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Induction Effects in Terrestrial Magnetism

Part III. Electric Modes

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It can be shown that the currents in the earth's core which give rise to the externally observable magnetic field do not form a complete set of solutions of the field equations. There exists a second set of solutions composed of the modes of the electric type which produce a magnetic field inside the metallic sphere, but appear at the outside only through an electric field too weak to be measured. For reasons of symmetry the most important terms among the electric modes are the quadrupoles. The theory of inductive coupling by fluid motion, developed previously, is here applied to the interaction of the magnetic and electric modes. The system again is non-conservative, and work is done on the field by the fluid, or *vice versa*. It is shown that the interaction between the magnetic dipole and electric quadrupole modes constitutes a basic amplifier mechanism which amplifies the quadrupole mode until the magneto-mechanical forces exerted by the field upon the fluid begin to slow down the motion, thus prohibiting further increase of the field. This internal quadrupole field is likely to be much larger than the ordinary magnetic dipole field. Further analysis leads one to interpret the couplings between the magnetic and electric modes as a feed-back amplifier whereby the field can be maintained through the power delivered to it by the fluid motion. A survey of possible sources of power for this process indicates that the power for the maintenance of the field is provided from the rotational energy lost by the earth as it is slowed down through the action of the lunar tide.

INTRODUCTION

THE analysis of Part I[1] and Part II[2] has led to an interpretation of the geomagnetic secular variation in terms of interactions between fluid motions in the earth's metallic core and electric currents in the core that are the sources of the magnetic field. This analysis suffers from the shortcoming that the current modes which give rise to a magnetic field outside the metallic sphere do not represent a complete set of solutions of the electromagnetic field equations. There exists a second set of solutions, representing modes of the electric type, whose magnetic field is confined to the interior of the conducting sphere. In the preceding parts these modes have been disregarded on the assumption that they cannot be excited. It has been found, however, that in the theory of inductive coupling by fluid motion there appear definite couplings between the two types of modes and that, therefore, the electric modes are an integral part of the field as described by this theory. It will appear in the course of this paper that from this viewpoint inductive coupling between the magnetic and electric modes is by far the most important feature of the earth's magnetic field.

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[1] W. M. Elsasser, Phys. Rev. **69**, 106 (1946), designated as Part I in the text.

[2] W. M. Elsasser, Phys. Rev. **70**, 202 (1946), designated as Part II in the text.

FREE ELECTRIC MODES

It is a known fact[3] that a sphere of macroscopic dimensions and of a conductivity of the metallic order has four distinct sets of electromagnetic modes of free oscillations. There are two high frequency modes, oscillations that correspond to wave-lengths comparable to the radius of the sphere. These modes are not of interest to us here. There are two low frequency modes which, in the case of a metallic sphere of macroscopic size, are completely aperiodic. The free aperiodic modes of the magnetic type have been extensively dealt with in Part I. The present account of the aperiodic modes of the electric type can therefore be brief; the procedure and symbols follow those of Part I. We first give a treatment in which the displacement current is omitted from the outset as has been done in the case of the magnetic modes. This method is extremely simple, but leads to an apparent inconsistency. The essential correctness of the result will then be demonstrated by deriving them from the more general solutions that include the displacement current.

[3] P. Debye, Ann. d. Physik **30**, 57 (1909).

The relation of the magnetic and electric modes may be described by saying that the symmetry types of the field vectors are interchanged. If again **T** and **S** designate the toroidal and poloidal vector fields defined in Part I, we have

$$\text{magnetic modes:}\quad \mathbf{B}\sim\mathbf{S},\quad \mathbf{E}\sim\mathbf{T}$$
$$\text{electric modes:}\quad \mathbf{B}\sim\mathbf{T},\quad \mathbf{E}\sim\mathbf{S}.$$

In analogy to (I, 26) we set now for the vector potential of any individual free mode, inside the sphere

$$\mathbf{A}^{(i)}=c\mathbf{S}e^{-\Lambda t}, \tag{1}$$

where all the pertinent relations of Part I, namely, (I, 8), (I, 11), (I, 24), (I, 27) remain unchanged. We quote the last-named

$$\Lambda=k^2/\mu\sigma. \tag{2}$$

We then find readily for the field vectors in the interior:

$$\mathbf{B}^{(i)}=cRk^2\mathbf{T}e^{-\Lambda t},\quad \mathbf{E}^{(i)}=c\Lambda\mathbf{S}e^{-\Lambda t}. \tag{3}$$

The field equations in the space external to the sphere are fulfilled if we set

$$\mathbf{B}^{(e)}=0,\quad \mathbf{E}^{(e)}=C\Lambda R^n\nabla(r^{-n-1}Y_n{}^m)e^{-\Lambda t}. \tag{4}$$

The boundary conditions are the same as in the case of the magnetic modes. They readily yield the relation

$$J_{n+\frac{1}{2}}(k_{ns}R)=0, \tag{5}$$

which is to be compared to the condition (I, 16) for the magnetic modes.

Again we wish to normalize these modes in analogy to the normalization of the magnetic modes which was given by (I, 21). Hence, we require

$$\int \mathbf{S}\cdot\mathbf{S}^*dV=1,$$

the integral extending over the interior of the sphere. The calculation shows[4] that the normalization of the spherical harmonics (I, 22) may be left the same, but the normalized radial functions are now, in place of (I, 23),

$$Z_n(k_sr)=2^{\frac{1}{2}}(2n+1)^{\frac{1}{2}}R^{-2}[(n+1)J_{n-\frac{1}{2}}(k_sR)$$
$$+nJ_{n+3/2}(k_sR)]^{-\frac{1}{2}}r^{-\frac{1}{2}}J_{n+\frac{1}{2}}(k_sr). \tag{7}$$

[4] By (I, 18); see also J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Section 7.13.

Since the tangential component of **E** is continuous, we can readily estimate the magnitude of the external electric multipole field given by the second Eq. (4). The electrostatic potential, U, of this field is of the order

$$U\approx RE\approx RC\Lambda\approx Rc\Lambda\approx B/\mu\sigma.$$

Taking the magnetic field to be 1 gauss $=10^{-4}$ m.k.s. units, we find U to be of the order of 10^{-4} volt. Such a field cannot, of course, be detected by electrostatic measurements at the surface of the earth.

