The Electromagnetic Field in Quantized Space-Time

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Relativistically invariant equations of motion for the electromagnetic field are set up in quantized space-time. These equations are solved by a process similar to a Fourier analysis.

IN a previous paper[1] it was shown that a Lorentz invariant space-time is not necessarily a continuum, and an example was given of a discrete Lorentz invariant space-time. This paper is a report on work done to determine whether relativistically invariant field equations could be introduced into quantized space-time, and whether such field equations are solvable. In continuous space-time the field quantities are taken to be functions of the space and time coordinates, and the field equations are partial differential equations. In quantized space-time, coordinates do not commute, thus one has difficulty in giving a general definition of functions of non-commuting variables. Also, since the space-time is discrete, partial derivatives are not definable in the ordinary sense. Now, in the transition from continuous space-time to quantized space-time, the space-time coordinates which were real variables become Hermitian operators; therefore, it is reasonable to suppose that field quantities which were functions of the space-time coordinates will become operators on the Hilbert space on which the coordinate operators operate. I assume that this is the case.

The next thing which must be done is to find replacements for partial derivatives. We note that the displacement operators[2] p_x, p_y, p_z, p_t defined in [*I*] have the same transformation properties as do ∂/∂_x, ∂/∂_y, ∂/∂_z, and ∂/∂_t, and that their commutators with x, y, z, and t approximate, with the exception of numerical factors, those of the partial differential operators with the x, y, z, and t of ordinary space. Consequently, I assume that if $A(x, y, z, t)$ is a field quantity of continuous space-time and if a term of the form $\partial A/\partial x$ appears in the field equations, this term will be replaced by $i[p_x, A]$ in the transition to quantized space-time. It is evident that if A is a Hermitian operator, then $i[p_x, A]$ will also be Hermitian, so that this replacement of partial derivatives preserves reality conditions. If we make the replacements suggested above into the usual form of the vacuum Maxwell's equations, we obtain[3] a relativistically invariant set of equations,

$$i[\mathbf{p}\cdot,\ \cdot\mathbf{E}]=4\pi\rho, \tag{1}$$

$$i[\mathbf{p}\times,\ \times\mathbf{H}]-i[p_t,\mathbf{E}]=4\pi\mathbf{S}, \tag{2}$$

$$i[\mathbf{p}\cdot,\ \cdot\mathbf{H}]=0, \tag{3}$$

$$i[\mathbf{p}\times,\ \times\mathbf{E}]+i[p_t,\mathbf{H}]=0, \tag{4}$$

in which the symbols $\mathbf{E}$, $\mathbf{H}$, ρ, and $\mathbf{S}$ have their usual meanings and transformation properties. We are using a system of units such that $\hbar=1$ unit of action, $c=1$ unit of velocity, and the vacuum has a unit dielectric constant.

Now, from the first two of Maxwell's equations, (1), (2), one can show that

$$i[p_t,\rho]+i[\mathbf{p}\cdot,\ \cdot\mathbf{S}]=0, \tag{5}$$

which is our analog of the differential equation for the conservation of charge.

By taking the scalar product of Eq. (2) by $\mathbf{E}$ on both right and left, the scalar product of Eq. (4) by $\mathbf{H}$ on both right and left, and performing other algebraic manipulations, one obtains

$$\tfrac{1}{2}i[p_t,\mathbf{E}^2+\mathbf{H}^2]+\tfrac{1}{2}i[\mathbf{p}\cdot,\ \cdot(\mathbf{E}\times\mathbf{H}-\mathbf{H}\times\mathbf{E})]+2\pi(\mathbf{S}\cdot\mathbf{E}+\mathbf{E}\cdot\mathbf{S})=0. \tag{6}$$

This result is our analog of the usual differential

[1] Hartland S. Snyder, Phys. Rev. **71**, 38 (1947). This paper will be referred to as [*I*] throughout the remainder of this paper.

[2] We will use the general forms for p_x, p_y, p_z, p_t, which are

$$p_x=\frac{\hbar}{a}\frac{\eta_1}{\eta_4}f\left(\frac{\eta_4}{\eta}\right),\ \cdots,\quad p_t=\frac{\hbar c}{a}\frac{\eta_0}{\eta_4}f\left(\frac{\eta_4}{\eta}\right)$$

as given in [*I*], reference 3.

