PHYSICAL REVIEW VOLUME 73, NUMBER 9 MAY 1, 1948

Quantum Theory of Localizable Dynamical Systems

P. A. M. DIRAC
Institute for Advanced Study, Princeton, New Jersey
(Received January 12, 1948)

A dynamical system is called localizable if its wave functions can be expressed in terms of variables, each referring to physical conditions at only one point in space-time. These variables may be at points on any three-dimensional space-like surface in space-time.

A general investigation is made of how the wave function varies when the surface is varied in any way. The variation of the wave function is given by equations of the Schrödinger type involving certain operators $H^n(u)$ which play the role of Hamiltonians. The commutation relations for these operators are obtained (Eqs. (50), (51)). The theory works entirely with relativistic concepts and it provides the general pattern which any relativistic quantum theory must conform to, provided the dynamical system is localizable.

I. INTRODUCTION

A *LOCALIZED* dynamical variable is a quantity which describes physical conditions at one point in space-time. Examples are field quantities and derivatives of field quantities to any order, also the position of a point particle at a particular time and the spin variable of a point particle at a particular time. Examples of dynamical variables that are not localized are the momentum of a point particle at a particular time in quantum theory and the spin variables of a body of finite size at a particular time, in classical or quantum theory.

A dynamical system in quantum theory will be defined as *localizable* if a representation for the wave functions can be set up in which all the dynamical variables are localized. Thus the system composed of a number of point particles interacting with fields is localizable. The representation will be called a *localized representation*.

To fix a state of motion of a dynamical system one usually specifies the wave function at a particular time. If the dynamical system is localizable and the representation is a localized one, each point in the domain of the wave function will refer to a set of points in space-time at this time. The set may be an infinite one and may be all points at this time. The wave function thus refers to points on a flat three-dimensional surface in space-time. The state of motion is then determined by physical information which refers to points on this flat three-dimensional surface.

From the general requirement of relativity that physical conditions at one point cannot influence the conditions at another point outside the light cone of the original point, one would expect the state of motion to be determinable also by physical information referring to points on any curved three-dimensional surface extending to infinity, provided this surface is everywhere space-like, i.e., the direction of the line joining any two points of it lies outside the light cone. Thus we should expect to be able to set up a wave function involving localized dynamical variables referring to points on any three-dimensional space-like surface S. This we should expect as a general characteristic of a localizable dynamical system.

The important question now presents itself. Are atomic systems in fact localizable dynamical systems in the above sense? This cannot be answered at the present time. It might very well be that states of atomic systems can be fixed only by wave functions on flat surfaces in space-time, owing to the atomic world containing some features which are not describable in terms of localized dynamical variables. It is true that all atomic models in present use that are relativistic are also localizable. But these models all lead to serious difficulties in their development. Many people think that the difficulties are directly caused by the models' being localizable and are trying to construct new theories in terms of quantities that are not accurately localizable. Heisenberg's[1] theory of the S-matrix is a general investigation on these lines.

It seems to be a very difficult and awkward

[1] W. Heisenberg, Zeits. f. Physik **120**, 513, 673 (1943).

problem to set up a non-localizable dynamical system which shall be relativistic, and no work on this subject has yet progressed very far. In view of these difficulties I do not think the localizable models should be abandoned at the present time.

In most of the work that has been done on localizable dynamical systems people have confined their attention to wave functions and dynamical variables on flat surfaces in space-time, which form instants of time in some Lorentz frame. The work is thus not in relativistic form. In order to show that the theory is relativistic it would be necessary to study how the wave functions and dynamical variables transform when one changes the direction of the flat surfaces in space-time, and to show that all physical results are independent of this direction. This was done by Heisenberg and Pauli[2] for the case of electrons interacting with the electromagnetic field. But for most theories of particles in interaction (e.g., the various meson theories) it has not been done, and one does not know if these theories are relativistic.

In the following work I have investigated localizable dynamical systems on the most general lines, working only with relativistic ideas. This work provides a test for whether any theory of particles in interaction is relativistic and gives the conditions which must be satisfied for new theories.

II. THE GENERALIZED SCHRÖDINGER AND HEISENBERG EQUATIONS

Let S denote any three-dimensional space-like surface extending to infinity. We suppose a state of our dynamical system can be fixed by a wave function $\psi(q)$ involving variables q which are all localized on S. The variables q may consist of certain discrete variables, denoting positions of particles on S and their spins, if they have any, and may also consist of three-fold infinities of variables, denoting field quantities on S. If any three-fold infinities occur, then ψ is to be understood as a functional. The wave function must be capable of being normalized, according to an equation which we may write schematically

$$\int |\psi(q)|^2 dq = 1. \tag{1}$$

If the variables q contain a three-fold infinity, we must make a transformation which reduces them to an enumerable set (e.g., a transformation to particle variables) in order to be able to give a meaning to the integration in (1).

The surface S plays the role of the time in the non-relativistic theory. We have, so to speak, very many time variables in the present theory, namely all those variables needed to fix S. We must study how the wave function changes when S is changed in any way, subject to the condition that it remains space-like.

Take two surfaces S, say S_1 and S_2, and let the wave functions on these two surfaces be $\psi_1(q_1)$ and $\psi_2(q_2)$, respectively. We have to use different variables q_1 and q_2 in the two wave functions because they denote different sets of dynamical variables, localized in S_1 and S_2, respectively. From the principle of superposition of quantum mechanics, each of the functions ψ_1 and ψ_2 determins the other according to a linear law, so the two functions are connected by an equation of the form

$$\psi_2(q_2) = R\psi_1(q_1), \tag{2}$$

where R is a linear operator that can operate on a function of the q_1 variables and turns it into a function of the q_2 variables. R depends only on the dynamical system concerned and the two surfaces S_1 and S_2, and must be unitary in order to preserve the normalization of the wave function.

If the operators R connecting any two surfaces are known, then a wave function $\psi_1(q_1)$ on a particular surface S_1 determines a wave function $\psi(q)$ on an arbitrary space-like surface S. This wave function may be looked upon as a function $\psi(qS)$ of two sets of variables, the variables needed to fix S and the variables q, the latter variables being themselves dependent on S. This function $\psi(qS)$ which depends on S forms the analogue of the time-dependent wave function of non-relativistic quantum theory.