The preceding derivation of the electric modes suffers from a flaw that might be stated as follows. Since the radial component of **E** vanishes when the boundary is approached from the inside, there must be a charge on the surface of the sphere (at least for any finite assumed value of the dielectric constant on the inside). As this charge must decay exponentially in time, there should be a current to or from the surface, in contradiction to the boundary condition for **E** just stated. Closer scrutiny indicates that this current is small of the second order in the neglected displacement-current term. To show this, we shall derive the aperiodic modes from the more general case of the oscillatory modes. The characteristic equation for the electric modes is[5] (assuming μ constant throughout)

$$\psi_n'(k_iR)/\psi_n(k_iR)=k_i\zeta_n'(k_eR)/k_e\zeta_n(k_eR), \tag{8}$$

where

$$\psi_n(x)=x^{\frac{1}{2}}J_{n+\frac{1}{2}}(x),\quad \zeta_n(x)=x^{\frac{1}{2}}H_{n+\frac{1}{2}}{}^{(1)}(x).$$

$H^{(1)}$ being the Hankel function of the first kind. The propagation constants, k_i and k_e, have the usual meaning and refer to the internal and external space, respectively. Both are given functions of the frequency. If now the right-hand side of (8) is reduced to the first term of a power series in k_eR, one obtains

$$\psi_n(k_iR)=-(Rk_e{}^2/nk_i)\psi_n'(k_iR)$$

which for small values of k_e goes over into $\psi_n(k_iR)=0$, identical with (5). If the roots of this equation are designated $(k_iR)_0$ we have in the next approximation

$$k_iR=(k_iR)_0(1-k_e{}^2/nk_i{}^2)_0.$$

[5] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), Section 9.22.

The second term in the parenthesis now gives rise to a small current normal to the boundary.

INDUCTIVE COUPLINGS

For the sake of completeness we start again from the field equations

$$\nabla \times \mathbf{B} = \mu\sigma\mathbf{E} + \mu\sigma\mathbf{v} \times \mathbf{B}, \tag{9}$$

$$\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = 0, \quad \nabla \cdot \mathbf{B} = 0. \tag{10}$$

We shall moreover require the energy integral. Apply to (9) and (10) the conventional procedure of deriving Poynting's theorem and neglect the surface integral of the Poynting vector when the surface over which the latter is taken tends to infinity. After some simple transformations we get

$$\tfrac{1}{2}(\partial/\partial t)\int \mathbf{B}^2 dV = \int \mathbf{v} \cdot (\mathbf{B} \times (\nabla \times \mathbf{B}))dV - (\mu\sigma)^{-1}\int (\nabla \times \mathbf{B})^2 dV, \tag{11}$$

where the integral on the left extends over all space, those on the right over the sphere only. The last term is proportional to the square of the *total* current as given by the right-hand side of (9), and represents the rate at which heat is generated. The first term on the right-hand side of (11) represents the change in the field energy produced by the motional induction. As this term does not, in general, vanish, the system is non-conservative; work is being done by the fluid on the field, or *vice versa.*

As we are primarily interested in the motional induction we shall assume $\mathbf{v}$ and σ so large that the free decay terms (and diffusion effects treated in Part II) become negligible. The transition to this limit does not involve any mathematical or physical difficulties. Equation (9) can then be written in the simple form

$$\partial\mathbf{A}/\partial t = \mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\nabla \times \mathbf{A}). \tag{12}$$

Following the developments of earlier parts, put

$$\mathbf{A} = \textstyle\sum_\gamma c_\gamma \mathbf{A}_\gamma, \quad \mathbf{v} = \sum_\alpha v_\alpha \mathbf{V}_\alpha \tag{13}$$

where the vectors with indices are of the orthogonal set, and where, moreover, the summation extends over both types of vectors, $\mathbf{T}$ and $\mathbf{S}$. From now on, we shall use *real* vectors in place of the complex vectors of the former papers; the coefficients in (13) are then real, and the normalizations (6) and (I, 21) are replaced by real conditions (increasing the normalization factor (I, 22) by $\sqrt{2}$, if $m \neq 0$).

Equation (12) becomes now

$$\dot{c}_\gamma = \sum_{\alpha\beta} v_\alpha c_\beta [\mathbf{V}_\alpha \cdot \mathbf{B}_\beta \times \mathbf{A}_\gamma], \tag{14}$$

which agrees with the former relations (I, 34) or (II, 10) except for the fact that the summation is now understood to extend over both types of modes. The expression (11) for the magnetic energy can be put in an analogous form. For any individual mode we have

$$\nabla \times \mathbf{B}_\gamma = \nabla \times \nabla \times \mathbf{A}_\gamma = k_\gamma{}^2 \mathbf{A}_\gamma$$

and hence (11) may be written, on neglecting the last integral,

$$dE_{\text{mag}}/dt = \sum_{\alpha\beta\gamma} v_\alpha c_\beta c_\gamma k_\gamma{}^2 [\mathbf{V}_\alpha \cdot \mathbf{B}_\beta \times \mathbf{A}_\gamma]. \tag{15}$$

On substituting from (14) and integrating, we find

$$E_{\text{mag}} = \tfrac{1}{2} \textstyle\sum_\gamma k_\gamma{}^2 c_\gamma{}^2. \tag{16}$$

Again, let $\mathbf{F}$ be the density of the magneto-mechanical forces exerted by the field upon the fluid. Let $\mathbf{F}_\alpha$ be an orthogonal vector set; then

$$\mathbf{F} = \mathbf{J} \times \mathbf{B} = -\sigma(\partial\mathbf{A}/\partial t) \times \mathbf{B} = -\mu^{-1} \sum_{\alpha\beta\gamma} c_\beta c_\gamma k_\gamma{}^2 [\mathbf{F}_\alpha \cdot \mathbf{B}_\beta \times \mathbf{A}_\gamma] \mathbf{F}_\alpha \tag{17}$$

the last equality by (1) and (2).

We now proceed to classify all the matrix elements appearing in (14) according to the types of vectors, $\mathbf{T}$ or $\mathbf{S}$, involved. In the "primary" mode (index β) the magnetic field itself enters, in the "secondary" mode (index γ), the vector potential appears instead. We may distinguish the following types:

Interactions among magnetic modes:

$$[\mathbf{T}_\alpha \cdot \mathbf{S}_\beta \times \mathbf{T}_\gamma], \quad [\mathbf{S}_\alpha \cdot \mathbf{S}_\beta \times \mathbf{T}_\gamma]. \tag{18a, b}$$

Interactions among electric modes:

$$[\mathbf{T}_\alpha \cdot \mathbf{T}_\beta \times \mathbf{S}_\gamma], \quad [\mathbf{S}_\alpha \cdot \mathbf{T}_\beta \times \mathbf{S}_\gamma]. \tag{19a, b}$$

Magnetic mode primary, electric mode secondary:

$$[\mathbf{T}_\alpha \cdot \mathbf{S}_\beta \times \mathbf{S}_\gamma], \quad [\mathbf{S}_\alpha \cdot \mathbf{S}_\beta \times \mathbf{S}_\gamma]. \tag{20a, b}$$

Electric mode primary, magnetic mode secondary:

$$[\mathbf{S}_\alpha \cdot \mathbf{T}_\beta \times \mathbf{T}_\gamma]. \tag{21}$$

In the case (21) there is only one type of interaction, since all matrix elements containing three toroidal vectors vanish. All the preceding matrix elements differ from the two types (18) only by permutations, with the exception of (20b). The types (18) have been described in the preceding parts and a table of all the elements containing only dipole and quadrupole vectors has been given.[6] The type (20b) is new; a table of the lowest elements will be found in the appendix. This type appears to be of minor physical significance, and we shall not encounter it later on. From the tables, or the analytical expressions given in the appendix, the values of *all* matrix elements (18)–(21) can be obtained either directly or by suitable permutations of the vectors or their indices.