[3] In these equations $[\mathbf{p}\cdot,\ \cdot\mathbf{E}]=\mathbf{p}\cdot\mathbf{E}-\mathbf{E}\cdot\mathbf{p}$, $[\mathbf{p}\times,\ \times\mathbf{H}]=\mathbf{p}\times\mathbf{H}+\mathbf{H}\times\mathbf{p}$, and the dot and cross products have their usual meanings. The use of the double dot and double cross notation inside the commutators helps to avoid confusion.

power equation for the electromagnetic field. Equation (6) is independent of the commutation properties of $\mathbf{E}$, $\mathbf{H}$, ρ, and $\mathbf{S}$.

As in the case of the ordinary Maxwell equations, one can introduce scalar and vector potentials in such a manner that Eqs. (3) and (4) become identities. We introduce the operators V, and $\mathbf{A}$ and set

$$\mathbf{E}=-i[\mathbf{p},\ V]-i[p_t,\mathbf{A}], \tag{7}$$

$$\mathbf{H}=i[\mathbf{p}\times,\ \times\mathbf{A}]. \tag{8}$$

It is readily seen that if the values of $\mathbf{E}$ and $\mathbf{H}$ given by (7) and (8) are substituted into (3) and (4), then (3) and (4) are satisfied identically. If we set $V=V'-i[p_t,\lambda]$, $\mathbf{A}=\mathbf{A}'+i[\mathbf{p},\lambda]$, in which λ is a scalar operator, we find that $\mathbf{E}$ and $\mathbf{H}$ are expressible in terms of V' and $\mathbf{A}'$ in exactly the same form as Eq. (7) and (8), with V' and $\mathbf{A}'$ replacing V and $\mathbf{A}$. This result gives the gauge invariance of Eqs. (1), (2), (3), and (4). As we can now see, all the derived relationships between the field quantities (such as (5)–(8)) which do not depend upon the commutation properties of the fields, could have been obtained from the relations derived from Maxwell's equation by replacing the operators $\partial/\partial x$, $\partial/\partial y$, $\cdots$, ∂/∂_t by the commutators $i[p_x,\quad]$, $\cdots$, $i[p_t,\quad]$. Thus, we find that the replacement of partial differential operators by these commutator brackets is a consistent procedure, as well as a relativistically invariant one.

A major question is whether such commutator equations as the above can be given meaning, and whether operators satisfying them can be found. The quantities, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{S}$, V, $\mathbf{A}$, and ρ, are supposed to be operators on the Hilbert space on which x, y, z, and t operate. If we denote a vector of the Hilbert space by the symbol, χ, then the scalar (inner) product of the two such vectors, χ and χ', is

$$(\chi',\chi)=\int\frac{d\mathbf{p}dp_t}{D(p_t,\mathbf{p})}\chi'^*(\mathbf{p},p_t)\chi(\mathbf{p},p_t), \tag{9}$$

if we use a wave number-frequency space $(\mathbf{p}, p_t)$ representation for the vectors of the Hilbert space. The function $D(p_t, \mathbf{p})$ is a calculable relativistically invariant function of its arguments which depends on $f(\eta_4/\eta)$ and which makes x, y, z, and t Hermitian operators. $D^{-1}(p_t, \mathbf{p})d\mathbf{p}dp_t$ is essentially the volume element of the hyper surface $\eta^2=$constant. The asterisk on a function means the complex conjugate. The meaning of all terms such as orthogonal, Hermitian, unitary, etc., is determined by the usual definitions, with (9) giving the meaning to the scalar product of the vectors of Hilbert space.

For the purpose of finding operators satisfying Eqs. (1), (2), (3), and (4), consider an operator, $A_\mathbf{k}$, with the property that

$$A_\mathbf{k}\chi(\mathbf{p},p_t)=\int d\omega f(\mathbf{p},p_t,\mathbf{k},\omega)\chi(\mathbf{p}-\mathbf{k},p_t-\omega) \tag{10}$$

for every vector $\chi(\mathbf{p}, p_t)$ of Hilbert space, with $f(\mathbf{p}, p_t, \mathbf{k}, \omega)$ an arbitrary given function of its arguments which is independent of the function $\chi(\mathbf{p}, p_t)$. It is not difficult to verify that the operator, $A_\mathbf{k}$, satisfies the commutation relation,