Let us consider a linear operator X_1 that can operate on a wave function $\psi_1(q_1)$ on the surface S_1 and turns it into another function of the

[2] W. Heisenberg and W. Pauli, Zeits. f. Physik 56, 1 (1929).

variables q_1, which can be looked upon as another wave function on the surface S_1. For example, X_1 may be an algebraic function of the variables q_1 and of the operators of differentiation with respect to the q_1's. Such a linear operator corresponds to a dynamical variable on the surface S_1. Of course, it need not be a localized dynamical variable, as it may involve q_1 variables localized at different points.

Suppose the linear operator X_1 changes the wave function $\psi_1(q_1)$ on S_1 into the wave function $\chi_1(q_1)$ on S_1. The wave function $\psi_1(q_1)$ on S_1 will determine an S-dependent wave function $\psi(qS)$ and, similarly, $\chi_1(q_1)$ will determine an S-dependent wave function $\chi(qS)$. We can now look upon X_1 as a linear operator which changes the S-dependent wave function $\psi(qS)$ into the S-dependent wave function $\chi(qS)$. Although X_1 refers to a particular S, namely S_1, it can in this way operate on wave functions which do not refer to any particular S, but are functions of a general S.

This way of looking upon dynamical variables on particular surfaces enables us to add and multiply dynamical variables on different surfaces. For example, the dynamical variable X_1 on the surface S_1 and the dynamical variable Y_2 on the surface S_2 can both be looked upon as linear operators operating on S-dependent wave functions $\psi(qS)$ to give other similar wave functions, and so their sum X_1+Y_2 and their products X_1Y_2, Y_2X_1 have an immediate interpretation as similar linear operators. The dynamical variables on different surfaces all form an algebra.

In order to study the consequences of the linear connection (2) between wave functions on different surfaces, let us take S_1 and S_2 to differ only by an infinitesimal of order ϵ. The surface S_2 is thus the result of applying any deformation of order ϵ to S_1. Equation (2) now gives

$$\psi_2(q_2)-\psi_1(q_1)=-i\epsilon A\psi_1(q_1), \tag{3}$$

where A is a linear operator connected with the previous R by

$$R=1-i\epsilon A. \tag{4}$$

The left-hand side of Eq. (3) is the sum of a function of the q_1's and a function of the q_2's and does not have any meaning as a single quantity. (E.g., it would have no meaning to say that the left-hand side of (3) vanishes.) For the purpose of developing the theory it is necessary to set up a connection between the domain of the function $\psi_1(q_1)$ and that of the function $\psi_2(q_2)$ so that two such functions can be added to produce another similar function. Given a function of the q_1's, we must be able to give a meaning to the *same* function of the q_2's.

The most obvious way of doing this is by introducing a general set of curvilinear coordinates u_1, u_2, u_3 on S_1, and a neighboring set of curvilinear coordinates, with the same names u_1, u_2, u_3, on S_2. This sets up a one-one correspondence between points on S_1 and points on S_2, and also between directions in S_1 at points on S_1 and directions in S_2 at points on S_2. Given a wave function on S_1 for which certain particles are at certain points with certain u-values and have certain spins, we now define the same function on S_2 as having the same particles at the points on S_2 with the same u-values, and having spins in the corresponding directions referred to the new system of coordinates. If the wave function on S_1 involves field variables, the same function on S_2 must involve in the same way the corresponding field variables with the same u-values.

In this way functions of the q_1's can be identified with functions of the q_2's, so that we can omit the suffixes and write them both as functions of the q's. The operator A becomes an operator that can be represented by a square matrix, which changes a function into another function of the same kind. The condition that R is unitary requires A to be Hermitian.

The method of identifying functions that we have introduced requires all the variables q occurring in the wave function $\psi_1(q)$ on S_1 to be defined with respect to the coordinate system u introduced on S_1. If the coordinate system is altered, the q's get altered in their meaning and so the wave function for a particular state gets altered. Equation (3) gives the law of change of the wave function $\psi_1(q)$ for any small change in the surface S_1 or in the coordinate system on it.

It becomes convenient at this stage to use the symbol S to denote a parametrized surface, i.e., a surface with a system of coordinates u_1, u_2, u_3 on it. A state of motion of a dynamical system is now fixed by a wave function $\psi_1(q)$ on any

parametrized surface S_1. It is the parametrized surface that really plays the role of the time in the non-relativistic theory—not the surface without the coordinate system.

We may take a small deformation of S_1 which does not change the surface but only changes the system of coordinates u on it. Such a deformation will cause the wave function $\psi_1(q)$ to change in a rather trivial way. Thus the operator A for such a deformation will be a trivial one and will not contain any terms corresponding to interaction between the various particles of the dynamical system. The operators A for deformations which change the surface will be the important ones and will include interaction terms.

A parametrized surface S_1 is fixed by specifying the four coordinates x_μ in space-time ($\mu=0, 1, 2, 3$) of any point on the surface as functions of the parameters $u_r (r=1, 2, 3)$, i.e., it is fixed by the four functions

$$x_\mu = x_\mu(u).$$

A small deformation of S_1 is fixed by specifying the variation δx_μ in the coordinates of a point with any given parameters u,

$$\delta x_\mu = \delta x_\mu(u). \tag{5}$$

We may resolve the displacement δx_μ into a part lying in the surface S_1 and a part normal to the surface S_1. The first part corresponds to a variation in the parameters u on the surface, say the variation, $\delta u_r = \epsilon a_r$. The second part is a displacement, of magnitude ϵa_n say, in the direction of the unit normal n_μ at each point u of the surface S_1, n_μ being defined by

$$n_\mu(\partial x^\mu/\partial u_r)=0, \quad n_\mu n^\mu = 1, \quad n_0 > 0. \tag{6}$$

The total displacement δx_μ is thus

$$\delta x_\mu = (\partial x_\mu/\partial u_r)\epsilon a_r + n_\mu \epsilon a_n. \tag{7}$$

The deformation of S_1 may be fixed by specifying ϵa_r and ϵa_n as functions of the u's. There are two advantages in using this specification instead of that provided by (5). Firstly, one may continually apply small deformations with a_r, a_n any differentiable functions of the u's to a space-like parametrized surface S_1 and it always remains space-like. Secondly, the ϵa_r, ϵa_n specification allows one immediately to separate out those deformations for which only the coordinate system u is changed, corresponding to a trivial change in the wave function.

