THE

PHYSICAL REVIEW

*A journal of experimental and theoretical physics established by E. L. Nichols in 1893*

SECOND SERIES, VOL. 72, NO. 8 OCTOBER 15, 1947

# Theory of the Synchro-Cyclotron

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(Received May 31, 1947)

In the synchro-cyclotron (or frequency-modulated cyclotron) the higher energies available are obtained at the expense of a decrease in the ion current compared with that available from the conventional cyclotron. This decrease results from the fact that during only a small fraction of the frequency-modulation cycle is it possible for ions to be captured into phase stable orbits that do not return to the center during the first phase oscillation. By solving the phase equation, it is possible to obtain a general expression for this fraction, which is defined as the capture efficiency. At a constant dee voltage and varying rate of frequency modulation, the capture efficiency has a maximum at an equilibrium phase angle of 30° (corresponding to an energy gain per turn equal to half the maximum available). For larger equilibrium phase angles the efficiency decreases as a result of the smaller range of phase stability, while for smaller phase angles it decreases as a result of return of particles to the center. The maximum efficiency is proportional to the square root of the dee voltage or alternatively to the square root of the rate of frequency modulation, and depends on the charge and mass of the ions only through the ratio of charge to mass. Comparisons of the theoretical expectations with available experimental data show satisfactory agreement. Capture efficiencies for present designs of synchro-cyclotrons are of the order of 0.1 to 2 percent.

## 1. INTRODUCTION

IN the conventional cyclotron,[1] the maximum attainable energy is limited by the fact that the ions ultimately fall out of step with the applied electric field as a result of the decrease of angular velocity as the speed of light is approached.[2,3] V. Veksler[4] and E. M. McMillan[5] have independently proposed a new accelerator utilizing a cyclotron-like combination of electric and magnetic fields in such a way that a theoretically infinite number of accelerations may be accomplished. The theory of this accelerator has been studied in some detail by Dennison and Berlin,[6] by Saxon and Schwinger,[7] by Rabinovich,[8] and by the present authors[9] in a paper hereafter referred to as A. In all of these papers, the basic principles of the new accelerator are discussed, and the equations of orbital motion are solved, primarily with reference to applications to the acceleration of electrons in a machine which has come to be known as the synchrotron.

In the synchro-cyclotron (or frequency-modulated cyclotron) the same basic idea is applied to the acceleration of ions. One important new problem arises in this connection, however, which has not been treated in the previous papers. This new problem results from the manner in

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which the ions are started. In the synchrotron the electrons are started at a large radius either by direct injection from an electron gun or by a preliminary operation of the device as a betatron. In the present design of the synchro-cyclotron it appears to be more practical to start the ions from rest at the center of the magnet as in the conventional cyclotron. However, in contrast to the conventional cyclotron, there is only a limited range of times in which ions starting in this way can enter into stable orbits which never return to the origin. This means, of course, that ions are accelerated intermittantly so that for a given dee voltage and gas pressure the average output is correspondingly reduced below that of a conventional cyclotron. We may define the *capture efficiency*, $\epsilon$, as just the ratio of the time available during a frequency modulation cycle for starting particles into stable orbits which do not return to the center to the total time for repetition of the frequency modulation cycle. The calculation of $\epsilon$ is the principal object of this paper.

## 2. QUALITATIVE DISCUSSION OF FACTORS DETERMINING CAPTURE EFFICIENCY

In the synchro-cyclotron the decrease of frequency of ionic rotation accompanying the increase in energy of the ion is compensated by a corresponding decrease of frequency of the applied dee voltage with time. This decrease is achieved by a periodic modulation of the applied frequency by some means such as the use of a rotating condenser in the oscillator circuit. One may readily observe that there is always one way for a particle to be started into an indefinitely accelerating orbit under these circumstances. It is only necessary that the ion start at the instant when the applied frequency is equal to the ionic rotation frequency at the center of the machine and that the phase of the voltage at the time that it crosses the accelerating gap be such that the resulting energy gain per turn causes a decrease of the ionic rotation frequency which exactly matches the decrease of the applied frequency with time. Such a particle will gain energy steadily, never getting out of phase with the applied voltage.

However, just because the applied frequency is changing with time very few ions can start during the instant of time when their rotation frequency matches the applied frequency. Furthermore, the ions will not, in general, start with the ideal phase described above, which will hereafter be referred to as the equilibrium phase. The fundamental principle which makes the synchro-cyclotron feasible under these circumstances is the fact that ions in orbits whose frequency and phase do not differ too widely from that of the equilibrium orbit will execute stable oscillations of phase and frequency about the equilibrium values and so undergo an indefinite acceleration. This property of *phase stability* of the orbits has been discussed in the papers referred to above in considerable detail.

For any given starting phase of an ion there will, nevertheless, be a maximum discrepancy between ionic rotation frequency and applied frequency at the time of starting consistent with phase stable oscillations. Thus ions starting too late or too early will be subject to limitations on the maximum attainable energy similar to those applying in the ordinary cyclotron and will consequently be lost. Hence the property of phase stability is achieved at the expense of a reduction in the length of time available for starting particles into indefinitely accelerating orbits, so that the higher energies attainable with the synchro-cyclotron as compared with the conventional cyclotron are accompanied by a decrease in the output current.