$$[\mathbf{p},A_\mathbf{k}]=\mathbf{k}A_\mathbf{k}. \tag{11}$$

Conversely, one can show that the general solution of (11) is an operator whose properties are given by Eq. (10). We note here that if two operators have the property given by (11), $[\mathbf{p}, A_\mathbf{k}]=\mathbf{k}A_\mathbf{k}$, $[\mathbf{p}, B_{\mathbf{k}'}]=\mathbf{k}'B_{\mathbf{k}'}$ then the product $A_\mathbf{k}B_{\mathbf{k}'}$ has the same property, $[\mathbf{p}, A_\mathbf{k}B_{\mathbf{k}'}]=(\mathbf{k}+\mathbf{k}')A_\mathbf{k}B_{\mathbf{k}'}$. Also, operators satisfying (11) necessarily possess complex adjoint operators. If $A_\mathbf{k}$ is an operator with a complex adjoint operator $A_\mathbf{k}^+$, then

$$(A_\mathbf{k}^+\chi',\chi)=(\chi',A_\mathbf{k}\chi).$$

Now,

$$A_\mathbf{k}^+\chi'(\mathbf{p},p_t)=\int d\omega f^+(\mathbf{p},p_t,\mathbf{k},\omega)\chi'(\mathbf{p}+\mathbf{k},p_t+\omega),$$

and

$$A_\mathbf{k}\chi(\mathbf{p},p_t)=\int d\omega f(\mathbf{p},p_t,\mathbf{k},\omega)\chi(\mathbf{p}-\mathbf{k},p_t-\omega),$$

then Eq. (9) requires that

$$D(\mathbf{p},p_t)f^+(\mathbf{p}-\mathbf{k},p_t-\omega,\mathbf{k},\omega) =D(\mathbf{p}-\mathbf{k},p_t-\omega)f^*(\mathbf{p},p_t,\mathbf{k},\omega).$$

This last equation is solvable for $f^*(\mathbf{p}, p_t, \mathbf{k}, \omega)$ with the consequence that every operator satisfying (11) has an adjoint.

We will now suppose that all of the field operators entering the field equations can be written as linear combinations of operators which satisfy

Eq. (11). For example, we assume that

$$\rho=\int d\mathbf{k}\rho_{\mathbf{k}}, \tag{12}$$

where

$$[\mathbf{p},\rho_{\mathbf{k}}]=\mathbf{k}\rho_{\mathbf{k}}.$$

This assumption is the equivalent of the assumption usually made for field theories in continuous space-time, that the field can be Fourier analyzed. We will call the process exemplified by Eq. (12) a wave number analysis.

If we choose the gauge of the electromagnetic potentials, V and $\mathbf{A}$, so that

$$[\mathbf{p}\cdot,\cdot\mathbf{A}]=0, \tag{13}$$

and if we make a wave number analysis of $\mathbf{A}$, then it can be written

$$\mathbf{A}=\int d\mathbf{k}\sum_{\lambda=1}^{3}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}A_{\lambda\mathbf{k}}, \tag{14}$$

in which we take the $\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}$ to be three mutually perpendicular unit vectors with $\boldsymbol{\varepsilon}_{3\mathbf{k}}$ in the direction of the vector $\mathbf{k}$. When we apply condition (13) to $\mathbf{A}$, we obtain

$$[\mathbf{p}\cdot,\cdot\mathbf{A}]=\int d\mathbf{k}kA_{3\mathbf{k}}=0. \tag{15}$$

This implies that $A_{3\mathbf{k}}=0$, whereas $A_{1\mathbf{k}}$ and $A_{2\mathbf{k}}$ are, insofar as this condition is concerned, arbitrary operators satisfying

$$[\mathbf{p},A_{\lambda\mathbf{k}}]=\mathbf{k}A_{\lambda\mathbf{k}},\quad \lambda=1,2.$$

This result corresponds to the usual result that the Fourier coefficients of the vector potential are perpendicular to the wave number vector if the divergence of the vector potential vanishes.

If the vector potential $\mathbf{A}$ satisfies (13), then by eliminating the electric field from (1) by the use of (7), we obtain the analog of Poisson's equation

$$[\mathbf{p}\cdot,\cdot[\mathbf{p},V]]=4\pi\rho. \tag{16}$$

If we make a wave number analysis of V and ρ, we find for the components, $V_{\mathbf{k}}$ and $\rho_{\mathbf{k}}$, the relation

$$k^2V_{\mathbf{k}}=4\pi\rho_{\mathbf{k}}. \tag{17}$$

Equation (17) is identical in form with that given for the connection between the Fourier components of the potential and the Fourier components of the charge density. The solution of (17) gives us the following solution of (16):

$$V=4\pi\int\frac{d\mathbf{k}}{k^2}\rho_{\mathbf{k}}+V_0, \tag{18}$$

in which V_0 is an operator which satisfies the condition $[\mathbf{p},V_0]=0$. The general form of V_0 is

$$V_0=\int d\omega\, V_{0\omega}, \tag{19}$$

where

$$V_{0\omega}\chi(\mathbf{p},p_t)=g(\mathbf{p},p_t,\omega)\chi(\mathbf{p},p_t-\omega).$$

The presence of the term V_0 in the solution of the "Poisson" equation corresponds to the fact that in continuous space-time an arbitrary function of the time can be added to the scalar potential. This term V_0 in V can be removed by a gauge transformation which does not affect the vector potential.