With the method of specifying deformations by the displacements ϵa_r, ϵa_n we are able, when we are given any deformation applied to a particular S, to give a meaning to the *same* deformation applied to any other S. We can then introduce a deformation operator $\epsilon \mathfrak{A}$ say, connected with this deformation. This deformation operator has the property that, if $F(S_1)$ is any function of S_1, i.e., any functional of the functions $x_\mu(u)$ that specify S_1, the same function $F(S_2)$ of the deformed S_2 will be given by

$$F(S_2) - F(S_1) = -i\epsilon \mathfrak{A} F(S_1) \tag{8}$$

to the first order in ϵ. If $F(S)$ and $\mathfrak{A}$ are treated as non-commutative quantities in an algebra, this equation must be replaced by

$$F(S_2) - F(S_1) = -i\epsilon\{\mathfrak{A}F(S_1) - F(S_1)\mathfrak{A}\}. \tag{9}$$

With the help of this notation, Eq. (3) becomes

$$\mathfrak{A}\psi_1(q) = A\psi_1(q).$$

If we let $\psi(qS)$ be the wave function on an arbitrary S, we get

$$\mathfrak{A}\psi(qS) = A\psi(qS), \tag{10}$$

in which $\mathfrak{A}$ is an operator which operates on the S variables and A is an operator which operates on the q variables. This equation is like the ordinary Schrödinger equation of non-relativistic quantum theory,

$$i\frac{\partial}{\partial t}\psi(qt) = \hbar^{-1}H\psi(qt), \tag{11}$$

with $\mathfrak{A}$ instead of $i\partial/\partial t$ and A instead of $\hbar^{-1}H$. There is, however, the important difference that non-relativistic quantum theory has just one Schrödinger equation (11), while the present theory has a very large number, one for each kind of deformation that can be applied to a parametrized surface S. The present theory may be compared more closely with the many-time theory of electrons in interaction with the electromagnetic field, which theory has many wave equations of the form (11), one for each electron. There is still the difference, though, that in this many-time theory all the operators $i\partial/\partial t$ commute with each other, while in the present theory

the various operators $\mathfrak{A}$ corresponding to different kinds of deformation do not in general commute.

A dynamical variable in the present theory corresponds to a linear operator which can operate on the variables q in the wave function and is independent of the variables that fix S. A linear operator which does depend on the variables that fix S is a more general thing, which is analogous to a function of dynamical variables involving the time explicitly in non-relativistic quantum theory.

The identification of functions of the variables q_1 on S_1 with functions of the variables q_2 on S_2 enables us to identify linear operators which operate on wave functions on S_1 with linear operators which operate on wave functions on S_2. This means that a dynamical variable on S_1 can be identified with a dynamical variable on S_2. Using language borrowed from non-relativistic quantum theory we may say that, if we have any dynamical variable "at the time S_1" we can give a meaning to the same dynamical variable at the time S_2.

We had previously a way of adding, and thus subtracting, dynamical variables at different times S_1 and S_2 by considering them as operators operating on the S-dependent wave function, (this process being independent of the parametrization). Let us use this to evaluate the difference between a dynamical variable at the time S_1 and the same dynamical variable at the time S_2.

Take a dynamical variable X and let X_1 be its value at the time S_1 and X_2 its value at the time S_2. X_1 changes the wave function ψ on S_1 to another wave function, which we may denote by $X\psi$, on S_1. From (2), X_1 must have the same effect on the S-dependent wave function as that operator which changes $R\psi$ on S_2 to $RX\psi$ on S_2, which is the same as changing $R\psi$ on S_2 to $RX_2\psi$ on S_2. Thus

$$X_1R\psi = RX_2\psi,$$

so that

$$X_1R = RX_2. \tag{12}$$

If S_2 differs from S_1 by an infinitesimal, we get, using (4),

$$X_2 - X_1 = -i\epsilon(X_1A - AX_1)$$

to the first order in ϵ. Introducing the commutator notation

$$XY - YX = i[X, Y],$$

we may write this result

$$X_2 - X_1 = \epsilon[X_1, A]. \tag{13}$$

It is the analogue of Heisenberg's equation of motion of non-relativistic quantum theory. It is of similar form to (9), except for a change of sign.

If we take a general quantity ξ_1 on S_1, depending on dynamical variables on S_1 and also on the variables that fix S_1, then on passing from S_1 to S_2, ξ_1 will undergo changes due to two causes, a change given by (13) arising from its involving dynamical variables and a change given by (9) arising from its explicit dependence on S. Thus the total change in ξ_1 will be

$$\xi_2 - \xi_1 = \epsilon[\xi_1, A - \mathfrak{A}]. \tag{14}$$

This corresponds to the equation

$$d\xi/dt = [\xi, H] + (\partial\xi/\partial t)$$

in non-relativistic quantum mechanics for a quantity ξ which is a function of dynamical variables and is also an explicit function of the time.

III. THE COMMUTATION RELATIONS

We shall suppose space-time is flat, as is usual in quantum theory. Embedded in this flat space we have a curved three-dimensional space-like surface S, parametrized by u_1, u_2, u_3. The square of an element of length in this surface is given by

$$-dx_\mu dx^\mu = -(\partial x_\mu/\partial u_r)(\partial x^\mu/\partial u_s)du_r du_s = \gamma^{rs}du_r du_s$$

where

$$\gamma^{rs} = -(\partial x_\mu/\partial u_r)(\partial x^\mu/\partial u_s). \tag{15}$$

Thus γ^{rs} determines the metric on the surface.

The geometrical properties of the surface (e.g., its curvature) are not completely determined by the γ^{rs}. Certain other quantities

$$\Omega^{rs} = (\partial n^\mu/\partial u_r)(\partial x_\mu/\partial u_s)$$
$$= -n^\mu(\partial^2 x_\mu/\partial u_r \partial u_s) = \Omega^{sr} \tag{16}$$

are needed as well. The Ω^{rs} are not independent of the γ^{rs}, but are connected with them by cer-

tain relations due to Gauss, Mainardi, and Codazzi.[3]

When S is deformed, the γ's and Ω's get changed. This means that the γ's and Ω's do not commute with the deformation operators like $\mathfrak{A}$. We shall proceed to work out the more important commutation relations.