Next, we try to simplify Eq. (14). We begin by imposing two restrictions on the fluid motion, namely, assuming rotational symmetry about the earth's axis and symmetry about the equatorial plane. The first condition restricts us to zonal vectors ($m=0$); this will later be somewhat relaxed. The second condition requires that the poloidal flow components, $\mathbf{S}_n$, must have even n, the toroidal flow components, $\mathbf{T}_n$, must have odd n. This is readily verified from the expressions (I, 13) and (I, 14) for these vectors. In Part I we derived certain selection rules for the matrix elements; from the formulas of the appendix these can now be generalized to apply to all the elements (18)–(21): For the elements (18b), (19b), and (20a), containing two vectors $\mathbf{S}$ and one vector $\mathbf{T}$, the sum of the three indices n must be even; for the remaining elements containing either one vector $\mathbf{S}$ and two vectors $\mathbf{T}$, or else three vectors $\mathbf{S}$, the sum of the three indices n must be odd. One finds now that under the symmetry assumptions made for the fluid motion, the entire set of electromagnetic modes is *reducible* into two sets that are not coupled to each other. One of these, the symmetrical set, has electric currents or vector potentials which have the same symmetry as the fluid motion, i.e., for $\mathbf{A}=\mathbf{S}_n$ the index n must be even, for $\mathbf{A}=\mathbf{T}_n$ it must be odd. The other, antisymmetrical set, is complementary to the symmetrical set; the electric currents or vector potentials are antisymmetrical about the equatorial plane. In order to prove this we need only insert the corresponding vectors into the matrix elements (18)–(21); it then appears that the couplings between the two sets vanish by virtue of the selection rules just enunciated. Since these selection rules do not depend on the magnitude of the tesseral index, m, we can at once generalize the result so that it applies to tesseral harmonics as well: Let the symmetrical set of vectors be composed of the $\mathbf{S}_n{}^m$ where n is even and of the $\mathbf{T}_n{}^m$ where n is odd; the antisymmetrical set is complementary to this. Then, when the fluid motion can be described as a linear aggregate of vectors of the symmetrical set, the electromagnetic field is reducible; modes whose electric currents (or vector potentials) are of the symmetrical set are not coupled to modes whose electric current are of the antisymmetrical set.

It is now furthermore found from (17) that the magneto-mechanical forces engendered by any one set of modes alone are vectors of the symmetrical set; the forces engendered by the interaction of a symmetrical and an antisymmetrical mode are vectors of the antisymmetrical set. It is clear that the primary mechanical forces acting upon the fluid, at least so long as they have rotational symmetry, must belong to the symmetrical set. This is evident for any forces caused by pressure differences between the polar and equatorial regions. The Coriolis force is of the form $\boldsymbol{\omega}\times\mathbf{v}$, where $\boldsymbol{\omega}$ is a vector of type $\mathbf{S}_1$ derived by (I, 14) from a generating function $\psi=(r/2R)\omega\cdot\cos\vartheta$. The development of $\boldsymbol{\omega}\times\mathbf{v}$ in terms of the fundamental vectors is then seen to yield only vector components of the symmetrical set. Thus we see that the forces, velocities and electric currents pertaining to the symmetrical set form a self-consistent system. This is not the case when the electric current modes are of the antisymmetrical set, since the resultant magneto-mechanical forces are symmetrical. Observation shows that the magnetic dipole modes, which are symmetrical vectors, are large, while the quadrupoles, which are antisymmetrical vectors, are very small. It will henceforth be assumed that the symmetrical set of electric

[6] Reference 2, p. 207.

current modes is the one that preponderates in the earth. This leads at once to the conclusion that for the modes of the electric type the quadrupoles must be large and the dipoles and octupoles small.

On relinquishing our original requirement of rotational symmetry for the field vectors, it will be appropriate to retain the assumption that the deviations from this symmetry are small. The simplest such case is that of an axis of rotation slightly inclined relative to the earth's axis. Let ϵ, ψ be the polar angles of the new axis in the old system and let ϑ' be measured from the new axis. From the addition theorem of the spherical harmonics we get, up to terms quadratic in ϵ,

$$P_n(\cos\vartheta') = (1-\tfrac{1}{2}\epsilon^2)P_n(\cos\vartheta) + \epsilon P_n^1(\cos\vartheta)\cos(\varphi-\psi) + (\epsilon^2/4)P_n^2(\cos\vartheta)\cos 2(\varphi-\psi) + \cdots, \quad (22)$$

where the P's are the ordinary (unnormalized) Legendre polynomials. From this formula the relations for the corresponding vectors containing normalized spherical harmonics are readily derived. In what follows, we shall confine ourselves to first-order terms in ϵ, that is to tesseral harmonics with $m=1$.

In order to achieve further simplification of Eq. (14) we shall now limit ourselves to the dipoles of the magnetic modes and the quadrupoles of the electric modes. We have, therefore, the following six vectors for the electric currents or vector potentials of the large modes

$$\mathbf{T}_1, \mathbf{T}_1^c, \mathbf{T}_1^s; \quad \mathbf{S}_2, \mathbf{S}_2^c, \mathbf{S}_2^s, \quad (23)$$

where the upper indices c and s represent the functions $\cos\vartheta$ and $\sin\vartheta$, respectively. Equation (14) reduces now to six sets (where the members of each set are distinguished only by the index of the radial eigenfunction). In place of the ordinary differential Eq. (14) we could form a set of *partial* differential equations for the two independent variables r and t; under the restriction (23) these would reduce to six simultaneous partial differential equations (or four simultaneous complex equations).

The assumption of rapid convergence of the spherical harmonic series is, of course, justified by the fact that the higher harmonic components are quickly damped out by free decay. There is no similar restriction on the components of the fluid motion. We can, however, limit ourselves to the vectors

$$\mathbf{T}_1, \mathbf{T}_1^c, \mathbf{T}_1^s; \quad \mathbf{S}_2, \mathbf{S}_2^c, \mathbf{S}_2^s; \quad \mathbf{T}_3, \mathbf{T}_3^c, \mathbf{T}_3^s \quad (24)$$

as it follows from the selection rules of Part I that, under the restriction (23) for the electric current vectors, only the velocity vectors (24) give rise to non-vanishing matrix elements.

