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Note on Angular Distributions in Nuclear Reactions*

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A proof is given of the physically plausible theorem that in a nuclear reaction produced with an unpolarized beam of given orbital angular momentum incident on an unpolarized target, the angular distribution of the outgoing intensity cannot be more complicated than that of the incoming intensity.

THE purpose of this note is to give a simple proof, based on symmetry arguments, of a conjecture made by Critchfield and Teller[1] concerning the angular distribution of the outgoing particles in a nuclear reaction. This conjecture, which is at the outset very plausible, may be stated as follows: *In a nuclear reaction produced with an unpolarized beam of given orbital angular momentum incident on an unpolarized target, the angular distribution of the outgoing intensity cannot be more complicated than that of the incoming intensity.* The statement appears plausible since the spins of the incident particle and target nucleus are unpolarized so the only angular effects which can occur are those produced by the incident wave. However, it seems worth while to give a rigorous proof because, in complicated cases, it is not a trivial matter to see how the angular dependence of the incoming wave fixes that of the outgoing intensity.[2]

The statement is obviously correct if the reaction is produced by an incident S-wave since then there is no fixed direction in space to which the angular distribution of outgoing particles could refer. It is also obvious if only two particles are produced in the reaction and if both initial and both final particles have zero spin. Then the incoming and outgoing relative orbital angular momenta must be equal.

When the particles have spin, compound states can be formed with J, the total angular momentum, larger than L, the orbital angular momentum of the incident wave. These states, in general, give rise to outgoing orbital angular momenta, l, larger than L. The content of our theorem is that, although $l>L$, the highest spherical harmonic required to describe the angular dependence of the outgoing intensity is of order $2L$ rather than $2l$ as might be expected. The proof follows.

First a remark on notation. ψ_j^m will be used to represent a function which transforms under rotations like a wave function with angular momentum, j, and magnetic quantum number m. The function is otherwise unspecified. Because of the appearance of sums over uncorrelated spins in the expressions to follow, a special notation for such a sum, $\mathfrak{S}$, is introduced. The implication is that

$$|\mathfrak{S}_m \psi_j^m|^2 = \sum_m |\psi_j^m|^2.$$

The wave function of the incident particle will, in general, contain terms corresponding to many orbital angular momenta. If the energy of the incoming particle is not too high, only the terms of low angular momentum will contribute to the reaction. We consider, in particular, the case for which the only term contributing appreciably corresponds to an orbital angular momentum, L. This L may be greater than zero as a consequence of a selection rule.

For an arbitrary coordinate system the spatial dependence of the incoming wave is given by some linear combination of the ψ_L^m, let us say,

$$\sum_m a_m \psi_L^m. \tag{1}$$

The spin dependence of the incident wave is given by

$$\mathfrak{S}_{m_N, m_I}\, \psi_N^{m_N} \psi_I^{m_I} \tag{2}$$

where the wave function $\psi_N^{m_N}$ is that of the target nucleus with total angular momentum, N,

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[1] C. L. Critchfield and E. Teller, Phys. Rev. 60, 10 (1941).

[2] An implicit proof is contained in the literature, R. D. Myers, Phys. Rev. 54, 361 (1938), but the direct proof given here appears to be somewhat simpler.

and the wave function $\psi_I^{m_I}$ is that of the incident particle with spin I. The angular momenta N and I may be combined to form a total spin angular momentum, S, where $S=N+I, \cdots |N-I|$. Then, as shown by Breit and Darling,[3] the function (2) may be replaced by

$$\mathfrak{S}_{S,m_S}\,\psi_S^{m_S}.$$

Combining this with (1), the incident wave function is found to have the form

$$\psi_i=\sum_m a_m\,\mathfrak{S}_{S,m_S}\,\psi_L^m\psi_S^{m_S}. \quad (3)$$

The products $\psi_L\psi_S$ appearing in Eq. (3) may be expressed in terms of the wave functions ψ_J associated with the total angular momenta J. Thus,

$$\psi_i=\sum_J \mathfrak{S}_{S,m_S}\sum_m a_m \times(L, m, S, m_S|J, m+m_S)\psi_J^{m+m_S} \quad (4)$$

where $(L, m, S, m_S|J, m+m_S)$ are the usual transformation coefficients.