Thus far, what has been said applies equally to the synchrotron and synchro-cyclotron. The additional complications in the synchro-cyclotron arise from the fact that ions start from rest at the center of the machine and consequently may return to the center during the course of a phase oscillation. The qualitative factors determining whether or not the ion returns to the origin can be seen quite easily with the aid of our picture of the motion as an oscillation of the actual orbit about an expanding equilibrium circle. If the expansion of the equilibrium circle is greater than the maximum inward swing of the radius of an ion in a phase stable orbit during the time when its phase becomes negative, the ion will not return to the center; otherwise it will. If it does not return to the center on the first phase oscillation, it will not do so on any subsequent oscillation because of the continual expansion of the equilibrium radius. Hence, whether or not the ion is caught and indefinitely accelerated is determined entirely during the first phase oscillation. Normally the first phase oscillation will not carry the ion far from the origin so that the capture process is determined only by the character of the oscillations near the origin.

We shall show that the phase at which an ion starts is not subject to external control and that therefore the catching efficiency is determined, for either fixed applied dee voltage or fixed rate of frequency modulation, by the equilibrium phase. If the equilibrium phase is too great, then the catching efficiency is limited by the narrow range of phase stability. On the other hand, if the equilibrium phase is too small, the capture efficiency is limited by the fact that ions return to the center of the machine. There exists therefore an intermediate value of the equilibrium phase for which the capture efficiency is a maximum.

### 3. THE EQUILIBRIUM ORBIT

The first quantity that we shall need to know is the energy gain per turn required to keep an ion in the equilibrium orbit. In this orbit the fractional change of energy, $\Delta E/E_s$ corresponding to a given fractional change of rotation frequency, $\Delta\omega/\omega_s$ is given by (A–15) as

$$-K\Delta E/E_s = \Delta\omega/\omega_s,$$

where $E$ is the total energy of the ion including its rest energy, $\omega$ is the frequency of rotation,

$$K = 1 + nc^2/v^2(1-n) \tag{1}$$

with $c$ the velocity of light, $v$ the velocity of the ion and

$$n = -r\partial H/H\partial r. \tag{2}$$

The subscript $s$ denotes those values of the quantities that correspond to exact resonance with the applied frequency. $H(r)$ is the vertical component of the magnetic field in the median plane at a radius $r$. Now, since the change in frequency per turn is $(2\pi/\omega_s)d\omega_s/dt$, the required energy gain per turn is

$$\Delta E/E_s = -(2\pi/K\omega_s^2)d\omega_s/dt$$

and is also equal to $eV\sin\varphi_s$, where $\varphi_s$ is the equilibrium phase and $V$ is the maximum possible energy gain per turn. (See A for exact definitions.) Hence the equilibrium phase is given by

$$eV\sin\varphi_s/E_s = -(2\pi/K\omega_s^2)d\omega_s/dt. \tag{3}$$

Thus $eV\sin\varphi_s$ is the average energy gain per turn of an ion in a phase stable orbit, and $\rho = \sin\varphi_s$ represents the ratio of the average to the maximum possible energy gain per turn for such an ion.

Because the catching process is determined only by the motion near the origin, it is possible to express $K$ with the aid of a power series for the magnetic field $H$, retaining terms only to the second order in $r$. With $H$ satisfying Laplace's equation, there can be no linear term in the expansion so we may write

$$H = H_0(1 - hr^2/2), \tag{4}$$

where $H_0$ is the field at the center. From this we obtain $n \simeq hr^2$, and since $n$ is small near the center we have

$$K = 1 + nc^2/v_s^2 = 1 + hc^2/\omega_s^2, \tag{5}$$

since $v_s = r\omega_s$.

This constancy of $K$ occurs strictly only for parabolic fields, and in most cyclotrons deviations from the parabolic field occur even for moderately small values of the radius so that $K$ has a higher order dependence on the radius. During the first phase oscillation, however, as has already been pointed out, these deviations are not generally important so that $K$ may be regarded as constant during the catching process. In Appendix II, the phase equation is derived for the case that $K$ is not constant, and limitations on the deviation of $H$ from parabolic form for validity of our results are discussed.

### 4. PHASE OSCILLATIONS

The oscillatory motions resulting from deviations of an ion from the equilibrium orbit are discussed in detail in A; we shall merely quote here the results which are needed in the solution of the catching problem.

The equation of motion of the phase is given by Eq. (A–17) as

$$\frac{d}{dt}\left(\frac{E_s}{\omega_s^2 K}\frac{d\varphi}{dt}\right)+\frac{eV}{2\pi}\sin\varphi=\frac{eV}{2\pi}\sin\varphi_s.$$

With the neglect of damping, a first order equation may be derived from the above by integration and is given by Eq. (A–18). However, the constant of integration there has been expressed in terms of $\varphi_m$, the maximum value of the phase during the phase oscillation. For our purpose it is more convenient to express it in terms of $\varphi_0$, the initial value of the phase $\varphi$, and $\dot{\varphi}_0$, the initial value of $\dot{\varphi}=d\varphi/dt$. We also approximate $E_s$ by $Mc^2$ and approximate $K$ by (5), thus obtaining

$$\dot{\varphi}^2=\dot{\varphi}_0^2+\frac{eV}{\pi Mc^2}(\omega_s^2+hc^2) \times[\cos\varphi-\cos\varphi_0+(\varphi-\varphi_0)\sin\varphi_s]. \quad (6)$$

