Non-Linear Invariants and the Problem of Motion

CHRISTOPHER GREGORY
University of Hawaii, Honolulu, Hawaii

(Received March 20, 1947)

The effect of adding quadratic invariants to the integrand of the usual variation principle in general relativity is considered in the light of a new approximation method. The case in which the quadratic invariant is a constant multiple of the square of the scalar curvature is treated up to the Newtonian approximation, insofar as the problem of motion is concerned. The Newtonian equations result for sufficiently large "distances" between particles regardless of the order of magnitude of the multiplying constant. The requirement that the "force function" be finite everywhere places restrictions on the number of "particles" p, comprising the system. When $p=1, 2$, the requirement is fulfilled. The less stringent requirement that the "force function" be finite when all "particles" coincide restricts p less severely. If the absolute values of the "masses" are not all equal there is no restriction on p, but if they are all equal p is restricted to certain integral values.

1. INTRODUCTION

By means of an ingenious approximation method initiated by Einstein, Infeld, and Hoffmann,[1] it has been shown that the motion of matter, represented as point singularities of the field, is sufficiently determined by the gravitational equations for empty space. The four differential identities make possible the equations of motion. The non-linear character of the field equations is directly responsible for the interaction terms present in the equations of motion. It is possible, however, to construct an infinite number of non-linear field equations involving the metric tensor, all of which satisfy four differential identities. The Hamiltonian derivative[2] of any invariant involving the $g^{\mu\nu}$ and their partial derivatives satisfies four differential tensor identities, and the equations are non-linear in character. It is a well-known fact that the field equations for empty space can be obtained by setting the Hamiltonian derivative of the scalar curvature, $R=g^{\mu\nu}R_{\mu\nu}$, equal to zero. The restriction to the invariant R seems to be dictated by the assumption that the field equations be second-order partial differential equations. This has engendered the use of phenomenological devices in subsequent generalizations of the field equations in the presence of matter.

It is of some interest to consider the effect of the addition of other invariant functions of the metric tensors and their partial derivatives to the scalar curvature on the equations of motion. The simplest generalization of the scalar R seems to be effected by the addition of terms which lead to partial derivatives of the fourth order of the metric tensor and quadratic terms involving the curvature tensor. Lanczos[3] has investigated quadratic invariants and has come to the conclusion that only two independent ones, namely, R^2 and $R_{\mu\nu}R^{\mu\nu}$, exist. In this paper, the effect of adding the additional term $\frac{1}{6}a^2R^2$, where a is a constant, will be studied up to, and including the Newtonian approximation.

The addition of a linear combination of the two quadratic invariants $\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}$ to R, may be thought of as representing the contribution of matter. Now the above may be written as $(\alpha-\frac{1}{3}\beta)R^2+\frac{1}{3}\beta(R^2-3R_{\mu\nu}R^{\mu\nu})$. It can be readily shown that the Hamiltonian derivative of the second part, when multiplied by $g_{\mu\nu}$ to form a scalar, is identically nil. This, however, is a property of the electromagnetic stress-energy-momentum tensor. It seems possible, then, that the equations of motion deduced from the scalar $R+\alpha R^2+\beta R_{\mu\nu}R^{\mu\nu}$ may include interaction terms of Lorentz' type. The details and the conclusions to be drawn from these considerations will be presented at a later date.

[1] A. Einstein, L. Infeld, and B. Hoffmann, Ann. of Math. **39**, 65 (1938); A. Einstein and L. Infeld, *ibid.*, **41**, 455 (1940). This method is referred to herein as the E.I.H. method.

[2] A. S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, England, 1940), p. 141.

[3] C. Lanczos, Ann. of Math. **39**, 842 (1938).

2. HAMILTONIAN DERIVATIVES OF R^2 AND $R_{\mu\nu}R^{\mu\nu}$

The process of Hamiltonian differentiation is equivalent to the determination of the coefficient of the variation, $\delta g_{\mu\nu}$, when the invariant under consideration is regarded as the integrand of a variation principle. A straightforward application of the calculus of variations leads to the result

$$I^{\mu\nu}=-2RR^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}R^2+2(g^{\alpha\beta}g^{\mu\nu}-g^{\mu\beta}g^{\nu\alpha})R_{;\alpha\beta} \quad (1)$$

for the Hamiltonian derivative of the invariant R^2. The symbol $(\ )_{;\,\alpha\beta\ldots}$ denotes co-variant differentiation. The Hamiltonian derivative of the invariant $R_{\mu\nu}R^{\mu\nu}$ turns out to be

$$J^{\mu\nu}=-R_\alpha{}^\mu R^{\alpha\nu}-g^{\nu\gamma}R^{\alpha\beta}R_{\alpha\beta\gamma}{}^\mu+\tfrac{1}{2}g^{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}$$
$$+(R^{\alpha\beta}g^{\mu\nu}+R^{\mu\nu}g^{\alpha\beta}-R^{\mu\beta}g^{\nu\alpha}-R^{\nu\alpha}g^{\mu\beta})_{;\,\alpha\beta}. \quad (2)$$

With the aid of the Bianchi identities, it can be shown that $I^{\mu\nu}{}_{;\,\nu}$ and $J^{\mu\nu}{}_{;\,\nu}$ are identically zero.