Take certain definite values for the parameters u, fixing a point $x_\mu(u)$ on the surface S. The deformation operator $\mathfrak{A}$ changes $x_\mu(u)$ by the amount

$$\delta x_\mu = \epsilon[\mathfrak{A}, x_\mu]$$

from (9). Equating this to the right-hand side of (7), we get

$$[\mathfrak{A}, x_\mu] = \frac{\partial x_\mu}{\partial u_r} a_r + n_\mu a_n. \tag{17}$$

Let us differentiate this equation with respect to u_s. The operator $\mathfrak{A}$ is not affected by the differentiation, since it has nothing to do with the values of u chosen above. The result is thus

$$\left[\mathfrak{A}, \frac{\partial x_\mu}{\partial u_s}\right] = \frac{\partial^2 x_\mu}{\partial u_r \partial u_s} a_r + \frac{\partial x_\mu}{\partial u_r}\frac{\partial a_r}{\partial u_s} + \frac{\partial n_\mu}{\partial u_s} a_n + n_\mu \frac{\partial a_n}{\partial u_s}. \tag{18}$$

Hence, from (15),

$$[\mathfrak{A}, \gamma^{rs}] = -\left[\mathfrak{A}, \frac{\partial x^\mu}{\partial u_r}\frac{\partial x_\mu}{\partial u_s}\right]$$

$$= -\frac{\partial x^\mu}{\partial u_r}\left\{\frac{\partial^2 x_\mu}{\partial u_t \partial u_s} a_t + \frac{\partial x_\mu}{\partial u_t}\frac{\partial a_t}{\partial u_s} + \frac{\partial n_\mu}{\partial u_s} a_n + n_\mu \frac{\partial a_n}{\partial u_s}\right\}$$

$$- \left\{\frac{\partial^2 x_\mu}{\partial u_t \partial u_r} a_t + \frac{\partial x_\mu}{\partial u_t}\frac{\partial a_t}{\partial u_r} + \frac{\partial n_\mu}{\partial u_r} a_n + n_\mu \frac{\partial a_n}{\partial u_r}\right\}\frac{\partial x^\mu}{\partial u_s}$$

$$(t = 1, 2, 3)$$

$$= \frac{\partial \gamma^{rs}}{\partial u_t} a_t + \gamma^{rt}\frac{\partial a_t}{\partial u_s} + \gamma^{ts}\frac{\partial a_t}{\partial u_r} - 2\Omega^{rs} a_n. \tag{19}$$

Let γ be the determinant formed by the γ^{rs}, and let γ_{rs} be the co-factor of the element γ^{rs} in this determinant, divided by γ. Then

$$[\mathfrak{A}, \gamma] = [\mathfrak{A}, \gamma^{rs}]\gamma_{rs}\gamma$$

$$= \frac{\partial \gamma}{\partial u_t} a_t + 2\gamma\left(\frac{\partial a_t}{\partial u_t} - \Omega_t{}^t a_n\right). \tag{20}$$

[3] See L. P. Eisenhart, *Riemannian Geometry* (Oxford University Press, 1926), p. 146.

The equation

$$\gamma^{tu}\gamma_{us} = \delta_s{}^t$$

leads to

$$[\mathfrak{A}, \gamma^{tu}\gamma_{us}] = 0$$

or

$$\gamma^{tu}[\mathfrak{A}, \gamma_{us}] + [\mathfrak{A}, \gamma^{tu}]\gamma_{us} = 0.$$

Thus

$$[\mathfrak{A}, \gamma_{rs}] = \gamma_{rt}\gamma^{tu}[\mathfrak{A}, \gamma_{us}]$$

$$= -\gamma_{rt}\gamma_{us}[\mathfrak{A}, \gamma^{tu}]. \tag{21}$$

Using (19), this gives

$$[\mathfrak{A}, \gamma_{rs}] = -\gamma_{rt}\gamma_{us}\left\{\frac{\partial \gamma^{tu}}{\partial u_v} a_v + \gamma^{tv}\frac{\partial a_v}{\partial u_u} + \gamma^{vu}\frac{\partial a_v}{\partial u_t} - 2\Omega^{tu} a_n\right\}$$

$$= \frac{\partial \gamma_{rs}}{\partial u_v} a_v - \gamma_{us}\frac{\partial a_r}{\partial u_u} - \gamma_{rt}\frac{\partial a_s}{\partial u_t} + 2\Omega_{rs} a_n \tag{22}$$

with the help of

$$\gamma_{us}\frac{\partial \gamma^{tu}}{\partial u_v} + \frac{\partial \gamma_{us}}{\partial u_v}\gamma^{tu} = 0.$$

We must now obtain the commutation relation of $\mathfrak{A}$ with n_μ. We get this by applying $\mathfrak{A}$ to Eqs. (6) which define n_μ. This gives

$$[\mathfrak{A}, n_\mu]\frac{\partial x^\mu}{\partial u_s} = -n_\mu\left[\mathfrak{A}, \frac{\partial x^\mu}{\partial u_s}\right]$$

$$= -n_\mu \frac{\partial^2 x^\mu}{\partial u_r \partial u_s} a_r - \frac{\partial a_n}{\partial u_s}$$

$$= \frac{\partial n_\mu}{\partial u_r}\frac{\partial x^\mu}{\partial u_s} a_r - \frac{\partial a_n}{\partial u_s},$$

and

$$[\mathfrak{A}, n_\mu] n^\mu = 0.$$

These four equations are sufficient to determine the four quantities $[\mathfrak{A}, n_\mu]$ and give

$$[\mathfrak{A}, n_\mu] = \frac{\partial n_\mu}{\partial u_r} a_r + \gamma_{rs}\frac{\partial x_\mu}{\partial u_r}\frac{\partial a_n}{\partial u_s}, \tag{23}$$

as is easily verified.

Let us now introduce another small deformation operator $\epsilon\mathfrak{B}$ which displaces each point

$x_\mu(u)$ on the surface S by the amount

$$\frac{\partial x_\mu}{\partial u_r}\epsilon b_r + n_\mu \epsilon b_n.$$

Thus,

$$[\mathfrak{B}, x_\mu] = \frac{\partial x_\mu}{\partial u_s} b_s + n_\mu b_n. \tag{24}$$

We shall now evaluate the commutator of $\mathfrak{A}$ and $\mathfrak{B}$.