AMPLIFICATION

We shall now study the individual interaction terms that involve vectors of the types (23) for the electric currents and (24) for the velocities. First, consider matrix elements that have full rotational symmetry. There are only four types of these

$$[\mathbf{S}_2 \cdot \mathbf{S}_1 \times \mathbf{T}_1], \quad [\mathbf{S}_2 \cdot \mathbf{T}_2 \times \mathbf{S}_2], \quad (25\text{a, b})$$

$$[\mathbf{T}_1 \cdot \mathbf{S}_1 \times \mathbf{S}_2], \quad [\mathbf{T}_3 \cdot \mathbf{S}_1 \times \mathbf{S}_2], \quad (26\text{a, b})$$

each element standing for an infinite sequence distinguished by the indices of the radial eigenfunctions. The elements (25a) represent interactions among the magnetic dipole components which have been discussed at some length in Part II. The elements (25b) represent interactions among the electric quadrupole components of a closely similar character. Our attention will be centered on the elements (26) which represent an induction from the magnetic dipole as primary to the electric quadrupole as secondary. These four types of elements exhaust the couplings of rotational symmetry among the rotationally symmetrical magnetic dipole and electric quadrupole modes. *There is, therefore, no reverse to the interactions (26).* This statement holds more generally in the sense that there are interactions of rotational symmetry between magnetic modes as primaries and electric modes as secondaries, but no interactions of rotational symmetry in the reverse direction. The proof can readily be deduced from the expressions of the matrix elements in the appendix.

For simplicity, let now the series of matrix elements resulting from the different radial eigenfunctions be represented symbolically by a single element; then Eq. (14) reduce to

$$dc_2/dt = [\mathbf{T}_1 \cdot \mathbf{S}_1 \times \mathbf{S}_2]v_1c_1 + [\mathbf{T}_3 \cdot \mathbf{S}_1 \times \mathbf{S}_2]v_3c_1. \quad (27)$$

If we assume that $\mathbf{v}_1$ and $\mathbf{v}_3$ are constants, and that the primary field is constant, the amplitude of the electric quadrupole increases linearly with time. The process (27) constitutes a basic mechanism of amplification. In somewhat more physical terms, this process might be described as follows: The primary dipole field, $\mathbf{S}_1$, is in the meridional plane, the velocity, $\mathbf{T}_1$ or $\mathbf{T}_3$, is perpendicular to this plane. The induced secondary current, being the vector product of the two, must again be in the meridional plane. The lines of magnetic force of the quadrupole thus generated are circles about the earth's axis. Hence, the secondary magnetic field is parallel to the velocity and there is no interaction between them which would produce a tertiary field. This is the reason why, in the presence of only the fields and velocities appearing in (27), Eq. (14) is *reducible* to the latter; the process of amplification of the quadrupole mode does not necessitate the simultaneous generation of other modes.

Now Eq. (27) is symbolical in the sense that in reality there is an infinite sequence of matrix elements corresponding to the different radial eigenfunctions of the primary as well as the secondary mode. But it is seen from the preceding geometrical consideration that on any circle $r=\text{const.}$, $\vartheta=\text{const.}$ the field of the secondary mode increases linearly with time, although the rate of increase will, in general, be a function of r and ϑ. This linear increase must continue until the magneto-mechanical forces exerted by the field upon the fluid become so large that they decelerate the fluid motion, thereby prohibiting further amplification. Equation (15) gives the amount of power delivered by the fluid motion to the field in the process of amplification. By (17) there corresponds to each matrix element a force component whose direction is everywhere opposite to the corresponding velocity component. In Part II the critical field strength for which the magneto-mechanical forces become equal in magnitude to the purely mechanical forces has been estimated. In view of (17), the last equation of Part II (preceding the appendix) may be modified to read

$$(B_1B_2)^{\frac{1}{2}} \sim (2\omega\rho/\sigma)^{\frac{1}{2}} = 12 \text{ gauss}, \qquad (28)$$

where B_1 and B_2 designate the primary and secondary field, respectively. For larger fields than given by (28) the magneto-mechanical reactions preponderate, for smaller fields, the purely mechanical driving forces. B_1 in the upper layers of the core is about 2–3 gauss which would make B_2 about 50–60 gauss. The force exerted upon the velocity components $\mathbf{T}_1$ and $\mathbf{T}_3$ is, however, not the largest force exerted by the electromagnetic field. It results from the interaction of the secondary current with the primary field; a larger force results from the interaction of the secondary current with the secondary field. The latter force component is of the type $\mathbf{S}_2$. Whether or not this force influences the fluid motions in such a way that it eventually tends to decrease the motion of the types $\mathbf{T}_1$ and $\mathbf{T}_3$ indirectly, could only be decided by a more extensive analysis. In order to fix our ideas we shall assume a specific figure for the mean value of the electric quadrupole field in the core, namely, 30 gauss. This is presumably fairly close to a lower limit for reasons which will appear later.

The effects just mentioned must have a rather profound influence upon the hydrodynamics of the motions in the core. Since the magneto-mechanical reactions are large in certain directions, the problem is different from one of ordinary hydrodynamics. Instead, we are dealing with a system where the coupling between the fluid motion and the magnetic field is extremely close and their mutual interactions cannot be neglected. It is interesting to note that, in the case of sunspots, Alfvén[7] has arrived at closely analogous conclusions. He finds that in sunspots the magneto-mechanical forces must be greatly in excess of all mechanical forces; hence the motion must be controlled by the former. The ideas of Gurevich and Lebedinsky[8] on sunspots are in substantial agreement with this view.

FEEDBACK

So far, we have dealt with interactions of rotational symmetry and we have seen that there is no feed-back mechanism whereby the large field of the quadrupole mode could in turn serve as the primary for amplification of the dipole. In order to obtain feedback in this sense we

[7] H. Alfvén, M.N.R.A.S. **105**, 3, 383 (1945).

[8] L. Gurevich and A. Lebedinsky, J. Phys. U.S.S.R. **10**, 327, 425 (1946).

must make use of interactions without rotational symmetry.

1. There is a vast number of matrix elements involving tesseral harmonics, and we shall not try to undertake their classification and discussion. A simple relation between zonal and tesseral harmonics is obtained by a tilt of the rotational axis, according to (22). If in the amplification process (27) the fluid motion takes place about an axis inclined relative to that of the earth, as is apparently the case in the observed field, Eq. (27) will be transformed so that it involves elements of type

$$[\mathbf{T}_1{}^c\cdot\mathbf{S}_1\times\mathbf{S}_2{}^c],\quad [\mathbf{T}_3{}^c\cdot\mathbf{S}_1\times\mathbf{S}_2{}^c],\qquad (29)$$

and two others where the upper indices c are replaced by s.