It should be noted that according to the definition of the phase given in (A-7), $\dot{\varphi}$ is just the difference between the frequency of rotation and the applied frequency,

$$\dot{\varphi}=\omega-\omega_s. \quad (7)$$

## 5. RANGE OF PHASE STABILITY; PENDULUM MODEL

We shall be interested here primarily in the behavior of the large phase oscillations since these are the ones which may lead to loss of particles. In A it is shown that the equation of motion of the phase is exactly the same as that of a pendulum acted upon by a restoring torque due to gravity and also by a constant torque of such magnitude that the position of stable equilibrium of the pendulum is at $\varphi=\varphi_s$ instead of at $\varphi=0$. Such a pendulum will execute stable oscillations about $\varphi=\varphi_s$ unless it reaches the point of unstable equilibrium $\varphi=\pi-\varphi_s$, in which case it will go into accelerated circular motion. For any given starting phase $\varphi_0$ there will then be a maximum starting angular velocity, $\dot{\varphi}_0$ which will just bring the pendulum to the position $\varphi=\pi-\varphi_s$. For larger values of $\dot{\varphi}_0$ the motion is unstable. The maximum $\dot{\varphi}_0$ corresponding to phase stability can be obtained quantitatively from Eq. (6) by setting $\dot{\varphi}=0$ and $\varphi=\pi-\varphi_s$. This gives just the condition that $\varphi$ reach its maximum amplitude at the point of unstable equilibrium. The result may be written

$$(\dot{\varphi}_0)^2_{\max}=\left(\frac{eV}{\pi Mc^2}\right)(\omega_s^2+hc^2) \times[\cos\varphi_0+\cos\varphi_s-(\pi-\varphi_s-\varphi_0)\sin\varphi_0] \quad (8)$$
$$=\frac{eV}{\pi Mc^2}(\omega_s^2+hc^2)F_1(\varphi_0,\varphi_s).$$

## 6. DETERMINATION OF CATCHING EFFICIENCY

With the results given above we are ready to calculate the catching efficiency with the neglect of return of ions to the center. According to the definition given in the introduction this is determined by the range of times in which ions can start in orbits which are phase stable and which do not return to the origin. Since the rate of decrease of applied frequency is practically constant over the first phase oscillation, this range of times is determined by the corresponding range of applied frequencies through the equation

$$\Delta t=\Delta\omega_s/|d\omega_s/dt|,$$

where $\Delta\omega_s$ is the range of $\omega-\omega_s$ with which an ion can start into an indefinitely accelerating orbit. If $\tau$ is the period of repetition of the cycle of frequency modulation, the efficiency is given by

$$\epsilon=\Delta t/\tau=\Delta\omega_s/\tau|d\omega_s/dt|. \quad (9)$$

The quantity $\tau|d\omega_s/dt|$ is independent of the rate of rotation of the condenser which modulates the frequency; hence for a given machine it is constant and the efficiency is proportional only to $\Delta\omega_s$.

The range of phase stability has already been calculated. According to Eq. (8) the maximum initial discrepancy between rotation and applied frequency consistent with stable motion is $\pm(\dot{\varphi}_0)_{\max}$. Consequently, since $\Delta\omega_s$ is the total range of $\omega-\omega_s$ during which capture in phase stable orbits is possible,

$$\Delta\omega_s=2(\dot{\varphi}_0)_{\max} =2\left[\frac{eV}{\pi Mc^2}(\omega_s^2+hc^2)F_1(\varphi_0,\varphi_s)\right]^{\frac{1}{2}}. \quad (10)$$

This formula depends upon the starting phase $\varphi_0$. In Appendix I, a discussion of the way in which ions start in the cyclotron is given and it

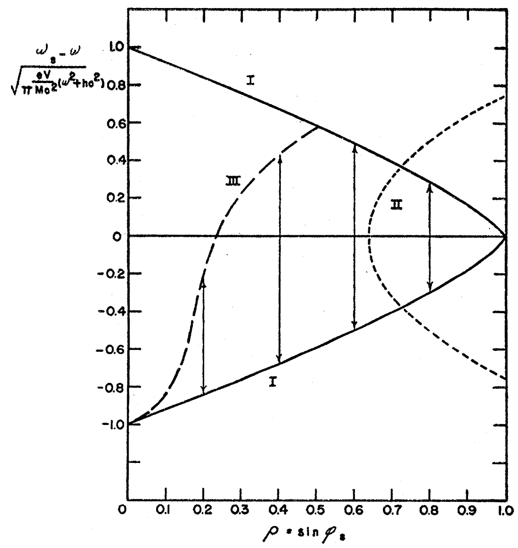


FIG. 1. Starting conditions for capture into stable orbits. Points lying between the two branches of curve I correspond to ions captured into phase stable orbits. Points lying within curve II correspond to ions which are never decelerated. Points lying below curve III correspond to ions which do not return to the origin, while points lying above curve III return to the origin during the first phase oscillation. The vertical arrows designate the range of frequencies for several values of $\rho$ for which ions are captured into phase orbits and do not return to the origin. The ordinate should be

$$\frac{\omega_s-\omega}{\sqrt{\frac{eV}{\pi Mc^2}(\omega_s^2+hc^2)}}$$

is shown that the starting phase is very close to 90°. Thus we shall replace $\varphi_0$ by 90° in all subsequent work.

In Fig. 1, curve I, is plotted the maximum deviation of $\omega_s$ from $\omega$ consistent with phase stability as a function of $\rho=\sin\varphi_s$ assuming $\varphi_0=90°$. It can there be seen that the range of starting frequencies corresponding to phase stability is zero for $\varphi_s=90°$, the reason being, of course, that with $\varphi_s$ close to 90°, the motion is always near the limit of phase stability. As $\varphi_s$ decreases, the range of admissable starting frequencies increases reaching its maximum at $\varphi_s=0°$. It is also noteworthy that the range of frequencies leading to stable phase oscillations is proportional to $[V(\omega_s^2+c^2h)]^{\frac{1}{2}}$, which shows that the range may be increased either by increasing the dee voltage or by increasing the curvature of the magnetic field at the origin.

