On the Behavior of Cross Sections Near Thresholds

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The energy dependence of the cross section for the formation of a product, near the threshold energy for that formation, is considered. It is shown that the cross section is, apart from a constant, in the neighborhood of the threshold the same function of energy, no matter what the reaction mechanism is, as long as the long-range interaction of the product particles is the same. The same must hold, because of the principle of detailed balance, for the back reaction, i.e., the reaction between particles with very low relative velocities. In this case, the cross section, as function of the energy, depends only on the long-range interaction of the reacting particles. The energy dependence of the cross section is determined for three types of interactions, *viz.* no interaction, Coulomb repulsion and Coulomb attraction. The rule for a $1/r^2$ interaction can be obtained from the first case. Reasons are adduced to show that two interactions, the difference of which goes to zero at least as fast as $r^{-2-\epsilon}$ with $(\epsilon>0)$, give the same energy dependence of the cross section. Hence, long-range interaction in the above connection should mean an interaction which, at large distances of the particles, does not go to zero faster than r^{-2}. The effect of small perturbations in the long-range interaction is discussed in general.

SECTION I

THE results of the present article are probably well known to a great many readers: they are concerned with the behavior of the various cross sections in the neighborhood of the threshold of a new mode of reaction. Examples for the rules to be derived can be found in almost every article which is concerned with reaction cross sections.[1] It is believed, however, that the generality of these rules has not been emphasized before, nor have they been derived with the same claim for universal validity as will be done in the present paper. The principal interest in such an investigation arises from the experimental difficulty in measuring threshold energies without the knowledge of how the yield of the product approaches zero with decreasing energy. From this point of view, it would be desirable to obtain also an expression for the yield of reactions in which the number of end products is three or more (e.g., an $(n, 2n)$ reaction). This, however, will not be attempted in the present article which deals only with processes in which only pairs of particles are formed.

The mathematical formalism will be that adopted in a previous article of L. Eisenbud and the present writer.[2] In fact, this formalism will be used in the simplest possible form (*viz.* Eq. (1), reference 2):

$$\varphi=\sum_\lambda \frac{\int X_\lambda(\mathrm{grad}\varphi)_n dS}{E_\lambda-E} X_\lambda. \qquad (1)$$

In this, φ is an arbitrary wave function with energy E, the E being part of the continuous spectrum; $\mathrm{grad}\varphi$ is the vector with components $(\hbar^2/2M)\partial\varphi/\partial x$, $\cdots$, etc. Only φ and $\mathrm{grad}\varphi$ in (1) depend on E. The X_λ and E_λ are the characteristic functions and characteristic values of the Hamiltonian, with the boundary condition that the normal derivative in the above sense of X_λ vanish on a surface S. This S is the boundary between the internal and the external region of the configuration space. The integration in (1) is over this surface. The external region of the configuration space is defined by the requirement that, at the energy E, in the external region (i.e., outside of S) single alternative wave functions

$$F_{sl}(r_s)P_l(\Omega_s)\psi_s=F_{sl}(r_s)\psi_{sl} \qquad (2)$$

form a complete set of solutions of the Schrö-

[1] It would be idle to attempt to give a full list of the pertinent papers. Suffice it to say, therefore, that, if there is no long-range interaction between the colliding particles, Born's approximation already gives the correct asymptotic behavior even in cases in which it is quite inaccurate as far as absolute magnitude is concerned. The importance of the potential type of interaction (cf. Eq. (5)) of the colliding particles at large distances was probably first recognized by P. M. Morse and E. C. G. Stueckelberg (Ann. d. Physik 9, 579 (1931)). Their method, if properly carried out, should give the asymptotic behavior correctly in all cases, even if the method itself is quite inaccurate for the calculation of the magnitude of the cross section.

[2] L. Eisenbud and E. P. Wigner, Phys. Rev. 72, 29 (1947).

dinger equation. The ψ_s in (2) is the product of the wave functions of a pair s of particles which may be formed by the reaction, r_s and Ω_s are the length and direction of the line connecting their centers of mass. P_l can be assumed to be a spherical harmonic of degree l, giving an angular momentum $l\hbar$ to the two particles of the pair s about their common center of mass. The $\psi_{sl}=P_l(\Omega_s)\psi_s$ are orthogonal to each other on the surface S, the letter s specifying not only the nature of the particles which form the pair s but also their state of excitation, the resultant of their spin angular momenta, and the projection of this into a given direction. The ψ_{sl} will be further assumed to be normalized if integrated over S

$$\int \psi_{sl}\psi_{s'l'}dS=\delta_{ss'}\delta_{ll'}. \tag{2a}$$

Let us assume that φ and $(\text{grad}\varphi)_n$, the component of $\text{grad}\varphi$ perpendicular to S, can be expanded on S in terms of the ψ_{sl}