The elements that provide feedback are of type (21). Limiting the permissible vectors to those indicated before, they are

$$[\mathbf{S}_2\cdot\mathbf{T}_2{}^c\times\mathbf{T}_1{}^s],\quad [\mathbf{S}_2{}^c\cdot\mathbf{T}_2\times\mathbf{T}_1{}^s],\qquad (30\text{a, b})$$

$$[\mathbf{S}_2{}^c\cdot\mathbf{T}_2{}^s\times\mathbf{T}_1],\qquad (31)$$

and three others obtained by interchanging the upper indices. (Note the different selection rule for the upper indices as compared to (29).) Only the elements of type (31) give a component of the secondary magnetic dipole along the earth's axis. It follows that both the fluid motion $\mathbf{S}_2$ and the quadrupole field $\mathbf{T}_2$ must be inclined relative to the earth's axis. The same should then be the case for the velocity components, $\mathbf{T}_1$ and $\mathbf{T}_3$, producing the quadrupole field. But the two vectors in (31) have a shift in phase (i.e. in geographical longitude) of 90°; hence by (29) there should also be such a phase shift between the fluid components of the **T** and **S** types. In order to establish a concrete picture we may imagine that this whole complicated system precesses slowly about the earth's axis, as the inclined axes can hardly remain fixed. Now estimate the angle of inclination by means of the observed inclination of the dipole field, $\epsilon=0.20$. Other things being equal, (31) is of the order ϵ^2 as compared to a rotationally symmetrical matrix element, hence quite small. An alternate feed-back mechanism derived from (30) is of the same order, and other feed-back mechanisms involving vectors that we have neglected here are of the order ϵ^3 or smaller. This type of feed-back mechanism is so complicated and artificial that it would hardly seem convincing.

2. *A more satisfactory feed-back mechanism* can be developed by taking into account the effect of turbulence upon the distribution of the magnetic field. It follows from general hydrodynamical principles that fluid motion in the core must be turbulent. From the viewpoint of the present theory the irregular features on any map of the magnetic field or its secular variation may be interpreted as the result of turbulent motion.

We shall first prove two useful theorems. Consider a surface of arbitrary shape, bounded by a contour C, located entirely inside the conducting fluid. We then get[9] from (9) and (10), on integration

$$(\partial/\partial t)\int \mathbf{B}_n dS=-\int \mathbf{E}\cdot d\mathbf{C}$$

$$=\int(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{C}-(\mu\sigma)^{-1}\int(\nabla\times\mathbf{B})\cdot d\mathbf{C}.$$

Now, if the first integrand on the right-hand side is written $\mathbf{B}\cdot(d\mathbf{C}\times\mathbf{v})$, the integral can be given a simple geometrical meaning: It becomes $-\int\mathbf{B}_n dS$ where the integration extends over the strip that the contour C subtends in its motion during the time dt. Since $\int\mathbf{B}_n dS=0$ for any closed surface, we find that

$$(d/dt)\int\mathbf{B}_n dS=(d/dt)\int\mathbf{A}\cdot d\mathbf{C}=O(\sigma^{-1})\qquad (32)$$

where the term on the right-hand side tends to zero as the free decay effects are neglected. The operator (d/dt) has the usual significance, giving the rate of change when the surface and contour move with the fluid. By means of (32) we can understand why certain simple theories of inductive amplification do not succeed. A theory of such a type was proposed for the sun by Larmor.[10] Apply (32) to a circle $r=$const. Any rotationally symmetrical motion transforms such

[9] This theorem is due to T. G. Cowling who communicated it to us some months ago. The writer is particularly indebted for Dr. Cowling's generous permission to reproduce it here and to use the results in the context of this paper.

[10] J. Larmor, Brit. Assoc. Adv. Sci., Bournemouth Meeting, 1919, p. 159 (1920).

a circle into a family of similar circles, and eventually the circle will return arbitrarily close to its initial position. Now (32) asserts that, neglecting free decay, the flux threaded by the circle remains constant. This flux is produced by the modes of magnetic type, as may readily be verified. On admitting that free decay can, in the mean, only decrease the magnetic flux it follows that the magnetic modes cannot be amplified by a stationary flow of rotational symmetry. This is an alternate proof[9] of Cowling's earlier theorem.[11]

A second integral theorem is obtained by extending the integration over an arbitrary volume instead of over a surface. This gives, by (9) and (10)

$$(\partial/\partial t)\int \mathbf{B}dV = -\int \nabla\times\mathbf{E}dV = -\int \mathbf{n}\times\mathbf{E}dS$$

$$= \int \mathbf{n}\times(\mathbf{v}\times\mathbf{B})dS - (\mu\sigma)^{-1}\int \mathbf{n}\times(\nabla\times\mathbf{B})dS$$

$$= \int \mathbf{v}B_n dS - \int \mathbf{B}v_n dS - O(\sigma^{-1}), \tag{33}$$

where dS is an element of the surface bounding the volume and $\mathbf{n}$ the normal to this surface, pointing outwards. Equation (33) can be rewritten as

$$(d/dt)\int \mathbf{B}dV = \int \mathbf{v}B_n dS - O(\sigma^{-1}), \tag{34}$$

where we shall again neglect the second term on the right-hand side.

This relation may now be used to determine what happens to the field of a fluid particle during the motion of the latter. Consider, in turn, the three basic forms of displacement: translation, rotation, and deformation. For a pure translation, $\mathbf{v}$ can be taken out of the surface integral in (34) and the latter vanishes. Hence, if the fluid particle is displaced parallel to itself, it carries the field with it unchanged, both in magnitude and direction. Next, let the particle rotate about an arbitrary axis, $\boldsymbol{\omega}$ being the angular velocity and $\mathbf{r}$ the distance from a point on this axis. Then (34) gives

$$(d/dt)\int \mathbf{B}dV = \boldsymbol{\omega}\times\int \mathbf{r}B_n dS.$$

Now apply the vectorial identity

$$\int \mathbf{r}B_n dS = \int \mathbf{r}(\nabla\cdot\mathbf{B})dV + \int (\mathbf{B}\cdot\nabla)\mathbf{r}dV.$$

Using the particular properties of $\mathbf{r}$ and $\mathbf{B}$, we find that the first integrand vanishes and the second reduces to $\mathbf{B}$. Hence

$$(d/dt)\int \mathbf{B}dV = \boldsymbol{\omega}\times\int \mathbf{B}dV \tag{35}$$

which shows that the magnetic field rotates together with the fluid particle.

In order to study deformation, let the volume be a cylinder with axis parallel to $\mathbf{B}$, bounded by two plane surface of area S perpendicular to the axis. Then, by (34)

$$(dt/dt)\int \mathbf{B}dV = (v_1+v_2)B\cdot S, \tag{36}$$

where v_1 and v_2 are the outward velocities across the end faces of the cylinder. Relation (36) is the integral theorem corresponding to the differential equation (II, 6) and leads again to the results derived in Part II: Convergence of the motion in a plane perpendicular to the field produces amplification, divergence of the motion in this plane produces de-amplification.

It is a well-known fact that, with respect to a *scalar* property of the fluid, turbulence acts like a greatly enhanced diffusion. Now we know that, in the absence of motion, the field equation (II, 3) for $\mathbf{B}$ reduces to an equation of diffusion for the *vectorial* property $\mathbf{B}$, describing the phenomena of free decay, skin effect, etc., with a coefficient of diffusion equal to $(\mu\sigma)^{-1}$. From the preceding results we can infer that turbulent motion of the fluid acts so as to accelerate the free decay to a value that corresponds to the increased rate of diffusion measured by the coefficient of turbulent mixing. The formal analogy would seem to be as complete as in the scalar case. We shall not dwell here upon the mathematical aspects. The periods of free decay,

[11] T. G. Cowling, M.N.R.A.S. 94, 39 (1934).

formerly found of the order of 20–50,000 years, may now be much smaller, of the order of the more rapid periods of the secular variation, say a thousand years or slightly less.