Equation (10) could be used to calculate the efficiency if one could neglect the fact that for small $\varphi_s$ some of the ions will return to the origin and thus be lost.

## 7. RETURN OF IONS TO THE ORIGIN

The question of whether or not a particle returns to the origin can be studied qualitatively with the aid of the pendulum model for the motion. In our case the pendulum starts at a phase of 90°. If the equilibrium phase is also close to 90°, a stable oscillation must remain within positive phases even in the most negative part of its swing so that the particle will never be decelerated. Hence there will be no possibility of loss of ions by return to the origin for large values of $\varphi_s$. On the other hand the efficiency will be low for large $\varphi_s$ because of the small range of phase stability, and it will certainly be advantageous to reduce $\varphi_s$ until loss of ions by return to the origin begins to outweigh the gain in range of phase stability.

As $\varphi_s$ is reduced the maximum downward swing of the pendulum corresponding to phase stability comes closer and closer to $\varphi=0$; and for small enough $\varphi_s$, negative phases will be reached in which ions suffer deceleration. The value of $\varphi_s$ for which the pendulum just reaches $\varphi=0$ can be calculated by finding the condition that $\dot{\varphi}=0$ at $\varphi=0$. If these values are substituted into Eq. (6) we obtain, for $\varphi_0=\pi/2$

$$\dot{\varphi}_0=\pm\left[\frac{eV}{\pi Mc^2}(\omega_s^2+hc^2)(\tfrac{1}{2}\pi\sin\varphi_s-1)\right]^{\frac{1}{2}}. \quad (11)$$

In Fig. 1, curve II, $\pm[\frac{1}{2}\pi\sin\varphi_s-1]^{\frac{1}{2}}=\pm[F_2(\varphi_s)]^{\frac{1}{2}}$ is plotted against $\rho=\sin\varphi_s$. Curve II crosses curve I at about $\rho=0.7(\varphi_s=45°)$ indicating that below this value of $\varphi_s$ some phase stable orbits involve deceleration of the ions. The fact that curve II crosses the $\rho$-axis at $\rho=0.63$ $(\varphi_s=39°)$ means that for smaller values of $\varphi_s$, all phase stable orbits involve some deceleration.

As $\varphi_s$ decreases still further, the maximum amplitude of the negative swing increases and particles suffer correspondingly more deceleration. There will finally be reached a critical value of $\varphi_s$ for which some particles within the range of phase stability begin to be decelerated all the way back to the origin so that they are lost. Calculations which will be discussed later indicate that this critical $\varphi_s$ is about 30°. Below this value (for constant dee voltage) more and more particles are lost by deceleration to the origin, so many in fact that the gain in range of phase stability is more than compensated by the increase in number of ions lost by return to the origin. Hence the maximum efficiency at constant dee voltage is at $\varphi_s=29°(\rho\cong 0.5)$. The range which can be caught below this point is bounded by curve III of Fig. 1. It may be noted that it goes to zero rapidly as $\rho$ approaches zero.

It is helpful to obtain a more detailed qualitative understanding of the factors determining the shape of curve III. As has already been stated, a particle will miss the origin only if the maximum inward swing of the radius occurring when the phase goes negative is less than the amount by which the equilibrium radius has expanded while the swing is taking place. The rate of expansion of the equilibrium radius is proportional to $d\omega_s/dt$ which, according to Eq. (3) is proportional to $\sin\varphi_s$. The total expansion of the equilibrium radius during the time interval $t_1$ in which the swing takes place will therefore be proportional to the product $t_1 \sin\varphi_s$.

Both $t_1$ and the maximum amplitude of the inward swing will depend on the starting phase and on the starting value of $\dot{\varphi}=\omega-\omega_s$. We are interested mainly in the character of the motion for small $\varphi_s$ since we have already seen that for large $\varphi_s$ the particle is not even decelerated so that the question of return to the origin does not arise. From the pendulum model it can be seen that when $\varphi_s$ is small a stable oscillation starting out at $\varphi_0=90°$ must reach a maximum negative phase somewhere between a little less than $-90°$ and a little less than $-180°$. Hence

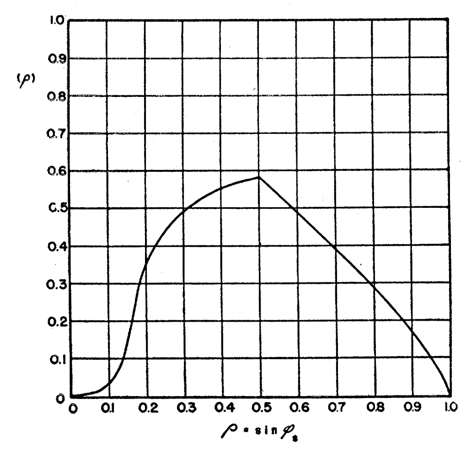


FIG. 2. The function $L(\rho)$ determining the capture efficiency as a function of equilibrium angle for constant dee voltage.

the amount of deceleration experienced by the particle and the maximum inward swing of the radius will depend strongly neither on $\varphi_s$ nor on $\dot{\varphi}_0$ as long as $\varphi_s$ is small. For qualitative purposes the amplitude of the swing may be regarded as constant. Hence the question of whether or not the particle misses the origin depends on whether the expansion of the equilibrium radius is greater than this constant. It is therefore adequate to consider only the manner in which the product $t_1 \sin\varphi_s$ depends on the variables involved.