Applying the operator $\mathfrak{A}$ to Eq. (24) and using (18) and (23), we get

$$[\mathfrak{A}, [\mathfrak{B}, x_\mu]] = \left[\mathfrak{A}, \frac{\partial x_\mu}{\partial u_s}\right] b_s + [\mathfrak{A}, n_\mu] b_n$$

$$= \left\{\frac{\partial^2 x_\mu}{\partial u_r \partial u_s} a_r + \frac{\partial x_\mu}{\partial u_r}\frac{\partial a_r}{\partial u_s} + \frac{\partial n_\mu}{\partial u_s} a_n + n_\mu \frac{\partial a_n}{\partial u_s}\right\} b_s$$

$$+ \left\{\frac{\partial n_\mu}{\partial u_r} a_r + \gamma_{rs}\frac{\partial x_\mu}{\partial u_r}\frac{\partial a_n}{\partial u_s}\right\} b_n.$$

Interchanging $\mathfrak{A}$ and $\mathfrak{B}$ here and subtracting, we get

$$[[\mathfrak{A}, \mathfrak{B}], x_\mu] = [\mathfrak{A}, [\mathfrak{B}, x_\mu]] - [\mathfrak{B}, [\mathfrak{A}, x_\mu]]$$

$$= \frac{\partial x_\mu}{\partial u_r}\left(\frac{\partial a_r}{\partial u_s} b_s - \frac{\partial b_r}{\partial u_s} a_s\right)$$

$$+ n_\mu\left(\frac{\partial a_n}{\partial u_s} b_s - \frac{\partial b_n}{\partial u_s} a_s\right)$$

$$+ \gamma_{rs}\frac{\partial x_\mu}{\partial u_r}\left(\frac{\partial a_n}{\partial u_s} b_n - \frac{\partial b_n}{\partial u_s} a_n\right). \tag{25}$$

The commutator of the two deformation operators $\epsilon\mathfrak{A}$ and $\epsilon\mathfrak{B}$ is another deformation operator, $\epsilon^2[\mathfrak{A}, \mathfrak{B}] = \epsilon^2\mathfrak{C}$ say, which according to (25) displaces each point $x_\mu(u)$ on the surface S by the amount

$$\frac{\partial x_\mu}{\partial u_r}\epsilon^2 c_r + n_\mu \epsilon^2 c_n,$$

where

$$c_r = \frac{\partial a_r}{\partial u_s} b_s - \frac{\partial b_r}{\partial u_s} a_s + \gamma_{rs}\left(\frac{\partial a_n}{\partial u_s} b_n - \frac{\partial b_n}{\partial u_s} a_n\right), \tag{26}$$

$$c_n = \frac{\partial a_n}{\partial u_s} b_s - \frac{\partial b_n}{\partial u_s} a_s. \tag{27}$$

The deformation operator $\mathfrak{C}$ is not of quite the same nature as $\mathfrak{A}$ and $\mathfrak{B}$ since, while the displacements a_r, b_r are functions of the u's only, the c_r are functions of the γ_{rs} as well as being explicit functions of the u's. The c_r do not, in general, commute with deformation operators, as shown by Eq. (22). Thus, the set of deformation operators of the type of $\mathfrak{A}$ and $\mathfrak{B}$ do not form a Lie group. If we restrict ourselves to operators $\mathfrak{A}$, $\mathfrak{B}$, which do not change the surface S_1 but only its parametrization, so that $a_n = b_n = 0$, then from (26) c_r is independent of the γ's, so that $\mathfrak{C}$ is another operator of the same nature. Thus, in this restricted case the deformation operators do form a Lie group. In the general case we can get a Lie group by taking all the deformations in which the displacements a_r, a_n are general functions of the γ's and Ω's and their derivatives, as well as being explicit functions of the u's.

The deformation operator $\mathfrak{A}$ must depend linearly on the functions $a_r(u)$, $a_n(u)$ which fix the displacement of each point $x_\mu(u)$ on the surface S. Thus $\mathfrak{A}$ must be connected with the functions $a_r(u)$, $a_n(u)$ by an equation of the form.

$$\mathfrak{A} = \int \{a_r(u)\Pi^r(u) + a_n(u)\Pi^n(u)\} d^3u, \qquad (d^3u = du_1 du_2 du_3), \tag{28}$$

where $\Pi^r(u)$, $\Pi^n(u)$ are operator functions of u. The Π's are the elementary deformation operators, in terms of which any deformation operator can be linearly expressed.

We should expect an equation of the form (28) to hold also for more general deformation operators $\mathfrak{A}$ for which the displacements a_r, a_n are functions of the γ's and Ω's. The question then arises as to the order in which the two factors a and Π are to be put on the right-hand side of (28), as these two factors no longer commute. In order that one shall be able to apply these more general deformation operators to functions of S in accordance with formula (8), it is necessary that the Π's in (28) should always be to the right of their coefficients a, as this will make formula (8) follow from formula (9) together with the condition that each operator Π applied to something independent of S produces zero. It should be noticed that we get a consistent scheme when we assume that the commutator of

two of these more general operators $\mathfrak{A}$, with the Π's to the right of their coefficients, is another such operator with the Π's to the right of their coefficients, since the extra terms in the commutator arising from the Π's not commuting with the a's will all be linear functions of the Π's with their coefficients on the left. The Π's, together with the γ's and Ω's, then form an algebra with self-consistent commutation relations.

The commutation relations for the elementary deformation operators Π may easily be obtained from our previous commutation relations involving integrals of the Π's. For example, substituting

$$\mathfrak{A}=\int\{a_r(u')\Pi^r(u')+a_n(u')\Pi^n(u')\}d^3u'$$

(with a_r, a_n independent of the γ's and Ω's) in (17) and equating coefficients of $a_r(u')$, $a_n(u')$, we get

$$[\Pi^r(u'),\, x_\mu(u)]=(\partial x_\mu/\partial u_r)\delta(u-u'), \quad (29)$$

$$[\Pi^n(u'),\, x_\mu(u)]=n_\mu\delta(u-u'). \quad (30)$$

Similarly from (19) we get

$$[\Pi^t(u'),\, \gamma^{rs}(u)]=\frac{\partial\gamma^{rs}}{\partial u_t}\delta(u-u')+\gamma^{rt}(u)\frac{\partial\delta(u-u')}{\partial u_s}+\gamma^{ts}(u)\frac{\partial\delta(u-u')}{\partial u_r}, \quad (31)$$

$$[\Pi^n(u'),\, \gamma^{rs}(u)]=-2\Omega^{rs}\delta(u-u'), \quad (32)$$

and from (20) we get

$$[\Pi^t(u'),\, \gamma(u)]=\frac{\partial\gamma}{\partial u_t}\delta(u-u')+2\gamma(u)\frac{\partial\delta(u-u')}{\partial u_t}, \quad (33)$$

$$[\Pi^n(u'),\, \gamma(u)]=-2\gamma\Omega_t{}^s\delta(u-u'), \quad (34)$$

and so on.