$$\varphi=\sum_{sl} V_{sl}\psi_{sl}, \tag{3a}$$

$$(\text{grad}\varphi)_n=\sum_{sl} \hbar D_{sl}\psi_{sl}. \tag{3b}$$

Equation (1) then becomes

$$V_{sl}=\sum_{s'l'} R_{sl;s'l'}D_{s'l'}, \tag{4}$$

with

$$R_{sl;s'l'}=\sum_{\lambda}\frac{\gamma_{\lambda sl}\gamma_{\lambda s'l'}}{E_\lambda-E}, \tag{4a}$$

where

$$\gamma_{\lambda sl}=\hbar^{\frac{1}{2}}\int X_\lambda\psi_{sl}dS. \tag{4b}$$

The above is a repetition of the line of argument of reference 2 with the difference that the boundary conditions on S are assumed in the simplest possible form, no attempt being made to make the E_λ, $\gamma_{\lambda sl}$, etc., independent of the position of S. The above equations are almost identical with Eqs. (13), (23), and (17) of the above reference, but the derivation is, of course, much simpler. All that will be used in the following is that the expansion coefficients of φ on S (as φ itself), are linear functions of the expansion coefficients of $(\text{grad}\varphi)_n$ on S (or of $(\text{grad}\varphi)_n$ itself) which has no singularity at the threshold to be investigated as none of the E_λ can be expected to coincide with that threshold.

SECTION II

The $F_{sl}(r_s)$ satisfy, in the external region, the differential equation

$$-\frac{\hbar^2}{2M_s}\left(\frac{d^2}{dr_s^2}+\frac{2}{r_s}\frac{d}{dr_s}-\frac{l(l+1)}{r_s^2}\right)F_{sl}+V_sF_{sl}=(E-\epsilon_s)F_{sl}. \tag{5}$$

The M_s, V_s, and ϵ_s are the reduced mass, potential energy in the external region, and proper energy of the pair s. We shall be particularly concerned with that solution E_{sl} of (5) which corresponds to an outgoing wave and which we shall assume to be normalized in such a way that $\mathrm{E}_{sl}\psi_{sl}$ correspond to unit flux. Assuming that, in the direction of configuration space which corresponds to separation into the pair s, the surface S lies at $r_s=a_s$, the condition of unit flux is that at $r_s=a_s$

$$\frac{\hbar}{2iM_s}\left(\mathrm{E}_{sl}^*\frac{d\mathrm{E}_{sl}}{dr_s}-\mathrm{E}_{sl}\frac{d\mathrm{E}_{sl}^*}{dr_s}\right)=1 \quad (\text{at } r_s=a_s). \tag{6}$$

Since it follows from (5) that the expression in the bracket of (6) is r^{-2} times a constant, the normalization adopted implies

$$i(\mathrm{E}_{sl}e_{sl}^*-\mathrm{E}_{sl}^*e_{sl})=a_s^2/r_s^2, \tag{6a}$$

where

$$e_{sl}=(\hbar/2M_s)d\mathrm{E}_{sl}/dr_s. \tag{6b}$$

If, for large r_s, the potential V goes to zero sufficiently fast, (6a) should mean that $(r_s/a_s)\mathrm{E}_{sl}$ is asymptotically equal to $(M_s/k\hbar)^{\frac{1}{2}}\exp ikr_s$.

The above definition of E_{sl} applies if the pair s is energetically possible at the energy considered. However, particularly just below a threshold, the expansion (3) requires also such ψ_{sl} which are energetically impossible. In this case, E_{sl} shall be that solution of (5) which goes to zero exponentially at $r_s=\infty$. The E_{sl}^* which is in this case not the conjugate complex of E_{sl} is any other linearly independent solution of (5). E_{sl}^* goes, of course, to infinity for large r_s. It is not necessary to specify E_{sl} and E_{sl}^* more closely in this case.

It is convenient to introduce a symbol for the reciprocal logarithmic derivative of E_{sl}:

$$\mathrm{E}_{sl}=q_{sl}e_{sl}, \tag{7}$$

which permits (6a) to be written in the form

$$i(q_{sl}-q_{sl}^*)\,|e_{sl}|^2=a_s^2/r_s^2. \tag{7a}$$

This shows that the imaginary part of q_{sl}^* is positive for every r_s and permits one to introduce

$$j_{sl}^2=i(q_{sl}-q_{sl}^*)=a_s^2/r_s^2e_{sl}e_{sl}^*. \tag{7b}$$

Actually, q_{sl} satisfies a Ricatti-type differential equation which follows from (5). The q_{sl} as defined by (7) is real for s which are energetically impossible and (7a) and (7b) will not be used for such s.