We now write

$$\mathbf{A}=\int d\mathbf{k}\sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}(\alpha_{\lambda\mathbf{k}}+\alpha_{\lambda\mathbf{k}}^{+}), \tag{20}$$

$$\mathbf{E}=i\int d\mathbf{k}\sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}(\alpha_{\lambda\mathbf{k}}-\alpha_{\lambda\mathbf{k}}^{+})-i[\mathbf{p},V], \tag{21}$$

in which equations $\alpha_{\lambda\mathbf{k}}^{+}$ is the complex adjoint to $\alpha_{\lambda\mathbf{k}}$, and in which V is given by (18). The $\alpha_{\lambda\mathbf{k}}$ are taken to satisfy $[\mathbf{p},\alpha_{\lambda\mathbf{k}}]=\mathbf{k}\alpha_{\lambda\mathbf{k}}$ with the consequence that $[\mathbf{p},\alpha_{\lambda\mathbf{k}}^{+}]=-\mathbf{k}\alpha_{\lambda\mathbf{k}}^{+}$. These particular forms for $\mathbf{A}$ and $\mathbf{E}$ guarantee that $\mathbf{E}$, $\mathbf{A}$, and $\mathbf{H}$ are Hermitian operators if ρ is Hermitian. The value of the magnetic field may then be computed by (8) using the above properties of $\alpha_{\lambda\mathbf{k}}$, and $\alpha_{\lambda\mathbf{k}}^{+}$ and is

$$\mathbf{H}=i\int d\mathbf{k}\,\mathbf{k}\times\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}(\alpha_{\lambda\mathbf{k}}-\alpha_{\lambda\mathbf{k}}^{+}). \tag{22}$$

If we make a wave number analysis of $\mathbf{S}$, and if we substitute the values of $\mathbf{E}$ and $\mathbf{H}$ given by (21) and (22) into Eq. (2), we find that

$$\int d\mathbf{k}\sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}\{k^2(\alpha_{\lambda\mathbf{k}}+\alpha_{\lambda\mathbf{k}}^{+})+k[p_t,\alpha_{\lambda\mathbf{k}}-\alpha_{\lambda\mathbf{k}}^{+}]-4\pi S_{\lambda\mathbf{k}}\}=0. \tag{23}$$

That part, the longitudinal part, of the current arising from $S_{3\mathbf{k}}$ is canceled out of the right-hand side of Eq. (2) by that part of $-i[p_t, \mathbf{E}]$, on the left-hand side of (2), which comes from the term $-i[\mathbf{p}, V]$ in $\mathbf{E}$ because of the charge conservation Eq. (5). If we choose $\boldsymbol{\varepsilon}_{1\mathbf{k}}=-\boldsymbol{\varepsilon}_{1-\mathbf{k}}$, $\boldsymbol{\varepsilon}_{2\mathbf{k}}=\boldsymbol{\varepsilon}_{2-\mathbf{k}}$, Eq. (23) is satisfied by

$$k^2\alpha_{\lambda\mathbf{k}}+k[p_t, \alpha_{\lambda\mathbf{k}}]=2\pi S_{\lambda\mathbf{k}} \tag{24}$$

$$k^2\alpha_{\lambda\mathbf{k}}^+-k[p_t, \alpha_{\lambda\mathbf{k}}^+]=2\pi(-)^\lambda S_{\lambda-\mathbf{k}}. \tag{25}$$

Equations (24) and (25) are complex adjoint equations of each other and require that $S_{\lambda\mathbf{k}}^+=(-)^\lambda S_{\lambda-\mathbf{k}}$, which is just the condition for the current operator $\mathbf{S}$ to be Hermitian.

When $\mathbf{S}=0$, then Eq. (24) states that the operators $\alpha_{\lambda\mathbf{k}}$ have the property

$$\alpha_{\lambda\mathbf{k}}\chi(\mathbf{p}, p_t)=h(\mathbf{p}, p_t, \mathbf{k})\chi(\mathbf{p}-\mathbf{k}, p_t+k), \tag{26}$$

with $h(\mathbf{p}, p_t, \mathbf{k})$ an arbitrary function of its arguments. This result is essentially the same as if an electromagnetic wave had a wave number vector $\mathbf{k}$; its frequency is k for empty space.