Now $t_1$ is determined principally by the initial value of $\dot{\varphi}$. If, for example, a particle starts with negative $\dot{\varphi}_0$, it will reach the negative phases more rapidly than if $\dot{\varphi}$ is initially positive, and it will therefore begin to be decelerated sooner. The more negative the initial value of $\dot{\varphi}$, the greater is the deceleration and the greater is the likelihood of striking the origin. Conversely, the most favorable catching conditions occur with the largest possible positive initial $\dot{\varphi}$ consistent with phase stability. For such particles will come close to the point of unstable equilibrium at $\varphi=\pi-\varphi_s$, where they will spend a long time thus allowing the equilibrium radius to expand a great deal before their phase swings negative.

As $\varphi_s$ is reduced, it is necessary that $t_1$ increase if the product $t_1\sin\varphi_s$ is to remain large enough to result in an expansion of the equilibrium orbit which exceeds the maximum inward swing resulting from the phase oscillation. As has already been stated, at $\varphi_s=30°$ some particles begin to be decelerated back to the origin; these will be the particles with the most negative $\varphi_0$ consistent with phase stability. At $\varphi_s=13°(\rho=0.225)$ calculations show that particles with $\dot{\varphi}_0=0$ will reach the origin and below this value only particles with positive $\dot{\varphi}_0$ can be caught. As $\varphi_s$ approaches zero the range that misses the origin is progressively narrowed to an ever smaller range of positive $\dot{\varphi}_0$ near the limit of phase stability. This explains the approach of curve III to zero as $\varphi_s$ approaches zero.

### 8. RESULTS FOR EFFICIENCY

The actual value of the efficiency taking account of return of ions to the center is calculated in Appendix C. The results will merely be quoted here:

### (a) Constant Dee Voltage

If one wishes to compare efficiencies at constant dee voltage when $\rho=\sin\varphi_s$ is varied by varying the rate of frequency modulation ($\rho$ is proportional to rate of rotation of the condenser), the efficiency is most conveniently expressed in the following form:

$$\epsilon=\frac{2}{|\tau d\omega_s/dt|}[eV(\omega_0^2+hc^2)/\pi Mc^2]^{\frac{1}{2}}L(\rho). \quad (12)$$

The function $L(\rho)$ is plotted in Fig. 2.

### (b) Constant Rate of Frequency Modulation

For the case where the rate of frequency modulation is held fixed and $\varphi_s$ is varied by varying the dee voltage, we write $V_s=V\sin\varphi_s$ so that $V_s$ is the actual voltage gain per turn and is determined by the rate of frequency modulation through Eq. (3). It is thus a constant for constant rate of frequency modulation and from (12) it follows that

$$\epsilon=\frac{2}{|\tau d\omega_s/dt|}[eV_s(\omega_0^2+hc^2)/\pi Mc^2]^{\frac{1}{2}} \times L(\rho)/\rho^{\frac{1}{2}}. \quad (13)$$

In Fig. 3, $L(\rho)/\rho^{\frac{1}{2}}$ is plotted against $\rho$. It should be noted that the maximum efficiency now occurs at $\rho$ in the neighborhood of 0.34. It must be remembered when considering this formula that the operation of the ion source is affected by the dee voltage so that the output current will depend upon the dee voltage both through the variation of the capture efficiency and through the ion source efficiency.

It should be noted that the maximum efficiency available with a particular rotating condenser is proportional to the square root of the dee voltage or alternatively to the square root of the rate of frequency modulation. If for a particular machine, one is limited as to the voltage that can be applied to the dees, then the maximum efficiency is obtained by adjusting the rate of rotation of the condenser so that the equilibrium phase is about 30°. It may be remembered that according to Eq. (3), the actual average energy gain per turn, $eV\sin\varphi_s$, is determined solely by $d\omega_s/dt$ and does not depend on the dee voltage. Hence the statement that $\varphi_s=30°$ means that the average energy gain per turn is half the maximum available with the given dee voltage. On the other hand, if for a particular machine, one is limited as to the maximum rate of rotation for the condenser, then the maximum efficiency is obtained by adjusting the dee voltage so that the equilibrium phase is about 20°, or that the ratio of the average to the maximum available energy gain per turn is 0.34.

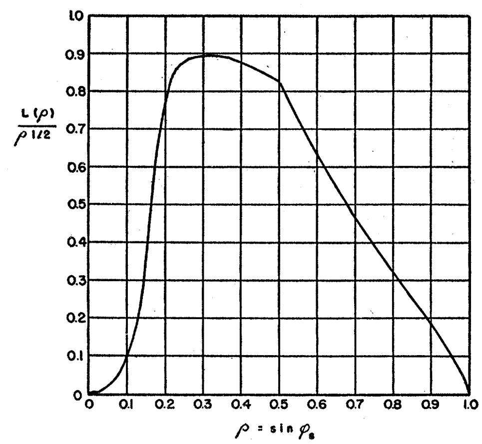


FIG. 3. The function $L(\rho)/\rho^{\frac{1}{2}}$ determining the capture efficiency as a function of equilibrium angle for constant rate of frequency modulation.