Equations (3) and (4) permit the determination of half of the coefficients of the most general solution

$$\varphi=\sum_{sl}\psi_{sl}(\alpha_{sl}\mathrm{E}_{sl}+\beta_{sl}\mathrm{E}_{sl}^*) \tag{8}$$

of the Schrödinger equation in the external region. The coefficient of ψ_{sl} in φ is

$$V_{sl}=\alpha_{sl}q_{sl}e_{sl}+\beta_{sl}q_{sl}^*e_{sl}^* \quad (r_s=a_s), \tag{9a}$$

while its coefficient in $(\mathrm{grad}\varphi)_n$ becomes

$$D_{sl}=\alpha_{sl}e_{sl}+\beta_{sl}e_{sl}^*, \quad (r_s=a_s). \tag{9b}$$

If φ is to be a permissible solution, the β_{sl} whose s is energetically impossible must vanish. It is, therefore, immaterial that the coefficients of these β_{sl} are not defined.

One can write (4) by means of (9) in terms of the α and β

$$q_{sl}e_{sl}\alpha_{sl}+q_{sl}^*e_{sl}^*\beta_{sl}=\sum_{s'l'}R_{sl;s'l'}(e_{s'l'}\alpha_{s'l'}+e_{s'l'}^*\beta_{s'l'}). \tag{10}$$

From this the collision matrix U, which gives the outgoing wave coefficients in terms of the incoming wave coefficients, can be obtained:

$$\alpha_{sl}=-\sum_{s'l'}U_{sl;s'l'}\beta_{s'l'}, \tag{11}$$

where

$$U=e^{-1}(q-R)^{-1}(q^*-R)e^*=e^{-1}e^*+e^{-1}(q-R)^{-1}(q^*-q)e^*. \tag{11a}$$

The e and q are diagonal matrices with diagonal elements $e_{sl}(a_s)$ and $q_{sl}(a_s)$, respectively, The summation over s', l' must be extended in (11) only over the energetically possible s'. This makes it possible to use the relation (7a) to give

$$U=e^{-1}e^*+ie^{-1}(q-R)^{-1}e^{-1}. \tag{12}$$

The simple form of (12) could be obtained by the use of the relation (7a) because the columns of U, which refer to energetically impossible states, are without significance. Since, on the other hand, the rows of U which refer to such states are without much interest—they give only the coefficients of the exponentially decaying solutions in the external region—it is permissible to make use of (7b) to eliminate $|e|$ from (12). For this purpose we define

$$e=|e|\omega^*, \tag{13a}$$

where ω is a unitary diagonal matrix defined by (13a). We obtain

$$U=\omega(1+ij(q-R)^{-1}j)\omega. \tag{13}$$

In this, j_{sl}^2 can be considered to be the imaginary part of $2q_{sl}^*$, which is zero if s is energetically impossible. For these s, the e_{sl} can be assumed to be real, $\omega_{sl}=1$. As a result, U will assume the form

$$U=\begin{Vmatrix}\bar{U} & 0\\ 0 & 1\end{Vmatrix}. \tag{13b}$$

$\bar{U}$, the reduced U, occupies the rows and columns which correspond to energetically possible pairs s, the 1 to the energetically impossible pairs. This form of U gives automatically zero cross section to those transitions which are energetically impossible. The unitary symmetric nature of (13b) can be easily proved directly by means of the relation (7b) $j=(i(q-q^*))^{\frac{1}{2}}$. Even though (12) and (13) are not fundamentally different from (38), reference 2, their form is not the same and they are more general by permitting the surface S to be drawn in close even just below a threshold (compare footnote 9, reference 2).

From (12) and (13) one obtains for the reaction cross sections (either $s\neq s'$ or $l\neq l'$)

$$\begin{aligned}\sigma_{sl;s'l'}&=\pi k_s^{-2}(2l+1)\,|U_{sl;s'l'}|^2\\&=\pi k_s^{-2}(2l+1)\,|e_{sl}e_{s'l'}|^{-2}\,|((q-R)^{-1})_{sl;s'l'}|^2\\&=\pi k_s^{-2}(2l+1)\,|j_{sl}j_{s'l'}|^2\\&\qquad\times|((q-R)^{-1})_{sl;s'l'}|^2.\end{aligned} \tag{14}$$

TABLE I. No interaction in external region. Asymptotic values of E, q, j, ω for small $k_s a_s$.