That term in Eq. (4) with a given total angular momentum, J, will interact only with those states of the compound nucleus with the same J. Thus each such term will contribute to the nuclear reaction to a different extent. The outgoing wave is then similar in form (insofar as transformation properties are concerned) to the incoming wave, Eq. (4), except for a change in the relative amplitudes of the terms with different J values. These relative amplitudes will be indicated by ρ_J. The intensity I_0 of the outgoing wave is proportional to the absolute square of the outgoing wave function or

$$I_0=\sum_{S,m_S}|\sum_J \rho_J \sum_m a_m \times(L, m, S, m_S|J, m+m_S)\psi_J^{m+m_S}|^2. \quad (5)$$

This expression contains terms of the type $(\psi_J^{m+m_S})^*\psi_{J'}^{m'+m_S}$ or, since $(\psi_J^m)^*=\pm\psi_J^{-m}$, of the type $\psi_J^{-(m+m_S)}\psi_{J'}^{(m'+m_S)}$. These products transform under rotations like a linear combination of ψ_j^μ where j may have any value from $j=|J-J'|$ to $J+J'$ and $\mu=m'-m$. However, m and m' both range from $-L$ to $+L$ so μ ranges from $-2L$ to $+2L$. Since we are using an arbitrarily oriented coordinate system, the coordinate system can be chosen in such a way that each possible value of μ occurs for a given value of j. In particular, the value $\mu=j$ will appear for every j. Since μ is never greater than $2L$, it follows that j cannot be greater than $2L$.

If there are only two products of the reaction and they both have zero spin, the last statement constitutes a proof of the theorem, since then the orbital angular momentum, l, of the products is equal to J, and their angular distribution is determined by the ψ_j^μ. If one of the two particles has a spin, s, then the ψ_J^M, with $M=m+m_S$, must be analyzed into products $\psi_l^{M'}\psi_s^{M-M'}$, and the spatial dependence of the outgoing intensity is determined by terms of the form $\psi_l^{M'}\psi_{l'}^{-M''}$. These products transform like a linear combination of $\psi_{j'}^{\mu'}$ and we wish to show that the largest value of j' which can occur is $j'=2L$. To demonstrate this, we note that the product $\psi_l^{M'}\psi_{l'}^{-M''}$ in the outgoing intensity is multiplied by a spin factor $\psi_s^{M-M'}\psi_s^{-(M-M'')}$. The latter product may also be analyzed into a linear combination of $\psi_{j''}^{\mu''}$. Then the over-all transformation properties of the outgoing intensity are given by a combination of terms of the form $\psi_{j'}^{\mu'}\psi_{j''}^{\mu''}$ and these may be analyzed into a series of ψ_j^μ with $j=j'+j'', \cdots |j'-j''|$. This final series is identical with our analysis in the preceding paragraph which was made without reference to the distribution of angular momentum between spin and orbit. Therefore, $j\leq 2L$. But $j=j_{\max}'+j_{\max}''$ always occurs in the series[4] where $j_{\max}'$ is the largest value of j', so we must have $j_{\max}'+j_{\max}''\leq 2L$ or finally $j_{\max}\leq 2L$ as was to be proved. It is to be noted that $j_{\max}''\leq 2L$ also but it does not appear to be possible to attach a simple physical significance to this result.

The proof for any number of product particles with arbitrary spins follows in similar manner. In general, one comes to a result of the form

$$j_{\max}'+j_{\max}''+j_{\max}'''+\cdots\leq 2L$$

so that $j_{\max}^i\leq 2L$.

[3] G. Breit and B. T. Darling, Phys. Rev. **71**, 402 (1947).

[4] $j_{\max}'+j_{\max}''$ occur only once so there can be no cancellation of this term as there may be for the lower values of $j'+j''$.