To get the commutator of two Π's, we notice that, from (26) and (27) and the rule that coefficients must always be put to the left of the operators Π,

$$[\mathfrak{A},\, \mathfrak{B}]=\int\left\{\left(\frac{\partial a_r}{\partial u_s}b_s-\frac{\partial b_r}{\partial u_s}a_s+\gamma_{rs}\frac{\partial a_n}{\partial u_s}b_n-\gamma_{rs}\frac{\partial b_n}{\partial u_s}a_n\right)\Pi^r+\left(\frac{\partial a_n}{\partial u_s}b_s-\frac{\partial b_n}{\partial u_s}a_s\right)\Pi^n\right\}d^3u.$$

Substituting the expression given by (28) for $\mathfrak{A}$ and

$$\int\{b_r(u')\Pi^r(u')+b_n(u')\Pi^n(u')\}d^3u'$$

for $\mathfrak{B}$ and picking out coefficients of $a_r(u)b_s(u')$, $a_r(u)b_n(u')$, $a_n(u)b_n(u')$, we get

$$[\Pi^r(u),\, \Pi^s(u')]=-\frac{\partial\delta(u-u')}{\partial u_s}\Pi^r(u')-\frac{\partial\delta(u-u')}{\partial u_r}\Pi^s(u), \quad (35)$$

$$[\Pi^r(u),\, \Pi^n(u')]=-\frac{\partial\delta(u-u')}{\partial u_r}\Pi^n(u), \quad (36)$$

$$[\Pi^n(u),\, \Pi^n(u')]=-\frac{\partial\delta(u-u')}{\partial u_s}\times\{\gamma_{rs}(u)\Pi^r(u)+\gamma_{rs}(u')\Pi^r(u')\}. \quad (37)$$

In this way we obtain all the properties of the Π's.

Corresponding to the form (28) for $\mathfrak{A}$, the operator A of the Schrödinger Eq. (10) must also be expressible linearly in terms of the functions $a_r(u)$, $a_n(u)$, say

$$A=\int\{a_r(u)P^r(u)+a_n(u)P^n(u)\}d^3u, \quad (38)$$

where $P^r(u)$ and $P^n(u)$ are operators operating on the variables q of the wave function. They are, of course, Hermitian. The Schrödinger Eq. (10) for a wave function ψ now gives

$$\int\{a_r\Pi^r+a_n\Pi^n\}d^3u\psi=\int\{a_rP^r+a_nP^n\}d^3u\psi. \quad (39)$$

Since this wave equation must hold for an arbitrary small deformation, corresponding to arbitrary functions $a_r(u)$, $a_n(u)$, the wave func-

tion ψ must satisfy

$$\Pi^r(u)\psi = P^r(u)\psi, \tag{40}$$

$$\Pi^n(u)\psi = P^n(u)\psi. \tag{41}$$

These are the wave equations expressed in terms of the elementary operators Π, P.

The operators $P^n(u)$ are the Hamiltonians of the present theory and Eq. (41) is the important wave equation. There is one of these equations for each point on the surface S. Equation (40) merely states how the wave function is changed by a change in the u coordinate system, and the operators $P^r(u)$ are a kind of momentum operator, which is simple and can easily be worked out for any given dynamical system.

Let us take as an example a dynamical system composed of a number of particles without spin, the mth particle having coordinates u^m. The variables u^m may be taken as the q's in the wave function, so that it reads $\psi(u^m)$. The change in the parametrization caused by the displacements $\epsilon a_r(u)$ results in the u coordinates of a fixed point on the surface S being reduced by $\epsilon a_r(u)$, so that the coordinates of a particle u_r^m get changed to $u_r^m - \epsilon a_r(u^m)$. The operator which simply causes this change in the variables in the wave function is $-\epsilon \sum_m a_r(u^m)(\partial/\partial u_r^m)$. However, this operator is not Hermitian. It must be replaced by

$$-\tfrac{1}{2}\epsilon \sum_m \left\{ a_r(u^m)\frac{\partial}{\partial u_r^m} + \frac{\partial}{\partial u_r^m} a_r(u^m) \right\}. \tag{42}$$

This operator not only changes the variable u^m in the wave function in the required way, but also multiplies the wave function by a suitable factor, so that the normalization is preserved under the change in parametrization. We can now put

$$\int a_r(u) P^r(u) d^3u = -\tfrac{1}{2} \sum_m \left\{ a_r(u^m)\frac{\partial}{\partial u_r^m} + \frac{\partial}{\partial u_r^m} a_r(u^m) \right\},$$

from which it follows that

$$P^r(u) = -\tfrac{1}{2} \sum_m \left\{ \delta(u-u^m)\frac{\partial}{\partial u_r^m} + \frac{\partial}{\partial u_r^m}\delta(u-u^m) \right\}$$

$$= \sum_m \left\{ -\delta(u-u^m)\frac{\partial}{\partial u_r^m} + \frac{1}{2}\frac{\partial \delta(u-u^m)}{\partial u_r} \right\}. \tag{43}$$

Equations (40) and (41) can be combined into the single equation

$$\{\Pi^\alpha(u) - P^\alpha(u)\}\psi = 0 \quad (\alpha = 1, 2, 3, n)$$

From this equation and

$$\{\Pi^\beta(u') - P^\beta(u')\}\psi = 0. \quad (\beta = 1, 2, 3, n)$$

we can infer

$$[\Pi^\alpha(u) - P^\alpha(u),\ \Pi^\beta(u') - P^\beta(u')]\psi = 0. \tag{44}$$