It will now be shown, as a check upon the derivation of Eq. (1), that $I^{\mu\nu}{}_{;\,\nu}\equiv 0$.

$$I^{\mu\nu}{}_{;\,\nu}=-2R^{\mu\nu}{}_{;\,\nu}R-2R^{\mu\nu}R_{;\,\nu}$$
$$+g^{\mu\nu}RR_{;\,\nu}+2g^{\alpha\beta}(R_{;\,\alpha\beta\nu}-R_{;\,\alpha\nu\beta}).$$

But,

$$R^{\mu\nu}{}_{;\,\nu}\equiv\tfrac{1}{2}g^{\mu\nu}R_{;\,\nu},$$

and

$$g^{\alpha\beta}g^{\mu\nu}(R_{;\,\alpha\beta\nu}-R_{;\,\alpha\nu\beta})\equiv g^{\alpha\beta}g^{\mu\nu}R_{;\,\gamma}R_{\alpha\beta\nu}{}^\gamma=R^{\mu\nu}R_{;\,\nu}.$$

Consequently,

$$I^{\mu\nu}{}_{;\,\nu}=-g^{\mu\nu}RR_{;\,\nu}-2R^{\mu\nu}R_{;\,\nu}$$
$$+g^{\mu\nu}RR_{;\,\nu}+2R^{\mu\nu}R_{;\,\nu}\equiv 0.$$

In a similar fashion $J^{\mu\nu}{}_{;\,\nu}\equiv 0$.

Upon lowering the indices by tensor multiplication, (1) and (2) may be written as

$$I_{\mu\nu}=-2RR_{\mu\nu}+\tfrac{1}{2}g_{\mu\nu}R^2+2(g_{\mu\nu}g^{\alpha\beta}-\delta_\mu{}^\alpha\delta_\nu{}^\beta)R_{;\,\alpha\beta}, \quad (3)$$

$$J_{\mu\nu}=-R^{\alpha\beta}R_{\alpha\beta\mu\nu}-g_{\nu\beta}R^{\alpha\beta}R_{\mu\alpha}+\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}$$
$$+g_{\mu\nu}R^{\alpha\beta}{}_{;\,\alpha\beta}+g^{\alpha\beta}R_{\mu\nu;\,\alpha\beta}$$
$$-g_{\nu\beta}R^{\alpha\beta}{}_{;\,\mu\alpha}-g_{\mu\alpha}R^{\alpha\beta}{}_{;\,\beta\nu} \quad (4)$$
$$=-R^{\alpha\beta}R_{\alpha\beta\mu\nu}-g_{\nu\beta}R^{\alpha\beta}R_{\mu\alpha}+\tfrac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}$$
$$+g^{\alpha\beta}R_{\mu\nu;\,\alpha\beta}+\tfrac{1}{2}g^{\alpha\beta}g_{\mu\nu}R_{;\,\alpha\beta}-R_{;\,\mu\nu}.$$

3. THE FIELD EQUATIONS DERIVED FROM THE INVARIANT $-R+\frac{1}{6}a^2R^2$

If one now considers the quadratic invariant

$$S=-(R+\tfrac{1}{6}a^2R^2), \quad (5)$$

the field equations become, upon applying (1) and recalling that the Hamiltonian derivative of $-R$ is $R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$,

$$S^{\mu\nu}=R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R+\tfrac{1}{6}a^2(2RR^{\mu\nu}$$
$$-\tfrac{1}{2}g^{\mu\nu}R^2-2g^{\alpha\beta}g^{\mu\nu}R_{;\,\alpha\beta}+2g^{\mu\beta}g^{\nu\alpha}R_{;\,\alpha\beta}) \quad (6)$$
$$=0,$$

which will be assumed equal to zero. Upon lowering indices and noting that on the basis of the assumption $S^{\mu\nu}=0$

$$g_{\mu\nu}S^{\mu\nu}=g^{\alpha\beta}R_{;\,\alpha\beta}+a^{-2}R=0, \quad (7)$$

Eq. (6) is equivalent to

$$S_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{6}g_{\mu\nu}R+\tfrac{1}{3}a^2R_{;\,\mu\nu}$$
$$-a^2g_{\mu\nu}R^2/12+\tfrac{1}{3}a^2RR_{\mu\nu}=0. \quad (8)$$