	$E_{sl}(r_s)$	$q_{sl}(a_s)$	$j_{sl}(a_s)$	$1+i^l\omega_{sl}(a_s)$
$l=0$	$[(M/\hbar k)^{\frac{1}{2}}]a/r$	$-2Ma/\hbar$	$2ka(M/\hbar k)^{\frac{1}{2}}$	$[i(ka)^3]/3$
$l=1$	$[i(M/\hbar k)^{\frac{1}{2}}]a/kr^2$	$-Ma/\hbar$	$[(ka)^2/3](M/\hbar k)^{\frac{1}{2}}$	$-[i(ka)^3]/6$
$l\geq 1$	$\dfrac{i^l 1.3.\cdots(2l-1)}{(\hbar k/M)^{\frac{1}{2}}(r/a)(kr)^l}$	$-\dfrac{2Ma}{(l+1)\hbar}$	$\dfrac{2(ka)^{l+1}(M/\hbar k)^{\frac{1}{2}}}{(l+1)1.3.\cdots(2l-1)}$	$\dfrac{-il(ka)^{2l+1}}{(l+1)(2l+1)(1.3.\cdots(2l-1))^2}$

TABLE II. $V=Ze^2/r>0$. Asymptotic values of E, q, j for small kr or ka.

	$E(r)$	$q(a)$	$j(a)$
$l=0$	$\dfrac{-\exp(\pi\alpha+i\alpha\ln 2kR)H_1(i\beta)}{(\hbar r/\pi Ma^2)^{\frac{1}{2}}\arg}$	$\dfrac{(4Ma/\hbar)H_1(i\beta)}{i\beta H_0(i\beta)-2H_1(i\beta)}$	$\dfrac{4(Ma/\pi\hbar)^{\frac{1}{2}}\exp(-\pi\alpha)}{i\beta H_0(i\beta)-2H_1(i\beta)}$
general	$\dfrac{(-)^{l+1}\exp(\pi\alpha+i\alpha\ln 2kR)H_{2l+1}(i\beta)}{(\hbar r/\pi Ma^2)^{\frac{1}{2}}\arg}$	$\dfrac{(4Ma/\hbar)H_{2l+1}(i\beta)}{i\beta H_{2l}(i\beta)-(2l+2)H_{2l+1}(i\beta)}$	$\dfrac{4(Ma/\pi\hbar)^{\frac{1}{2}}\exp(-\pi\alpha)}{\lvert i\beta H_{2l}(i\beta)-(2l+2)H_{2l+1}(i\beta)\rvert}$

The scattering cross sections become

$$\sigma_{sl;sl}=\pi k_s^{-2}(2l+1)\,|\,1-(-)^l U_{sl;sl}\,|^2$$
$$=\pi k_s^{-2}(2l+1)\,|\,1-(-)^l\omega_{sl}{}^2$$
$$-(-)^l i\omega_{sl}{}^2 j_{sl}{}^2((q-R)^{-1})_{sl;sl}\,|^2. \quad (15)$$

SECTION III

Table I gives E_{sl}, q_{sl}, j_{sl}, ω_{sl} for a free particle ($V_s=0$ in (5)) in terms of $k_s=(2M_s(E-\epsilon_s))^{\frac{1}{2}}/\hbar$, assuming that $k_s a_s$ is very small.

These are the quantities to be used in Eqs. (12) to (15) in case of neutral particles. It will be noted that the j_{sl} cannot be obtained from the q_{sl} of Table I by means of (7b) because the q_{sl} are real in the approximation used in this table. It is necessary, rather, to calculate the q_{sl} to the first imaginary term or, alternately, to obtain its imaginary part by means of (7b) and the e_{sl}. Table II contains the same quantities for Coulomb wave functions if both charges have the same sign. In this and Table III

$$\eta=2MZe^2/\hbar^2,$$
$$\alpha=\tfrac{1}{2}\eta/k=MZe^2/\hbar^2 k=Ze^2/\hbar v, \quad (16a)$$

$\beta=2(\eta r)^{\frac{1}{2}}=(8MZe^2 r)^{\frac{1}{2}}/\hbar$ in the expression for E and $\beta=2(\eta a)^{\frac{1}{2}}$ in the expressions for q and j. The Z is the product of the charges on the reaction products to which the E, q, and j apply. The ηa is not assumed to be small in the derivation of the above formulae, while the ka is. On the other hand, $\alpha=\eta a/2ka$ is a large number. R is the distance outside of which the electrostatic potential ceases to act. It is well known that, because of the complicated asymptotic behavior of the Coulomb wave functions at large distances, the introduction of such a range for the Coulomb force is necessary in order to obtain a definite phase shift, etc., $\arg=\Pi(i\alpha)/|\,\Pi(i\alpha)\,|$ is the phase factor of $\Pi(i\alpha)=(i\alpha)!$ The H_{2l} and H_{2l+1} are defined as these quantities with the upper index 1 are defined by Jahnke-Emde. Watson's $K_n(z)=i^{n-1}(2/\pi)H_n(iz)$. The above formulae were calculated on the basis of the outgoing Coulomb wave function[3]