### (c) Efficiency with Different Ions

In comparing the efficiency of the same machine for operation with different ions, we may consider two cases; operation with the same rotating condenser and oscillator circuit for both ions and operation with different rotating condensers for the two ions. In the first case $|\tau d\omega_s/dt|$ is the same for both ions, and since the applied frequency must be the same, the field $H$ must be adjusted so that $\omega_0=eH/Mc$ is the same for both. Hence with the same dee voltage and equilibrium phase in the two cases, we see that the efficiency is proportional to $(e/M)^{\frac{1}{2}}$. Thus

$$\epsilon\alpha(e/M)^{\frac{1}{2}}. \quad (14)$$

Hence the efficiency depends only on the ratio $e/M$ for the two ions. The efficiency should therefore be the same for alpha-particles and deuterons but higher by a factor of $\sqrt{2}$ for protons than for these ions.

In the second case where different condensers are used, it is probably fair to assume that the condensers will be built so as to give a change of frequency with angle of rotation proportional to the total shift in frequency required in the acceleration. Now the fractional change in frequency during the acceleration is equal to the fractional change in energy during the acceleration. If the ions are accelerated to the same radius, then the latter quantity is given for non-relativistic energies by $H^2e^2r^2/2M^2c^4$, where $r$ is the radius attained. Since the frequency is proportional to $eH/Mc$, it follows that $|\tau d\omega_s/dt|$ will be proportional to $H^3e^3r^2/2M^3c^5$, and the efficiency to

$$(M/e)^{5/2}[(eH/Mc)^2+hc^2]^{\frac{1}{2}}, \qquad (15)$$

for the same dee voltage and equilibrium phase angle. Again the efficiency depends only on $e/M$, but in this case is proportional to $(M/e)^{5/2}$, if the decrease in magnetic field with radius is primarily responsible for the decrease in angular velocity, and to $(M/e)^{\frac{3}{2}}$ if the increase of mass with velocity is primarily responsible for the decrease in angular velocity with radius. The efficiency for $\alpha$-particles and deuterons is again the same, but is lower in this case for protons. It should be noted in this comparison that the actual output current will depend not only on the catching efficiency but on other factors (source efficiency, loss of ions by scattering and striking the dees, deflector efficiency, etc.) as well, which will in general be different for different ions, even in the same machine.

## 9. COMPARISON WITH EXPERIMENT

In this concluding section we shall attempt to compare our results with the small amount of experimental data on the California 37-inch synchro-cyclotron and the California 184-inch synchro-cyclotron.

### (a) California 37-inch Synchro-Cyclotron

This machine was intended as a model for the 184-inch synchro-cyclotron. In order to simulate the large relativistic increase of ion mass occurring in the larger machine, the magnetic field was designed to decrease approximately parabolically by a factor of about 15 percent between the center of the magnet and the radius of the final orbit. In this machine the magnetic field is very accurately parabolic near the origin and from field measurements there was obtained the value $h=0.000138\ \text{cm}^{-2}$.

While no systematic study was made of the absolute value of the catching efficiency, on a typical run a value of about 2 percent was obtained. The conditions on this run were $V=23$ kev, $(-d\omega_s/dt)/\omega_0=1380\ \text{sec}^{-1}$, $\tau=0.00053$ sec

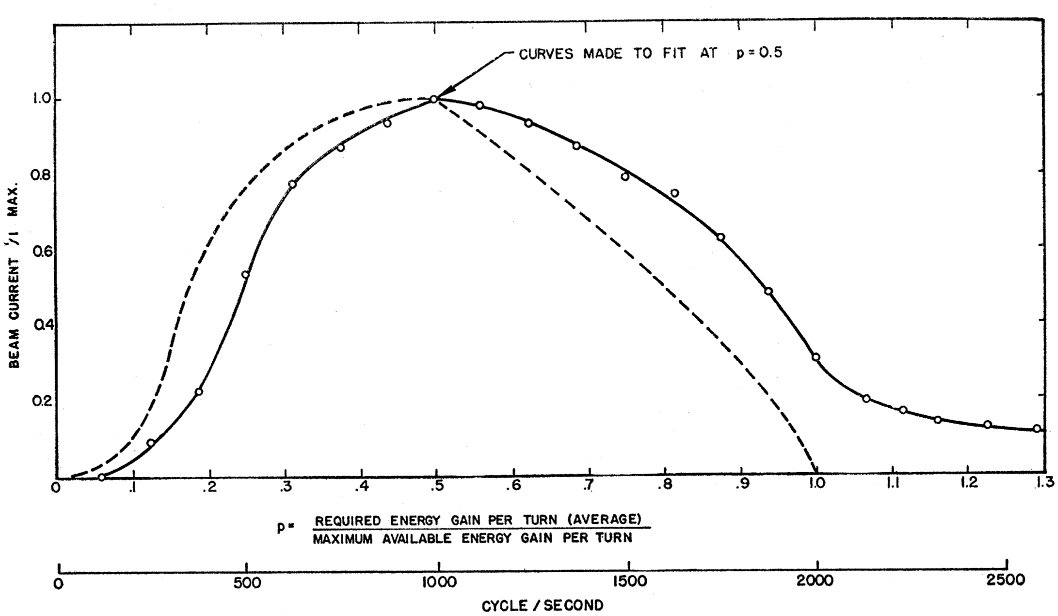


Fig. 4.

and $i_0=\omega_0/2\pi=11$ Mc/sec. At the best phase of 30°, the theoretical value for these conditions is 1.8 percent which is in satisfactory agreement with the rough experimental value.