(8) and (7) may be written as

$$S_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{6}g_{\mu\nu}R+\tfrac{1}{3}a^2R_{1\mu\nu}-\tfrac{1}{3}a^2R_{1\alpha}\left\{{\alpha \atop \mu\nu}\right\}$$
$$-a^2g_{\mu\nu}R^2/12+\tfrac{1}{3}a^2RR_{\mu\nu}=0, \quad (9)$$

and

$$a^2R_{1ll}-R=a^2R_{100}+a^2R_{1\alpha}\left[\left\{{\alpha \atop ll}\right\}-\left\{{\alpha \atop 00}\right\}\right]$$
$$+a^2h^{\alpha\beta}R_{1\alpha\beta}-a^2h^{\alpha\beta}R_{1\gamma}\left\{{\gamma \atop \alpha\beta}\right\}, \quad (10)$$

where $(\ )_{1\mu\nu\ldots}$ denotes ordinary partial differentiation; the h's, the deviation of the metric tensors from their Gailean values, and the Latin indices take the spatial values 1, 2, 3, while Greek indices range from 0 to 3. Upon making transformations of the type h to γ as in E.I.H. (reference 1), the field equations are equivalent to

$$\Phi_{00}+2\Lambda_{00}+\tfrac{1}{3}a^2R_{1ll}+\tfrac{1}{3}R+\tfrac{1}{3}a^2R_{100}$$
$$-\tfrac{1}{3}a^2\left[\left\{{\alpha \atop 00}\right\}+\left\{{\alpha \atop ll}\right\}\right]R_{1\alpha}+\tfrac{1}{6}a^2R^2$$
$$+\tfrac{1}{3}a^2(\Phi_{00}+2\Lambda_{00})R-\tfrac{1}{3}\gamma_{00}R-\tfrac{1}{6}a^2\gamma_{00}R^2$$
$$=\Phi_{00}+2\Omega_{00}=0, \quad (11)$$

$$\Phi_{0n}+2\Lambda_{0n}+\tfrac{1}{3}2a^2R_{10n}-\tfrac{1}{3}\gamma_{0n}R-\tfrac{1}{3}2a^2R_{1\alpha}\left\{ {\alpha \atop 0n} \right\}$$
$$-\tfrac{1}{6}a^2\gamma_{0n}R^2+\tfrac{1}{3}a^2(\Phi_{0n}+2\Lambda_{0n})R$$
$$=\Phi_{0n}+2\Omega_{0n}=0, \quad (12)$$

$$\Phi_{mn}+2\Lambda_{mn}-\tfrac{1}{3}\delta_{mn}R-\tfrac{1}{3}a^2\delta_{mn}R_{1ll}+\tfrac{1}{3}2a^2R_{1mn}$$
$$+\tfrac{1}{3}a^2\delta_{mn}R_{100}-\tfrac{1}{3}\gamma_{mn}R-\tfrac{1}{3}2a^2\Big[\left\{ {\alpha \atop mn} \right\}$$
$$-\tfrac{1}{2}\delta_{mn}\left\{ {\alpha \atop ll} \right\}+\tfrac{1}{2}\delta_{mn}\left\{ {\alpha \atop 00} \right\}\Big]R_{1\alpha}-\tfrac{1}{6}a^2\delta_{mn}R^2$$
$$+\tfrac{1}{3}a^2(\Phi_{mn}+2\Lambda_{mn})-\tfrac{1}{6}a^2\gamma_{mn}R^2$$
$$=\Phi_{mn}+2\Omega_{mn}=0, \quad (13)$$

where $\Omega_{\mu\nu}$ is accordingly defined. The Φ's and Λ's are defined in precisely the same manner as in the E.I.H. paper. If one compares (11), (12), and (13) with the transformed field equations treated there, it is readily seen that the new approximation method is applicable to the present equations with little change. The equations involving the $\Lambda_{\mu\nu}$ in E.I.H. have simply to be replaced by the $\Omega_{\mu\nu}$ implicitly defined in (11), (12), and (13). The equations to be solved at each stage of the approximation are similar, with the exception of the additional inhomogeneous wave equation (10).