$$E(r)=-(M/\hbar k)^{\frac{1}{2}}\eta a(\eta r)^l\exp(i\alpha\ln(2kR))$$
$$\times(\exp(\tfrac{1}{2}\pi\alpha)/\Pi(i\alpha))(1-\exp(-2\pi\alpha))$$
$$\times\int(y^2+k^2r^2)^{-l-1}$$
$$\exp[-y+i\alpha\ln((y+ikr)/(y-ikr))]dy, \quad (16)$$

where the path of integration runs from $y=\infty$ (where the value of the ln is zero) counter-clock-

[3] W. Gordon, Zeits. f. Physik **48**, 180 (1928). See also A. Sommerfeld and G. Schur, Ann. d. Physik **4**, 409 (1930); M. Stobbe, *ibid.* **7**, 661 (1930). H. Bethe, *Geiger-Scheel's Handbuch der Physik* (Berlin, 1933), 2nd edition, Volume XXIV/1, p. 289 ff. and N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, London, 1933). The remark that the Coulomb wave functions of the continuous spectrum reduce to Bessel functions for low values of the energy is due to J. G. Beckerley, Phys. Rev. **67**, 11 (1945). I could find no reference for the numerical coefficient of the Bessel function at a given asymptotic behavior of the wave function for large r, i.e., the essential part of Tables II and III. See, however, L. W. Nordheim, Proc. Roy. Soc. **A121**, 628 (1928), and Th. Sexl, Zeits. f. Physik **56**, 72 (1929).

TABLE III. $V=Ze^2/r<0$. Asymptotic values of E, q, j for small $k_s a_s$ or kr.

	$E_{sl}(r)$	$q_{sl}(a_s)$	$j_{sl}(a_s)$
$l=0$	$\dfrac{\exp[i\alpha\ln(2kR)]iH_1(\bar\beta)}{(\hbar r/\pi Ma^2)^{\frac{1}{2}}\arg}$	$\dfrac{(4Ma/\hbar)H_1(\bar\beta)}{\bar\beta H_0(\bar\beta)-2H_1(\bar\beta)}$	$\dfrac{4(Ma/\pi\hbar)^{\frac{1}{2}}}{\lvert\bar\beta H_0(\bar\beta)-2H_1(\bar\beta)\rvert}$
general	$\dfrac{(-)^l\exp[i\alpha\ln(2kR)]iH_{2l+1}(\bar\beta)}{(\hbar r/\pi Ma^2)^{\frac{1}{2}}\arg}$	$\dfrac{(4Ma/\hbar)H_{2l+1}(\bar\beta)}{\bar\beta H_{2l}(\bar\beta)-(2l+2)H_{2l+1}(\bar\beta)}$	$\dfrac{4(Ma/\pi\hbar)^{\frac{1}{2}}}{\lvert\bar\beta H_{2l}(\bar\beta)-(2l+2)H_{2l+1}(\bar\beta)\rvert}$

wise around the singularity at $y=-ikr$ back to $y=\infty$ (where the value of the ln is $2\pi i$ now).

Table III, which applies if the potential is attractive, is calculated on the basis of the same formula. The η and $\alpha=\eta/2k$ are negative in this case, and $\bar\beta=2(-\eta r)^{\frac{1}{2}}$ is real and positive. The most essential difference between these formulae and those of Table II is the absence of the factor $\exp(\pi\alpha)$ or $\exp(-\pi\alpha)$. It is to be remembered that $\alpha=Ze^2/\hbar v$ goes to infinity near the threshold. This factor is the most important part of the "penetration factor" and its absence in the case of the attractive potential of Table III is quite natural. Mathematically, the difference can be said to arise from the $1-\exp(-2\pi\alpha)$ factor in (16): the 1 is dominant if $\alpha>0$; the exponential term, which cancels a similar factor which always occurs at small kr is dominant if $\alpha<0$. This also accounts for the differences in sign. In addition, the $H=J+iN$ has a real argument in case of the attractive potential of Table III, an imaginary argument in case of the repulsive potential of Table II. In the latter case, it goes to zero exponentially for large $\beta=2(\eta a)^{\frac{1}{2}}$. This partly compensates for the $\exp(\pi\alpha)$ factor and is particularly important for large a in accord with the fact that only the part of the barrier outside a has to be penetrated. Strictly speaking, however, this factor is irrelevant from the point of view of the present paper since it is independent of energy. It is clear, however, that it influences the absolute magnitude of the cross section even if it leaves its energy dependence unaffected.