The relative efficiency was measured systematically as a function of $\rho$ at constant voltage.[10] The results are plotted in Fig. 4; the ordinates are the efficiencies and the abcissae are $\rho=\sin\varphi_s$. Because the absolute values of the efficiencies were not known, the theoretical curve was adjusted to agree at $\varphi_s=30°(\rho=0.5)$. The agreement between theory and experiment is quite satisfactory from $\rho=0$ to $\rho=0.85$. In particular the appearance of the maximum at $\rho=0.5$ is a direct confirmation of the basic assumption of the theory that losses by return of ions to the origin limit the efficiency at small equilibrium phase angles.

It should be noted that the measured efficiency does not approach zero as $\rho$ approaches unity but instead actually has non-zero values when $\rho$ is greater than unity. This seems to indicate that particles are being indefinitely accelerated even though they do not gain enough energy per turn to allow the rate of decrease of rotation frequency to keep up with the rate of decrease of applied frequency. By pulsing the ion source and examining the output current from the cyclotron with an oscilloscope, it was determined that some of the ions which left the source during one frequency modulation cycle did not strike the probe until two or three frequency modulation cycles later. It thus appears that some ions which cannot gain enough energy per turn to remain in synchronism with the applied frequency are accelerated out to some radius. When they there fall out of step with the applied frequency they successively lose and gain energy but on the average remain at about this radius until at the appropriate instant during a succeeding frequency modulation cycle they may again be picked up and further accelerated to a high energy.

### (b) California 184-inch Synchro-Cyclotron[11]

In this machine the magnetic field is parabolic only near the center. From a radius of about 8 inches the field then drops off linearly almost to the edge of the pole faces. The radius reached during the first phase oscillation is about 9 inches as calculated for typical operating conditions so that the theory developed here is approximately valid (see Appendix II). The acceptance time for ions was measured under typical operating conditions by applying pulses of voltage to the ion source and measuring the current by means of a probe as a function of the time at which the voltage pulse was applied. From the speed of rotation of the condenser it was thus possible to obtain an experimental value for the capture efficiency.

In order to obtain the theoretical capture efficiency, one would in general have to know the dee voltage. No accurate measurements of this quantity are available at present. To obtain the maximum efficiency at a fixed dee voltage, however, it suffices to know the rate of frequency modulation only. Hence in the measurement the rate of rotation of the condensor was adjusted to give maximum yield. Under typical operating conditions for the acceleration of deuterons ($H_0=14{,}760$ gauss, $h=1.57\times10^{-5}$, $\omega_0=7\times10^7$ radians/sec; $d\omega_s/dt=8.8\times10^9$ radians/sec$^2$, $\tau=0.0114$ sec) the value 0.33 percent was obtained for the theoretical efficiency. The measured acceptance time was 35 microseconds which combined with $\tau=0.0114$ sec, yields an experimental value for the efficiency of 0.31 percent in satisfactory agreement with the predicted value.

Because of the change in the field from parabolic to linear dependence on radius at about 8 inches, it will be seen from Eq. (2) that $K$ will be a decreasing function of the radius beyond this radius. As a consequence, if the dee voltage and frequency modulation rate do not change during the acceleration period, it follows from Eq. (3) that the equilibrium phase will increase with radius. This will result in a loss of captured particles, since as the equilibrium phase increases, the range of phase stability decreases; hence ions which were initially in phase stable orbits near the limit of stability will pass into unstable orbits and be lost. This effect can be very serious if the equilibrium phase should increase to a value close to 90° since there the region of phase stability is very narrow. Actually, in this ma-

[10] J. R. Richardson, K. R. MacKenzie, E. J. Lofgren, and B. T. Wright, Phys. Rev. 69, 669L (1946).

[11] The measurements on the 184-inch cyclotron were made by a group under the direction of Duane Sewell.

chine, the dee voltage is known to rise somewhat (about 15 percent) during the acceleration period. This tends to prevent the equilibrium phase from rising quite so far. From plots of the magnetic field, the variation of dee voltage with frequency, and the (slight) variation of frequency modulation rate with frequency, it was possible to calculate the required energy gain per turn as a function of radius under typical operating conditions. The result is shown in Fig. 5. It will be noted that the rise in required energy gain per turn is not too serious. Its decrease again at large radii is due to the fact that at these radii the magnetic field again falls off more rapidly than linearly with the distance. The rise in equilibrium phase could, of course, be prevented by continuing the parabolic field dependence to the edge of the pole pieces. The resultant gain in output current, however, would then be obtained at a loss in the maximum energy to which ions could be accelerated, since the initial rate of parabolic decrease is determined by vertical focussing considerations and cannot be appreciably lessened without the sacrifice of particles due to their striking the dees.

### ACKNOWLEDGMENTS

It is a pleasure to express our gratitude to Professor E. O. Lawrence for his encouragement of this work. We are also very grateful to Professor Robert Serber, Professor E. M. McMillan, Professor Robert Thornton, and many other members of the staff of the Radiation Laboratory who contributed many helpful discussions and suggestions. This work was carried out under the auspices of the Atomic Energy Commission under Contract No. W-7405-Eng-48 with the University of California.

### APPENDIX I. MOTION DURING FIRST FEW TURNS

During the first few turns several of the approximations leading to the phase equation (A-17) are not valid. In the first place the motion is not close enough to circular that it can be described as a circle with small deviations. Secondly, the energy gain per turn is not equal to $eV\sin\varphi$, as this requires that the diameter of the orbit be greater than the spatial extent of the accelerating field.