4. EQUATION OF MOTION IN NEWTONIAN APPROXIMATION

Upon introducing the E.I.H. expansions for the field quantities and proceeding in a somewhat similar manner, the equations of motion in the Newtonian approximation turn out to be

$$m_k\left(\frac{\partial^2\xi}{\partial t^2}\right)_k^{(n)}$$
$$+\sum_{j\neq k}^{p}(m_jm_kV_{jk}+a^2r_{0j}r_{0k}W_{jk}/12)_{,n} \quad (14)$$
$$j,\,k=1,\,2,\,\cdots,\,p,$$

where p denotes the number of "particles"; the m_k their masses; $(\xi_k^{(1)},\ \xi_k^{(2)},\ \xi_k^{(3)})$ the cartesian coordinates of the kth particle; the r_{0k} arbitrary constants associated with the kth particle; and $(\ )_{,n}$ denotes partial differentiation with respect to $\xi_k^{(n)}$. The V_{jk} and W_{jk} are given by

$$\begin{aligned} V_{jk}&\equiv 1/r_{jk}; \quad W_{jk}\equiv V_{jk}e^{-r_{jk}/a};\\ r_{jk}&\equiv[(\xi_j^{(s)}-\xi_k^{(s)})(\xi_j^{(s)}-\xi_k^{(s)})]^{\frac{1}{2}}. \end{aligned} \quad (15)$$

The equation differs from the ordinary Newtonian equation by the appearance of an additional term in the summand of (14). If $a>0$, it is seen that for large "distances" between "particles" $(r_{jk}\gg a)$ (14) reduces to the ordinary equations of motion *irrespective of the order of magnitude of a*. It is believed that this result is of importance in that it does not require one to treat the additional term in the variation principle as a perturbation.

5. ON RENDERING "FORCES" FINITE

The appearance of the arbitrary constants r_{0k}, associated with the kth particle in the equations of motion, leads one to speculate as to the possibility of rendering the "forces" finite everywhere in this order of approximation. To carry out this investigation, it is only necessary to examine the "potential function" corresponding to the equation of motion, namely,

$$U\equiv\sum_{j\neq k}^{p}[m_jm_kV_{jk}+a^2r_{0j}r_{0k}W_{jk}/12]. \quad (16)$$

If one defines

$$m_k\equiv p_k\mu_k, \quad p_k=\pm 1, \quad \mu_k>0; \quad (17)$$

and assumes

$$(a^4r_{0k}^2/12)^{\frac{1}{2}}=q_k\mu_k, \quad q_k=\pm 1; \quad (18)$$

then (15) becomes

$$U=\sum_{j\neq k}^{p}\mu_j\mu_k(p_jp_kV_{jk}+q_jq_kW_{jk}). \quad (19)$$

Thus, in order that U be finite everywhere for arbitrary μ's

$$p_jp_k+q_jq_k=0, \quad j\neq k. \quad (20)$$

It is possible to satisfy these conditions only when $p=2$.

The imposition of the requirement that the U be finite when all "particles" coincide places no restriction on the number of particles p if the μ's are not all equal. In this case, it is only necessary that

$$\sum_{j\neq k}^{p}\mu_j\mu_k(p_jp_k+q_jq_k)=0. \quad (21)$$

However, if the μ's are all equal, (21) degenerates into

$$\sum_{j\neq k} (p_j p_k + q_j q_k) = 0. \tag{22}$$

Since the p's and q's take on values of ± 1 only, it is clear that (22) will not hold for abitrary p. Simple calculations show that if α_+ and α_- denote the number of positive and negative p's, respectively, and β_+ and β_- the number of positive and negative q's, respectively, then in order that (22) hold,

$$\begin{aligned} &\alpha_+ = m^2+n^2+m, \quad \beta_+ = m^2+n^2+n, \\ &\alpha_- = m^2+n^2-m, \quad \beta_- = m^2+n^2-n, \\ &p = 2(m^2+n^2), \quad m, n = 0, 1, 2, \cdots, \end{aligned} \tag{23}$$

for p even. If p is odd

$$\begin{aligned} &\alpha_+ = m^2+n^2-n, \qquad \beta_+ = m^2+n^2-m, \\ &\alpha_- = m^2+n^2-2m-n+1, \\ &\qquad \beta_- = m^2+n^2-m-2n+1, \\ &p = 2(m^2+n^2-m-n)+1, \quad m, n = 0, 1, 2, \cdots. \end{aligned} \tag{24}$$

Other solutions may be generated by interchanging α_+, α_-, β_+, and β_- in a suitable fashion. For these cases $|\alpha_+ - \alpha_-|$ and $|\beta_+ - \beta_-|$ yield any positive integer.

6. CONCLUSION

The E.I.H. method seems to be applicable to the field equations derived by the procedure of adding the invariant $\frac{1}{6}a^2R^2$ to the integrand of the usual variation principle. It is found that insofar as the Newtonian approximation is concerned, for sufficiently large "distances" between "particles," a reduction to the ordinary Newtonian equations ensues regardless of the order of magnitude of $a(>0)$. The investigation regarding the condition for the existence of finite "forces" between particles *may* be of physical significance to the theory of nuclear structure assuming, of course, that the invariant studied possesses physical significance. It is hoped that this study at least increases the conceptual possibilities insofar as field theories are concerned.