The important point to be noted is that $q-R$ is, in first approximation, independent of energy. This means that $q-R$ assumes a definite value for the threshold energy ϵ_n and that its derivative is finite at that point. The same will be true of the matrix $(q-R)^{-1}$. It then follows from (14) that the factors $|e_{sl}e_{s'l'}|^{-2}$ or $|j_{sl}j_{s'l'}|^2$ will determine the behavior of the reaction cross section near ϵ_n. Even these factors will be independent of E in first approximation if ϵ_n is not a threshold for the formation of the pair s or s'. They will contain the relevant dependence on k if it is. It should be noted that only the dependence on k is meaningful in Tables I, II, III because R or $(q-R)^{-1}$ will depend on the a_s in such a way as to compensate for the dependence of e_{sl} or j_{sl} on these quantities.

The rules which follow can be best described by using, instead of s, the letters p or n for pairs which are or are not energetically possible below the threshold. These will also be denoted as old or new type of particles, respectively. The e_{pl}, j_{pl} will be considered as independent of E.

A. Reaction Cross Sections

1. *Production of new type of particles with angular momentum l from old type.* The behavior of the cross section depends in this and all other cases on whether the new type of particles, in the present case the reaction products, are charged or uncharged. We have to distinguish, therefore, three cases:

Neutral particles: $\sigma\sim k_n^{2l+1}$;
Coulomb repulsion:
$$\sigma\sim\exp(-2\pi M_nZ_ne^2/\hbar^2k_n);$$
Coulomb attraction: $\sigma\sim 1$.

In the last case, the cross section sets in with a finite value right at the threshold, even for higher angular momenta.

2. *Reaction of new type of particles with angular momentum l to form old type of particles.* This is the typical case of the reaction of particles with very low kinetic energy. Again, three cases have to be distinguished, depending on the charged or uncharged character of the reacting particles.

Neutral particles: $\sigma\sim k_n^{2l-1}$;
Equal charges:
$$\sigma\sim k_n^{-2}\exp(-2\pi M_nZ_ne^2/\hbar^2k_n);$$
Opposite charges: $\sigma\sim k_n^{-2}$.

These rules can be obtained, of course, from the above ones by the principle of detailed balance. According to this principle, the ratio of the cross sections of forward and back reaction is equal to the ratio of the square of the relative momenta of the particles in the final state. Since the relative momentum of an old type of pair is considered to be constant near the threshold, the quantities just enumerated are just k_n^{-2} times the quantities enumerated above.

The best known example for the above rule is the $1/v$ law of the absorption of slow neutrons. The angular momentum l can always be assumed to be zero in this case since any excess angular momentum can be carried off, if necessary, by the emitted light quantum. In fact, the above consideration is a more rigorous derivation of this law in the case of reactions like $B^{10}+n\rightarrow Li^7+He^4$ or $Li^6+n\rightarrow He^4+He^3$ than the one originally given.[4]

A greater variety of examples can be found, e.g., in the article of Schwinger and Teller or of Hamermesh and Schwinger.[5] Even these examples show, however, that the total cross section almost always follows the behavior given above for $l=0$. The reason for this is that an excess angular momentum can always be supplied or carried away by the l of the "old type" of particles.

3. *Reaction of new type to new type (depolarization at low energies) $n, l\rightarrow n, l'$.*

Neutral particles: $\sigma\sim k_n^{2l+2l'}$;
Equal charges:
$$\sigma\sim k_n^{-2}\exp(-4\pi M_n Z_n e^2/\hbar^2 k_n);$$
Opposite charges: $\sigma\sim k_n^{-2}$.

Naturally, all these reactions have finite cross sections only if they are consistent with the conservation rules for angular momentum and for parity.

B. Scattering Cross Sections

The only important case here is the scattering of a new type of particles $n, l\rightarrow n, l$.

Neutral particles: $\sigma\sim k_n^{4l}$;
Coulomb repulsion: $\sigma\sim k_n^{-2}$;
Coulomb attraction: $\sigma\sim k_n^{-2}$.