Now the commutation relations (35), (36), (37) may be written

$$[\Pi^\alpha(u), \Pi^\beta(u')] = \int \kappa_\gamma{}^{\alpha\beta}(u, u', u'')\Pi^\gamma(u'')d^3u'', \quad (\gamma = 1, 2, 3, n) \tag{45}$$

where the κ's are certain functions of the variables u, u', u'', which may involve the γ^{rs} at these points. From (44), (45), and the further wave equations

$$\{\Pi^\gamma(u'') - P^\gamma(u'')\}\psi = 0,$$

we get

$$\left\{ \int \kappa_\gamma{}^{\alpha\beta}(u, u', u'')P^\gamma(u'')d^3u'' - [\Pi^\alpha(u), P^\beta(u')] - [P^\alpha(u), \Pi^\beta(u')] + [P^\alpha(u), P^\beta(u')] \right\}\psi = 0. \tag{46}$$

All the four terms in the $\{\}$ here operate only on the q variables in ψ, not on the S variables. This is evident for the first and last terms, and for the others it is easily proved. Taking the second term, for example, we have

$$[[\Pi^\alpha(u), P^\beta(u')], x_\mu(u'')] = [[\Pi^\alpha(u), x_\mu(u'')], P^\beta(u')] = 0$$

from (29) and (30), showing that $[\Pi^\alpha(u), P^\beta(u')]$ commutes with all the quantities $x_\mu(u'')$ which define S, and hence it does not operate on the S variables. Thus, Eq. (46) is a condition on the wave function ψ for just one S.

Usually in quantum mechanics the wave function at one particular time is arbitrary. There are certain special dynamical systems for which it is not arbitrary, and the restrictions on the wave function at a particular time are then known as supplementary conditions. Let us sup-

pose there are no supplementary conditions in the present theory. There are then no restrictions on the wave function for one particular S, and so the operator in (46) must vanish. Thus,

$$\int \kappa_\gamma{}^{\alpha\beta}(u, u', u'')P^\gamma(u'')d^3u'' - [\Pi^\alpha(u), P^\beta(u')] - [P^\alpha(u), \Pi^\beta(u')] + [P^\alpha(u), P^\beta(u')] = 0. \quad (47)$$

This result may be written, with the help of (45),

$$[\Pi^\alpha(u) - P^\alpha(u), \Pi^\beta(u') - P^\beta(u')] = \int \kappa_\gamma{}^{\alpha\beta}(u, u', u'')\{\Pi^\gamma(u'') - P^\gamma(u'')\}d^3u''. \quad (48)$$

It can now be expressed in words, *the commutation relations between the quantities* $\Pi^\alpha(u) - P^\alpha(u)$ *are the same as those between the* $\Pi^\alpha(u)$. The argument by which this result was obtained is similar to the argument by which one infers that all the Hamiltonians must commute in the many-time theory of electrons interacting with the electromagnetic field, using the condition that all the $\partial/\partial t$ operators commute.

Equation (47) now gives, with the values of the κ's provided by (35), (36), (37),

$$[P^r(u), P^s(u')] + [\Pi^s(u'), P^r(u)] - [\Pi^r(u), P^s(u')] = \frac{\partial\delta(u-u')}{\partial u_s}P^r(u') + \frac{\partial\delta(u-u')}{\partial u_r}P^s(u), \quad (49)$$

$$[P^r(u), P^n(u')] + [\Pi^n(u'), P^r(u)] - [\Pi^r(u), P^n(u')] = \frac{\partial\delta(u-u')}{\partial u_r}P^n(u), \quad (50)$$

$$[P^n(u), P^n(u')] + [\Pi^n(u'), P^n(u)] - [\Pi^n(u), P^n(u')] = \frac{\partial\delta(u-u')}{\partial u_s}\{\gamma_{rs}(u)P^r(u) + \gamma_{rs}(u')P^r(u')\}. \quad (51)$$

Equations (49) give the commutation relations between the P^r's, and ought to be satisfied automatically if the P^r's have been correctly calculated. It may easily be checked that they are satisfied for Eq. (43). Equations (50) and (51) concern the Hamiltonians $P^n(u)$, and are *the fundamental commutation relations of the present theory.*

In non-relativistic quantum mechanics, for a dynamical system not acted on by external forces the Hamiltonian operator must not depend on the time. There must be a corresponding condition in the present theory. At first sight one would think this condition should be that the operators P^r, P^n are all independent of the S variables. This will not do, however, since it leads to a contradiction. It would make all quantities of the form $[\Pi^\alpha(u), P^\beta(u')]$ vanish, so that it would make the left-hand side of Eq. (51) independent of the S variables, while the right-hand side is not independent of them as it involves the γ_{rs}. We must take the weaker condition, that the operators P^r, P^n can involve only those functions of the S variables that describe geometrical properties of the parametrized surface S and are independent of its position and orientation in space-time. Thus they can involve the γ's and Ω's and their derivatives, but must not involve the variables $x_\mu(u)$, $\partial x_\mu(u)/\partial u_r$ except insofar as these variables are contained in the γ's and Ω's.

IV. SCALARS AND TENSORS

Let us consider a dynamical variable X on S. It varies with a variation of S according to the Heisenberg equation (13). It may be that X remains unchanged for all changes of S such that the surface and its parametrization are unchanged in the neighborhood of a particular point u'. In this case the dynamical variable X is *localized* at the point u' in S, or at the point $x_\mu(u')$ in space-time. We have here the mathematical condition for a dynamical variable to be localized at a particular point.

It may be further that X remains unchanged for all changes of S such that the point $x_\mu(u')$ remains fixed, i.e., such that S always passes through the fixed point in space-time where X is localized and the parameters u' of this point are unchanged, although the direction of S at this point may be altered. In this case X is called a *scalar.* In general, a dynamical variable X localized at a point $x_\mu(u')$ will undergo changes when S is changed subject to the condition that the point $x_\mu(u')$ remains fixed. X will then be a component

of a tensor, referred to the u's and the normal to the surface S as coordinate system, the tensor character of X being determined by its law of transformation under these changes of S.