If we make a frequency analysis of $\alpha_{\lambda\mathbf{k}}$ and $S_{\lambda\mathbf{k}}$,

$$\alpha_{\lambda\mathbf{k}}=\int d\omega\alpha_{\lambda\mathbf{k}\omega}, \quad S_{\lambda\mathbf{k}}=\int d\omega S_{\lambda\mathbf{k}\omega}, \tag{27}$$

in which $[p_t, \alpha_{\lambda\mathbf{k}\omega}]=\omega\alpha_{\lambda\mathbf{k}\omega}$ and $[p_t, S_{\lambda\mathbf{k}\omega}]=\omega S_{\lambda\mathbf{k}\omega}$, we obtain from (24) and (27)

$$k(k+\omega)\alpha_{\lambda\mathbf{k}\omega}=2\pi S_{\lambda\mathbf{k}\omega}. \tag{28}$$

When we use (27) and (20), the solution of (28) gives the following for the vector potential:

$$\mathbf{A}=2\pi\int d\mathbf{k}\sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}\int d\omega\frac{S_{\lambda\mathbf{k}\omega}+S_{\lambda\mathbf{k}\omega}^+}{k(k+\omega)} + \int d\mathbf{k}\sum_{\lambda=1,2}\boldsymbol{\varepsilon}_{\lambda\mathbf{k}}(\alpha_{\lambda\mathbf{k}}^0+\alpha_{\lambda\mathbf{k}}^{0+}). \tag{29}$$

In (29) the integral over ω is taken in the sense of its principal value for the neighborhood of $\omega=-k$. The operator $\alpha_{\lambda\mathbf{k}}^0$ is as follows:

$$\alpha_{\lambda\mathbf{k}}^0\chi(\mathbf{p}, p_t)=h(\mathbf{p}, p_t, \mathbf{k})\chi(\mathbf{p}-\mathbf{k}, p_t+k)$$

for arbitrary vectors $\chi(\mathbf{p}, p_t)$ of the Hilbert space, and $h(\mathbf{p}, p_t, \mathbf{k})$ is an arbitrary function of its arguments.

As a summary of what we have obtained, it may be said that in quantized space-time, relativistically invariant field equations may be written. In these equations the field quantities are treated as operators, and partial differential operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$, and $\partial/\partial t$ of continuous space-time are replaced by the commutators $i[p_x, \quad]$, $\cdots$, $i[p_t, \quad]$. It has been shown that this replacement process is a consistent one. Operators satisfying the commutator field equations have been found. These operators are expressed in terms of wave number-frequency components of the operators, a process which is quite analogous to the usual Fourier analysis of fields. When the solutions of the Maxwell equations are expressed in terms of the wave number-frequency analysis, the solutions are of exactly the same form as those obtained by the Fourier analysis in the case of continuous space-time. In fact, the general procedure which we have used here applies equally as well to continuous space-time as it does to quantized space-time. However, in the continuous space-time case additional limitations are placed on the field operators. The wave number-frequency components $A_{\mathbf{k}\omega}$ of an operator, A, are restricted for this case so that

$$A_{\mathbf{k}\omega}\chi(\mathbf{p}, p_t)=f(\mathbf{k}, \omega)\chi(\mathbf{p}-\mathbf{k}, p_t-\omega),$$

as compared with the more general form we have used,

$$A_{\mathbf{k}\omega}\chi(\mathbf{p}, p_t)=f(\mathbf{p}, p_t, \mathbf{k}, \omega)\chi(\mathbf{p}-\mathbf{k}, p_t-\omega).$$

The above restriction guarantees for the continuous space-time case that the operator A may be written as a function of x, y, z, and t. I am not certain what restriction should be used in the quantized space-time case, although it is probably connected with the normalizing function $D(p_t, \mathbf{p})$. The essential differences between continuous and quantized space-time lie in the change of definition of the scalar products of the vectors of the Hilbert space, and in the value regions of wave number-frequency four-vectors.

Although we have dealt in this paper only with the vacuum Maxwell equations with given charge and current distributions, the same procedure can be applied to the Klein-Gordon equation, the Dirac equation, the Proca equation, and others. At the present time work is being done to determine appropriate limitations on the field operators, and to determine whether field operators can be quantized.