Even in the case of neutral particles the cross section is finite for $l=0$ at very low energy. In this case, the $\omega^2 j^2(q-R)^{-1}$ term of (15) is the dominant one. On the other hand, in the case of charged particles, the first two terms within the absolute sign do not compensate and most of the scattering is due to these. In this case, the potential scattering is of the order of the maximum possible scattering and this situation is not altered by the nuclear scattering, and our rules become rather meaningless. All these rules are valid, of course, only if $k_s a_s \ll 1$ where a_s is a distance outside of which (5) can be expected to be valid. This condition assures, at the same time, the accuracy of the asymptotic expressions of Tables I and II. In addition to this, it is necessary that $(q-R)^{-1}$ or $q-R$ do not change substantially in the energy interval considered. For q, this condition amounts to $\Delta(k^2a^2)\ll 1$ but although this has to be valid for all s (rather than only for the particular s the formation or reaction of which we are considering) it will hardly limit the validity of the above rules. On the other hand, the condition that $\Delta E(dR/dE)\ll R$ effectively limits the validity of the above rules for the behavior of the cross sections to an energy interval which is small compared with the distance of the next resonance. It is clear, of course, that such a limitation must exist. The inclusion of the formulae for higher l into the tables can be of value only if the angular distribution of the reaction products can be investigated. The total cross section will be proportional to the expression given for the lowest l which can be formed.

SECTION IV

The above tabulations which form the essential result of the present note, give the energy dependence of the dominant term for the various cross sections just above the threshold. They give, of course, the cross section only up to a factor the determination of which cannot be carried out by the methods used in this article. It may be worth while to remark, however, that the next term in all cross sections is proportional

[4] G. Beck and H. L. Horsley, Phys. Rev. **47**, 510 (1935); H. A. Bethe, Phys. Rev. **47**, 747 (1935); F. Perrin and W. Elsasser, Comptes Rendus **200**, 450 (1935).
[5] J. Schwinger and E. Teller, Phys. Rev. **52**, 286 (1937); M. Hamermesh and J. Schwinger, Phys. Rev. **69**, 145 (1946); compare, in particular, Figs. 1 and 2 of the former article.

to $|k_n|$ and that there is such a term in all cross sections, even in those which do not involve the new mode of reaction. Furthermore, a term of this nature is present both above and also below the threshold. As a result, for example, all reaction and scattering cross sections, which are possible below the threshold energy of the new mode, show a cusp at the threshold energy for the new mode, i.e., have a functional dependence on E both for $E<\epsilon_n$ and for $E>\epsilon_n$ like $a+b|E-\epsilon_n|^{\frac{1}{2}}$ +higher order terms in $E-\epsilon_n$. In fact, the coefficients b can be determined if the R matrix (i.e., essentially the coefficients of the dominant term) is known at $E=\epsilon_n$. However, it does not seem that these cusps are particularly important and they will not be discussed further.[6]

It may be more important to consider small deviations from the potentials considered above. Indeed, it can hardly be expected that it is possible to choose the surface S so far out that the interaction outside of it be represented accurately by Ze^2/r (Z positive, negative, or zero). One can approximate such a condition by choosing the a_s very large but one thereby increases the internal region and thus also the fluctuations of R in a given energy range. It seems more natural, therefore, to investigate directly how a small potential outside S will affect the quantities of Table I. Such an investigation shows that if the small potential is twice integrable to infinity, i.e., if it drops as fast as r^{-n} with $n>2$, the asymptotic behavior will not be affected. This shows then that the centrifugal potential, which causes the differences between the different rows of Table I, is just about the weakest potential to change the asymptotic behavior. Indeed, a comparison with Tables II and III shows that the much stronger electrostatic potential completely washes out the effect of the centrifugal potential. On the other hand, while a potential which is small in the above sense does not alter the asymptotic behavior of the cross sections, it can substantially influence both the absolute magnitude of it and also the range within which the asymptotic behavior is valid.

As is well known, the emerging wave function, $\bar{\mathrm{E}}$, for a potential $V+\hbar^2P/2M$ can be obtained from the emerging wave function E for the potential V by means of the formula

$$\bar{\mathrm{E}}(r)=\mathrm{E}(r)+\mathrm{E}(r)\int_r^\infty (r'\mathrm{E}(r'))^{-2}dr' \times\int_{r'}^\infty r''\mathrm{E}(r'')r''\bar{\mathrm{E}}(r'')P(r'')dr''. \quad (17)$$

This can be rewritten as

$$\frac{\bar{\mathrm{E}}(r)}{\mathrm{E}(r)}=1+\int_r^\infty \frac{dr'}{r'^2\mathrm{E}(r')^2}\int_{r'}^\infty (r''\mathrm{E}(r''))^2 \times\bar{P}(r'')dr'', \quad (17a)$$