The changes in a dynamical variable discussed above require a departure from the usual concept of a "dynamical variable at a particular time." A dynamical variable is defined in the first place as a linear operator operating on a wave function on a particular S, S_1, say. By interpreting the linear operator as an operator on the S dependent wave function, the linear operator becomes dependent on S_1 and we get Heisenberg's picture of dynamical variables. In Heisenberg's picture in non-relativistic quantum mechanics, a dynamical variable localized at a particular point P in space-time is a well-defined operator. In the present theory it is not, unless it is a scalar. In general it will depend on the direction of the surface S_1 through the point P where the dynamical variable is localized, its variation with variations in this direction being given by an equation of the form of Heisenberg's equation of motion. It may also depend on the parametrization of S_1 in the neighborhood of P, but this will involve rather trivial variations. It may also even depend on the curvature of S_1 at the point P. From the point of view of Heisenberg's picture, it would be more reasonable to count the different linear operators corresponding to a dynamical variable localized at P referred to different S's through the point P as different dynamical variables, *but this would involve a departure from the usual connection between Heisenberg's picture and Schrödinger's.*

Let

$$\mathfrak{A}^*=\int\{a_r\Pi^r+a_n\Pi^n\}d^3u,$$
$$A^*=\int\{a_rP^r+a_nP^n\}d^3u, \tag{53}$$

for functions a_r, a_n, which vanish at one particular point u in their domain. The tensor character of a dynamical variable X localized at the point $x_\mu(u)$ is determined by its commutator with A^*, in accordance with (13). Thus a scalar commutes with A^*. Any function ξ of dynamical variables and of variables which determine S, which are all localized at $x_\mu(u)$, will similarly have its tensor character determined by its commutator with $A^*-\mathfrak{A}^*$, in accordance with (14). Thus a scalar ξ commutes with $A^*-\mathfrak{A}^*$,

$$[\xi, A^*-\mathfrak{A}^*]=0. \tag{54}$$

The four quantities $\partial x_\mu/\partial u_s$, n_μ for a particular value of μ with $s=1, 2, 3$ form the four components of a covariant vector with respect to the u coordinate system. Their commutators with $A^*-\mathfrak{A}^*$ are, from (18) and (23)

$$\left[\frac{\partial x_\mu}{\partial u_s}, A^*-\mathfrak{A}^*\right]=\frac{\partial x_\mu}{\partial u_r}\frac{\partial a_r}{\partial u_s}+n_\mu\frac{\partial a_n}{\partial u_s},$$
$$[n_\mu, A^*-\mathfrak{A}^*]=\gamma_{rs}\frac{\partial x_\mu}{\partial u_r}\frac{\partial a_n}{\partial u_s}.$$

Any four ξ's with similar commutation relations are the four components of a covariant vector. Thus the commutation relations for a general covariant vector ξ^s, ξ^n are

$$[\xi^s, A^*-\mathfrak{A}^*]=\xi^r\frac{\partial a_r}{\partial u_s}+\xi^n\frac{\partial a_n}{\partial u_s},$$
$$[\xi^n, A^*-\mathfrak{A}^*]=\gamma_{rs}\xi^r\frac{\partial a_n}{\partial u_s}.$$

The commutation relations for other tensor characters may readily be obtained from this one.

From (20)

$$[\gamma, A^*-\mathfrak{A}^*]=2\gamma\partial a_t/\partial u_t.$$

Thus the determinant γ is not a scalar according to the present definition. A quantity p with the same commutation relation as $\gamma^{\frac{1}{2}}$, namely

$$[p, A^*-\mathfrak{A}^*]=p\partial a_t/\partial u_t,$$

may be called a scalar density, (in a three-dimensional sense) because the integral of such a quantity over all u_1, u_2, u_3 is independent of the parametrization of S. Any scalar multiplied by $\gamma^{\frac{1}{2}}$ gives a scalar density. Similarly, a tensor multiplied by $\gamma^{\frac{1}{2}}$ may be called a tensor density.

One would be inclined to assume that the operators P^r, P^n are the components of a second-rank tensor density T, P^r being the component T^{rn} and P^n the component T^{nn}. This assumption would lead to some further simple commutation

relations between the P^r and P^n. However, I do not see any justification for this assumption in general. It leads to difficulties when there are particles with spin half a quantum present, owing to such spins not being directly describable with reference to oblique axes.

V. DISCUSSION

Any relativistic quantum theory of a localizable dynamical system will provide an example of linear operators $P^n(u)$ satisfying the commutation relations (50), (51), (provided there are no supplementary conditions). Conversely, *any example of linear operators $P^n(u)$ satisfying* (50), (51) *will provide a relativistic theory of a localizable dynamical system.* If one accepts the assumption that atomic systems are localizable, the problem of getting a relativistic theory of them becomes the problem of obtaining examples of sets of operators $P^n(u)$ satisfying (50), (51). If no suitable examples can be found, free from inconsistencies and sufficiently complicated to correspond to interaction between the elementary particles, one could infer that atomic systems are not localizable.

A field theory comparable with the present theory has been given by P. Weiss.[4] Weiss starts with a classical Lagrangian function involving field variables q, which provides an action integral bounded by any space-like surface S. Weiss then introduces momentum variables p on S, which are conjugate to the q's, also introduces Hamiltonians, and finally passes over from the classical theory to the analogous quantum theory.

Weiss' theory applies to dynamical systems which are localizable and are thus subject to the present theory, and his theory provides a method for obtaining Hamiltonians P^n satisfying the required commutation relations. The Hamiltonians obtainable in this way are not of the most general kind, but are restricted by two conditions. Firstly, the method applies only to dynamical systems describable in terms of dynamical variables satisfying the standard commutation relations of canonical q's and p's—it thus would not apply to electrons or other particles satisfying Fermi's statistics, for which the field quantities satisfy anticommutation relations. Secondly, the q's in Weiss' theory must be scalars, independent of the direction of the surface S. Only the p's may depend on this direction.[5]

Another quantum theory of fields has recently been given by Tomonaga.[6] This theory, like the present one, deals with wave functions on arbitrary space-like surfaces, but uses a different standard for identifying wave functions on different surfaces, based on the wave equations for particles without interaction. A wave function $\psi_1(q_1)$ on the surface S_1 counts as equal to a wave function $\psi_2(q_2)$ on the surface S_2 in Tomonaga's theory if the wave function $\psi_1(q_1)$ on S_1 leads to the wave function $\psi_2(q_2)$ on S_2 according to the wave equation for particles without interaction. This method of identifying wave functions avoids the need for parametrizing the surfaces S. A wave function will now change on passing from one surface S to another only on account of the interaction energy, so the Hamiltonians of Tomonaga's theory involve only the interaction energy. Tomonaga has developed his theory only for the case when all the Hamiltonians commute, but probably his method can be generalized.

[4] P. Weiss, Proc. Roy. Soc. **A169**, 102 (1938).

[5] P. Weiss, Proc. Roy. Soc. **A169**, 107 (1938), see bottom of page.

[6] S. Tomonaga, Progress of Theoretical Physics **1**, 27 (1946).