If the elements of turbulent flow rotate, the magnetic field vector rotates with them. For the electric quadrupole mode, **B** is perpendicular to the meridional plane. By rotating the fluid particles we get at once a component of **B** in this plane. In order that the *average* component in this plane may be finite, it is clearly necessary that the axes of the turbulent rotations be not entirely random in direction, but have some correlation either with the axis of the earth, or with the axis of the magnetic field. In order to visualize such a correlation in a very crude way, we may simply assume that a rotational component in the same sense as the earth's rotation is more probable than a rotational component in the opposite sense (or *vice versa*). If the magnetic field is small, the motion is controlled by the Coriolis forces, if it is large, the motion is controlled by the magneto-mechanical forces. It is therefore quite likely that the turbulent motion exhibits a correlation with the axis of the earth or with the axis of the electric quadrupole field, as the case may be. These qualitative considerations do not determine the sign of the correlation coefficient which must be assumed such that, in the mean, regenerative feedback results. We can now attribute the observed rapid change of the magnetic dipole terms, referred to above, to a turbulent feed-back mechanism of this kind rather than to a mean motion of the fluid. Since the observed dipole moment decreases, it appears that feedback is degenerative at the present time, at least in the surface layers. If the field maintains itself in the long run, the average feedback must, of course, be regenerative.

This concept of turbulent feedback can be expressed more precisely as follows. The rate of change of the dipole field is

$$dc_1/dt = [\mathbf{v} \times \mathbf{B} \cdot \mathbf{T}_1]. \qquad (37)$$

$\mathbf{T}_1$ has only a φ-component which may be written $a(r)\sin\vartheta$ where $a(r)$ is normalized. Hence

$$dc_1/dt = \int (\mathbf{v} \times \mathbf{B})_\varphi a(r) \sin\vartheta dV.$$

This form of the equations of motion brings out more clearly the significance of the matrix elements as coefficients of correlation for the turbulent motion and field. If **v** and **B** in (37) are developed in series of the orthogonal vectors, the right-hand side becomes a linear combination of matrix elements, as in (14), the first term being precisely the element (31). But there will now be an entire series of such elements involving higher harmonics. It should be pointed out that this feed-back effect, like any other, is a second-order effect in the sense of perturbation theory, but on purely physical grounds it would appear far more likely than the one sketched in the beginning of this section.

There is a lower limit to the size of the turbulent elements which can effectively contribute to changes of the field. By (I, 30) the lowest mode of free decay of a spherical particle is $\mu\sigma R^2/\pi^2$, or numerically, $0.13R^2$ when R is in meters. According to the evidence from the secular variation, significant changes of the field take place in periods of the order of a thousand years $= 3\cdot 10^{10}$ sec. Equating to the decay period just mentioned, this gives $R = 500$ km. For turbulent elements of smaller size we can expect the effects of free decay to outweigh the turbulence so that the field is smoothed out before the correlation required for feedback becomes established. The large turbulent vortices, evidenced by the major traits of the secular variation, are several thousand km in diameter, and they will be quite effective in producing the phenomenon of feedback.

We are now in a position to estimate the magnitude of the electric quadrupole field. A statistical mechanism of feedback, as described, must involve a large amount of cancellation of random components and cannot be highly efficient. The magnitude of the dipole field in the upper strata of the core is 2–3 gauss; a value of 30 gauss for the quadrupole field would then keep the rate of feedback below 10 percent. The quadrupole field might of course be larger than this value. We shall, later on, derive an upper limit for the field.

Before leaving this topic, we might point to the unusual degree of statistical fluctuation that can be expected in a coupled magneto-mechanical system of this kind. Not all the fluctuations are rooted in properties of the fluid motion. The theory of motional inductance might be said to be the theory of the differential Eq. (12). If one asks for the possible eigenvalues and eigenfunctions of this equation (for stationary $\mathbf{v}$) he will assume solutions of the form $A(r, \vartheta, \varphi) \exp(\gamma t)$, where γ is a complex constant. We know of one case where such a solution exists, and γ is purely imaginary. This occurs when the sphere rotates as if solid, and the field rotates with the sphere. But this is a trivial solution. The studies of the present writer have given little support to the hope that eigenvalues exist for more general types of stationary fluid motion. The operator appearing on the right-hand side of (12) is in general *skew*, being neither purely symmetrical nor purely antisymmetrical (see Part I). If a set of eigenvalues exist, they must therefore in general be complex numbers. The solution given by the amplifier mechanism (27) is not in the nature of an eigenfunction, since the primary field remains unchanged. If the presumption of the writer, namely, that the operator in (12) does not, in general, have eigenvalues, is correct, then one would expect fluctuations of the field of macroscopic magnitude, even if a system of feed-back amplification could be constructed on the basis of a stationary motion of the fluid. It seems questionable, however, whether the mechanism of feedback by turbulent correlation can be expressed in terms of strictly stationary motion; we have not investigated this point. We need hardly dwell on the familiar dynamical instability of fluid motion in large dimensions, giving rise to large-scale turbulence. One can therefore not be too surprised at the extraordinary amount of fluctuation, both in space and in time, which is exhibited by the observed field.

ENERGETICS

In the present theory the power required to maintain the field is much larger than one could have expected heretofore, from a consideration of the magnetic modes alone. An exhaustive, or even fairly complete analysis of the possible sources of power is beyond the scope of this paper. A satisfactory answer to the questions raised will no doubt be dependent upon a study of the features of the existing field and its variation. Inasmuch as the observed secular variation permits inferences about the character of the fluid motion, it might be possible to arrive at a decision as between the mechanisms outlined below.

If the mean field in the core is assumed to be 30 gauss, the energy density is 3.6 joule/m^3. This amount of energy must be supplied once during the time of free decay, say every 1000 years. Although we might expect large fluctuations in the magnitude of the field it is, in the absence of other information, perhaps best to assume that a field of this magnitude has in the mean existed throughout the geological history of the earth. If the matter in the core is taken to be mainly iron and its specific heat as $4R$ per mole (somewhat in excess of the Dulong-Petit value in order to allow for a contribution of the conductivity electrons at the high temperatures involved) the heat generated by the field is readily computed. Over a period of 5.10^8 years, comprising most of the known geological past, the rise in temperature of the core owing to this cause is 0.3 degrees.

