\left(\frac{n+\delta-l+2}{2}\right)}{\Gamma(1/2)}$$
$$\times\frac{\Gamma(l+\frac{3}{2})}{\Gamma\left(\frac{n+\delta+l+3}{2}\right)\Gamma\left(\frac{n+\delta+l+4}{2}\right)}\frac{d^{l+1}}{d(-\sigma^2)^{l+1}}$$
$$\times F\left(\frac{n+\delta-l+1}{2},\frac{n+\delta-l+2}{2},\tfrac{1}{2},-\sigma^2\right)$$
$$=(-1)^{l+1}2^{2l+1}\frac{\Gamma(n+\delta-l+1)\Gamma(l+\frac{3}{2})}{(\pi)^{\frac{1}{2}}\Gamma(n+\delta+l+3)}\frac{d^{l+1}}{d(\sigma^2)^{l+1}}$$
$$\times\{(1-i\sigma)^{-(n+\delta-l+1)}+(1+i\sigma)^{-(n+\delta-l+1)}\}.$$

Therefore, we have

$$G(\nu,\beta,\delta,l,\sigma)$$
$$=(-1)^{l+1}\frac{2^{2l+2}}{(\pi)^{\frac{1}{2}}}\Gamma(\delta-l+1)\sigma^l\frac{d^{l+1}}{d(\sigma^2)^{l+1}}$$
$$\times\left\{\left(\frac{2}{1+i\sigma}\right)^{\delta-l+1}F\left(-\nu,\delta-l+1,\beta,\frac{2}{1+i\sigma}\right)\right.$$
$$\left.+\left(\frac{2}{1-i\sigma}\right)^{\delta-l+1}F\left(-\nu,\delta-l+1,\beta,\frac{2}{1-i\sigma}\right)\right\}.$$

A similar expression for G_1 can be obtained from (7), using

$$F(-l,l+1,\gamma,x)=\frac{\Gamma(\gamma)}{\Gamma(\gamma+l)}x^{1-\gamma}(1-x)^{\gamma-1}\frac{d^l}{dx^l}$$
$$\times(x^{l+\gamma-1}(1-x)^{l-\gamma+1}).$$