where $\bar{P}=P(\bar{\mathrm{E}}/\mathrm{E})$. Now a changed asymptotic behavior would entail a change of j^2, or of $|e|^2$, at $k=0$, by an infinitely large or infinitely small factor. The same would then hold for E. It thus appears that the asymptotic behavior of the cross section can change only if the perturbation P is such that the integral on the right side of (17a) diverges at $k=0$. Since however, $r\mathrm{E}(r)$ is, at large r, proportional to $\exp(ikr)$ this condition is equivalent with the condition that the second integral of $\bar{P}$ diverge. This certainly will be true if the second integral of P diverges. On the other hand, if the second integral of P converges, and goes to zero for large r, then $\bar{\mathrm{E}}/\mathrm{E}$ goes to 1 at large r and $\bar{P}$ becomes equal to P under the same condition. Hence, $\bar{P}$'s second integral will converge or diverge if P's second integral converges or diverges. It appears to follow that the asymptotic behavior of the cross section near a resonance remains unaltered by the addition of a "small" potential P whose second integral to infinity converges. Although this derivation of the insensitivity of the asymptotic behavior of the cross sections is certainly not rigorous (it can be made rigorous easily if P drops to zero exponentially or faster), the writer feels convinced that its result is correct.

Even if the above surmise should prove to be correct, there remains the question of the restriction of the region of validity of the limiting laws above derived for the cross sections. This region is determined, in the absence of the additional potential $\hbar^2P/2M$, by the more rigorous of the two conditions enumerated before: (a) The region must be small as compared with the distance of resonances E_λ of (4a) from each other.

[6] It may be mentioned, though, that these cusps are not given by either of the two methods of reference 1.

(This is the condition which apparently limits the validity of the $1/v$ law for slow neutron absorption.) (b) In the case of uncharged particles $k_n a_n \ll 1$, and in the case of charged particles also $\alpha = Ze^2/\hbar v \gg 1$.

It is clear, however, that the additional potential can further reduce the region in which the asymptotic behavior is followed. An obvious example for this is a low but extended repulsive potential of constant magnitude of say $\hbar^2 k_0^2/2M$. The $|e|^2$ for a potential of this height and of extension l outside a is

$$|e|^2 = (\hbar/4Mka^2)[(\mathrm{Ch}\kappa l + \kappa a\ \mathrm{Sh}\kappa l)^2 + (k^2/\kappa^2)(\mathrm{Sh}\kappa l + \kappa a\ \mathrm{Ch}\kappa l)^2] \quad (18)$$

$$\kappa = (k_0^2 - k^2)^{\frac{1}{2}}.$$

Hence, for small k,

$$j(a)^2 = |e(a)|^{-2} \approx (4Mka^2/\hbar) \times (\mathrm{Ch}k_0 l + k_0 a\ \mathrm{Sh}k_0 l)^{-2} \quad (18a)$$

+terms proportional to k^2 and higher powers of k. The dependence of the j on k is, according to (18a), the same as given in the first row of Table I. This illustrates the point that a potential which goes at $r = \infty$ sufficiently fast to zero does not affect the asymptotic behavior of the cross sections. However, even for $k = k_0$ the j is equal to $[1 + k_0^2(l+a)^2]^{-\frac{1}{2}}$ which is already very different from (18a) and shows that the asymptotic behavior does not extend to $k = k_0$. This was, of course, to be expected, since the potential $\hbar^2 k_0^2/2M$ for $a < r < a + l$, which is important for $k = 0$, becomes insignificant for $k \gg k_0$. The condition that the potentials be insignificant for all k, and therefore do not affect the region of validity of the asymptotic law for the cross section, is that the integral of (17a) with P instead $\bar{P}$, be not only finite but also small compared to 1. This is equivalent with the condition that the second integral P_2 of P from a to infinity be small compared with 1. In practice this is equivalent with the condition that the *WKB* integral be small.

As an example, the magnetic interaction of a neutron with a nucleus may be considered. The P is in this case

$$P \approx \frac{2M}{\hbar^2}\left(\frac{eh}{2Mc}\right)^2 \frac{\mu_1\mu_2}{r^3},$$

and the second integral of this is

$$P_2 \approx \frac{e^2\mu_1\mu_2}{4Mc^2 r}.$$

For $r = a \sim e^2/mc^2$ this is of the order of 10^{-3}, and shows that the magnetic interaction should be without appreciable effect on slow neutron processes.

For $k \gg k_0$ the κ of (18) becomes imaginary and $|e|^2$ fluctuates with a small amplitude around the value of Table I. If the other conditions of the asymptotic behavior of the cross section remain valid, there will then be at $k \gg k_0$ a second region in which the reaction cross section is proportional to k but the constant of proportionality will be different from the constant of proportionality at $k \ll k_0$. A similar second region at somewhat larger k, in which the asymptotic behavior is followed, may exist even if P goes so slowly to zero at $r = \infty$ that it changes the asymptotic behavior at very low k. Its existence depends more on the maximum of P than on its behavior at large r, and its appearance can be judged from the decrease of the integral in (17a) to a very low value at such a low k that the other conditions for the validity of the asymptotic behavior still hold.