We may now first assume that the power for the maintenance of the field is produced by thermal sources, for instance through radioactivity, inside the earth. It is possible to estimate the total heat required for this purpose. A hydrodynamical theorem states that hydrostatic equilibrium obtains when the surfaces of constant temperature coincide with the surfaces of constant mechanical (gravitational plus centrifugal) potential. Only the deviations, ΔT say, from the mean temperature of an equipotential surface can be utilized to generate motion. Since the fluid motion itself continually redistributes the heat throughout the core, ΔT can hardly amount to more than a few degrees, whatever the origin and shape of the variations. If an amount Q of heat is converted into energy of motion, the amount $QT/\Delta T$ must flow irreversibly from the regions of positive ΔT to the regions of negative ΔT, by the second law of thermodynamics. T is certainly of the order of several thousand degrees, so that $T/\Delta T$ should be at least 10^3; in practice the ratio of irreversible to reversible thermal effects is more likely to be about 10^4. Hence, the thermal sources must be of such magnitude as to be capable of heating the core by several hundred degrees in the course of $5 \cdot 10^8$ years.

Evidently this line of reasoning is not too attractive, even if it should prove possible to admit of radioactive sources of this magnitude. We shall not pursue the question further since a more plausible source of power for the fluid motions and the magnetic field can be found.

In this alternate view the power is a by-product of the change in the earth's speed of rotation caused by the lunar tide.[12] From astronomical observations

$$d\omega/dt = -2.5\cdot 10^{-22}\ \text{sec.}^{-2},$$

corresponding to a lengthening of the day by 1 second in 120,000 years. The accuracy of this figure may be estimated as about 10 percent. The angular momentum lost by the earth reappears in the orbital motion of the moon, as the latter gradually recedes from the earth. The power required for removal of the moon is small, and the existing theory[12] indicates that most of the kinetic energy of the earth's rotation is dissipated by tidal friction in the oceans. For the core alone, the average kinetic energy lost per unit volume is

$$[dE_{\text{kin}}/dt]_{\text{Av}} = (\theta/V)\omega\cdot d\omega/dt, \tag{38}$$

where V and θ are the volume and moment of inertia of the core. Numerically, $V = 1.75\cdot 10^{20}$ m^3 and $\theta = 8.4\cdot 10^{36}$ kg. m^2, obtained from the known density distribution.[13] Then (38) is equal to $2.7\cdot 10^{-2}$ joule/m^3 per year, an amount large enough to maintain an average field of 80 gauss against a mean decay period of 10^3 years. On the hypothesis that all the kinetic energy of the core instead of being dissipated by the tide, can be converted into magnetic energy, the figure may be interpreted as an upper limit for the magnitude of the internal field.

Since the retarding torque of tidal friction attacks the solid mangle of the earth, the central parts of the core will have a tendency to rotate faster than the mantle. In the stationary state there will be an increase of angular velocity with depth in the core, and the distribution of angular velocity will be such that angular momentum is carried from the inside out by frictional shear at a rate prescribed by the slowing down process. This condition of inhomogeneous rotation is a sufficient prerequisite for the functioning of the amplifier (27). Mathematically speaking, it is only one of two possible solutions. If the core is inhomogenous, as seems indicated by seismic observations,[14] the density increasing rather rapidly below a radius of $0.4R$, then the central part of the core can itself be subject to a rather large tidal deformation owing to the direct gravitational influence of the moon. If this tide slows down the central part of the core at a rate more rapid than that at which the solid mantle is slowed down by the oceanic tide, the velocity distribution would be the opposite of the one just described: the angular velocity would decrease with depth in the core and the transport of angular momentum by frictional shear would be towards the central part where the main tidal deceleration would take place. Without entering into details, it may be remarked that there are certain qualitative indications in the observed secular variation to the effect that the fluid core rotates somewhat slower than the solid body of the earth. Specifically, the local features of the magnetic field and the *foci* of the secular variation show a mean drift motion from east to west, well beyond the limits of observational errors.[15] Whether this effect can be interpreted along the lines sketched, could only be decided by an extensive study of the dynamics of these motions.

Next we can estimate the kinetic energy of the fluid motion. Let the velocity of a point in the core be written as $\boldsymbol{\omega}\times\mathbf{r}+\mathbf{v}$, where $\mathbf{v}=0$ if the core rotates synchronously with the solid mantle. If the fluid moves, the density of kinetic energy relative to the state of synchronous rotation is

$$\begin{aligned} E_{\text{kin}} &= (\rho/2)(\boldsymbol{\omega}\times\mathbf{r}+\mathbf{v})^2-(\rho/2)(\boldsymbol{\omega}\times\mathbf{r})^2 \\ &= \rho(\mathbf{r}\times\mathbf{v})\cdot\boldsymbol{\omega}+(\rho/2)\mathbf{v}^2. \end{aligned} \tag{39}$$

Assuming a typical value, $v = 0.01$ cm/sec. as inferred from the secular variation,[16] the second term on the right-hand side is about 10^{-4} joule/m^3, exceedingly small compared to the magnetic energy density. The first term, however, is of the order of 150 joule/m^3, and thus is ample. Now the first term is nothing but the angular momen-

[12] H. Jeffreys, *The Earth* (The Macmillan Company, New York, 1929), second edition, chapter 14.
[13] K. E. Bullen, Bull. Am. Seismol. Soc. **32**, 19 (1942).
[14] Reference 2, Appendix.
[15] See the maps by E. H. Vestine *et al.*, Carnegie Inst. of Washington Publ. 578 (1947).
[16] Reference 2, p. 209.

tum of the fluid particle about the earth's axis multiplied by ω; hence its integral over the sphere does not vanish when the motion is a zonal flow of type $\mathbf{T}_n$, as required by (27), and as provided by the mechanism of tidal deceleration just described.

This source of power for the maintenance of the magnetic field would appear to be characteristic of the earth's core; it cannot readily be generalized to apply to other celestial bodies. If it should be undertaken to explain the magnetic field of the sun by a mechanism of feed-back amplification, the inhomogeneous rotation must be produced by thermal effects and the energy required for amplification must ultimately be of thermal origin. Clearly, the thermal energy available in the sun is so vast that no direct comparison with the conditions in the earth's core is possible.

ELECTROMOTIVE FORCES

While the theory of feed-back amplification indicates how a magnetic field can be increased from small beginnings until the magneto-mechanical reaction prohibits further growth, it does not tell us how large the initial field can be. The theory would no doubt be more satisfactory if it could be demonstrated, on physical grounds, that such an initial field is likely to exist, and that it need not be too minute. We shall now proceed to show that electromotive forces acting in the core (for instance thermoelectric forces) can produce a small internal magnetic field of the electric quadrupole type.

According to the theory of the earth's figure the eccentricity of the core's boundary is nearly the same as that of the earth's surface.[13] It is therefore conceivable that a slight temperature difference exists between the polar and equatorial regions of this boundary, giving rise to a differential thermoelectric potential between its pole and its equator. As the electric conductivity of the solid mantle is likely to be much smaller than that of the core, this will not give rise to an appreciable current density, but merely to an electrostatic charge at the boundary. Now assume that a similar differential thermoelectric force exists at the internal transition layer or boundary located just below $r=0.4R$, which separates the iron from the heavier metals below.[14] For the sake of the present calculation, we shall idealize this by a well-defined boundary. The matter to both sides of the boundary will be assumed a good conductor; the differential e.m.f. then produces electric currents that flow from the pole to the equator in the upper stratum and from the equator to the pole in the lower stratum, or *vice versa*. Such a current system can be represented by a linear aggregate of vectors $\mathbf{S}_n$ (n even). We shall consider a single harmonic component of this field.

Let $r=r_1$ be the inner boundary, $r=R$ the boundary of the core, and let the three regions be distinguished by the indices (1), (2), and (3), counting from the inside out. The electric field is a vector of type $\mathbf{S}_n$, but in the static case it can as well be represented by

$$\mathbf{E}=\nabla\psi.$$

By means of (I, 10) we can then write for the magnetic field

$$\mathbf{B}=\mu\sigma(n+1)^{-1}(\mathbf{E}\times\mathbf{r}) \quad \text{for } \psi=r^n Y_n, (\vartheta, \varphi),$$

$$\mathbf{B}=\mu\sigma n^{-1}(\mathbf{E}\times\mathbf{r}) \quad \text{for } \psi=r^{-n-1} Y_n, (\vartheta, \varphi),$$

thereby fulfilling the field equation

$$\nabla\times\mathbf{B}=\mu\sigma\mathbf{E}.$$

In region (3) the magnetic field vanishes, since $\sigma=0$.

We now set in the three regions

$$\psi^{(1)}=ar^n Y_n,$$

$$\psi^{(2)}=(br^n+cr^{-n-1})Y_n,$$

$$\psi^{(3)}=dr^{-n-1}Y_n.$$

The boundary conditions at the inner surface require

$$\psi^{(2)}-\psi^{(1)}=\text{const.}\,Y_n=\phi_n,$$

where ϕ_n is the impressed electromotive potential. They require, moreover, continuity of $\mathbf{B}$, which also provides continuity of the radial component of the current. At the outer boundary $\mathbf{B}$ must vanish and the tangential components of $\mathbf{E}$ are continuous. We shall confine ourselves to writing down the solution for region (2). If σ_1 and σ_2 are the conductivities of regions (1) and (2), we find

$$\psi^{(2)}=\frac{(n+1)(r/R)^n+(2n+1)(R/r)^{n+1}(\sigma_2/\sigma_1)}{(2n+1)(\sigma_2/\sigma_1)+(n+1)(1-\sigma_2/\sigma_1)(r_1/R)^{2n+1}}\left(\frac{r_1}{R}\right)^{n+1}\phi_n.$$

Now assume for simplicity $\sigma_1=\sigma_2$. If r_1 and r are not too close to R, we can neglect the first term in the numerator and the second term in the denominator, and get

$$\psi^{(2)}=(r_1/r)^{n+1}\phi_n$$

valid in the region just above the inner boundary. For the magnetic field of the quadrupole mode we find at $r=r_1$

$$\mathbf{B}=-(\mu\sigma/2)\sin\vartheta\cdot\phi.$$

If we put $\phi=1$ millivolt, $\sin\vartheta=\frac{1}{2}$, we get a field of about 3 gauss. Thermoelectric potentials are of the order of a few microvolts per degree temperature difference. Taking the acting e.m.f. to be 10 microvolts, we require an over-all amplification of about a thousand from the initial field to the final magnitude of the electric quadrupole mode.

APPENDIX

The following expressions for the matrix elements simplify and generalize those of Part I. Let $Z(r)$ designate a normalized radial eigenfunction. We define the following integrals where the prime denotes differentiation with respect to r.

$$F=R\int Z_\alpha Z_\beta Z_\gamma r dr,$$
$$G_\alpha=R^2\int (rZ_\alpha)' Z_\beta Z_\gamma dr,$$
$$H_\alpha=R^3\int Z_\alpha(rZ_\beta)'(rZ_\gamma)' r^{-1}dr.$$

We furthermore define integrals over the sphere

$$K=\int Y_\alpha Y_\beta Y_\gamma dS,$$
$$L_\alpha=\int\frac{Y_\alpha}{\sin\vartheta}\left[\frac{\partial Y_\beta}{\partial\vartheta}\frac{\partial Y_\gamma}{\partial\varphi}-\frac{\partial Y_\beta}{\partial\varphi}\frac{\partial Y_\gamma}{\partial\vartheta}\right]dS.$$

We may write this L, for short, since

$$L_\alpha=L_\beta=L_\gamma.$$

The last relation may be proved by means of integrations by parts. If now, as in Part I, the product of the normalization factors of the spherical harmonics is designated by N we obtain, by (I, 13) and (I, 14)

$$[\mathbf{S}(\alpha)\cdot\mathbf{T}(\beta)\times\mathbf{T}(\gamma)]=n_\alpha(n_\alpha+1)NFL,$$

$$[\mathbf{S}(\alpha)\cdot\mathbf{S}(\beta)\times\mathbf{T}(\gamma)]$$
$$=-\tfrac{1}{2}n_\alpha(n_\alpha+1)[n_\beta(n_\beta+1)+n_\gamma(n_\gamma+1)-n_\alpha(n_\alpha+1)]NG_\beta K$$
$$+\tfrac{1}{2}n_\beta(n_\beta+1)[n_\alpha(n_\alpha+1)+n_\gamma(n_\gamma+1)-n_\beta(n_\beta+1)]NG_\alpha K,$$

$$[\mathbf{S}(\alpha)\cdot\mathbf{S}(\beta)\times\mathbf{S}(\gamma)]$$
$$=[n_\alpha(n_\alpha+1)H_\alpha+n_\beta(n_\beta+1)H_\beta+n_\gamma(n_\gamma+1)H_\gamma]NL.$$

As a supplement to the tables in Part II, p. 207, we give below all the non-vanishing matrix elements containing dipole and quadrupole vectors of the type **S** only. To conform with the previous tables we use complex notation:

$$[\mathbf{S}_1\cdot\mathbf{S}_1^1\times\mathbf{S}_1^{-1}]=-4\pi i(2/3)N(H_\alpha+H_\beta+H_\gamma),$$
$$[\mathbf{S}_1\cdot\mathbf{S}_2^1\times\mathbf{S}_2^{-1}]=-4\pi i(2/5)N(H_\alpha+3H_\beta+3H_\gamma),$$
$$[\mathbf{S}_1\cdot\mathbf{S}_2^2\times\mathbf{S}_2^{-2}]=-4\pi i(4/5)N(H_\alpha+3H_\beta+3H_\gamma),$$
$$[\mathbf{S}_1^1\cdot\mathbf{S}_2\times\mathbf{S}_2^{-1}]=4\pi i(2/5)N(H_\alpha+3H_\beta+3H_\gamma),$$
$$[\mathbf{S}_1^1\cdot\mathbf{S}_2^1\times\mathbf{S}_2^{-2}]=4\pi i(2/5)N(H_\alpha+3H_\beta+3H_\gamma).$$