Just after the particle starts, the motion consists of an expanding spiral within the region of the accelerating electric field. By the time the orbit becomes appreciably larger than the accelerating region, however, the above approximations will apply. It is therefore necessary to integrate the equations of motion without these approximations only over the first few turns. During this time $\omega_s$, $\omega$, and $H$ change so little that they may be set equal to their initial values. The equations of motion then take the form

$$M\ddot{y}=(eH_0/c)\dot{x}-eE_0(\sin\omega_s t+\alpha), \qquad (16)$$

$$M\ddot{x}=-(eH_0/c)\dot{y}. \qquad (17)$$

Here $E_0$ is the electric field at the center, which may also be regarded as essentially uniform in this region, and $\alpha$ represents a d.c. bias which may be applied to the dees to improve operation of the oscillator. The quantity $\alpha$ is seldom larger than $\frac{1}{2}$.

The initial conditions are $x=\dot{x}=y=\dot{y}=0$ at $t=t_0$. The solution may be written in the form

$$y=-\frac{eE_0}{M\omega_0^2}\left\{\alpha-\alpha\cos\omega_0(t-t_0)-\frac{\omega_0}{\omega_0+\omega_s}\cos\omega_s t_0\cos\omega_0(t-t_0)\right\}+\frac{2eE_0}{M(\omega_0^2-\omega_s^2)}\sin\left[(\omega_s-\omega_0)\left(\frac{t-t_0}{2}\right)\right]\times\cos\left[\omega_s t+\frac{(\omega_0-\omega_s)}{2}(t-t_0)\right], \qquad (18)$$

$$x=-\omega_0\int_{t_0}^{t} y\,dt. \qquad (19)$$

Since $\omega_s$ is always close to $\omega_0$, it is clear that the last term on the right will become large; that is, it is the "resonant" term. After only four turns, for example it is about 25 times larger than the remaining terms. This means that the main part of the motion corresponds to an expanding spiral with an almost constant fractional increase of radius per turn. As soon as the radius reaches an appreciable size the relative increase becomes small and the motion becomes nearly circular. The remaining terms contribute only small perturbations causing the motion to differ

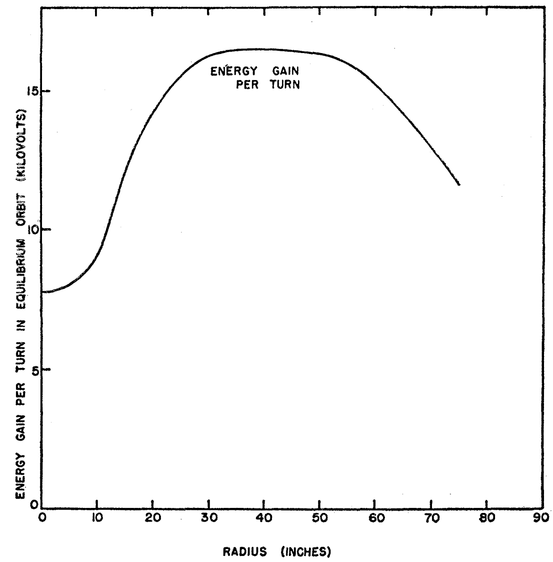


FIG. 5. Variation of energy gain per turn as a function of radius for 184-inch synchro-cyclotron.

slightly from circularity, but in a way which cancels out over a turn. When $(\omega_0-\omega_s)(t-t_0)/2$ is small, as it will be during the first few turns, $y$ is proportional to $\cos\omega_s t$. This means that the ion will cross $y=0$ when $\omega_s t=(n+\frac{1}{2})\pi$ or when the accelerating force is a maximum. Therefore the phase as defined in A is $\pi/2$ and the effect of the accelerating force is always to start ions in a phase close to that at which they gain energy at the maximum rate.

## APPENDIX II. PHASE EQUATION AT SMALL RADII

In A the phase equation was derived with the aid of the approximation that the fractional change of energy, momentum, and radius occurring during a phase oscillation is small. Because the particle starts at zero radius and from rest in the synchro-cyclotron, this approximation will break down at the start just when the catching process is taking place. In this section we shall derive the phase equation without this approximation and show that for a parabolic variation of magnetic field with radius, the exact phase equation is the same as the approximate equation obtained in A. If the magnetic field deviates from a parabolic form the phase equation becomes much more complex and, as we shall see, no longer possesses an elementary first integral analagous to (5). The catching problem can then be solved only by numerical integration. Fortunately, the magnetic field is nearly parabolic near the origin in most cyclotrons although it does deviate from parabolic form at larger radii. In order to decide whether these deviations will affect the catching process it is necessary to estimate the range of radii covered during the first phase oscillation. This will be done after the derivation of the phase equation. An approximation will also be given for treating the case in which the first phase oscillation carries the particle into non-parabolic regions of the magnetic field.

We begin with the well known relation between ionic rotation frequency and energy

$$\omega=ecH(r)/E. \tag{20}$$

From this we calculate the energy gain per turn which is $\Delta E=2\pi(dE/dt)/\omega$. Differentiation of (20) yields

$$d\omega/\omega dt=-dE/Edt+(\partial H/H\partial r)dr/dt. \tag{21}$$

To eliminate $dr/dt$, we write

$$r=v/\omega=c[1-(Mc^2/E)^2]^{\frac{1}{2}}/\omega, \tag{22}$$

so that

$$dr/rdt=-d\omega/\omega dt+\frac{(Mc^2/E)^2}{1-(Mc^2/E)^2}(dE/Edt).$$

Equation (21) then yields

$$-(d\omega/\omega dt)(1+r\partial H/H\partial r)$$
$$=(dE/Edt)\left[1-(r\partial H/H\partial r)\frac{(Mc^2/E)}{1-(Mc^2/E)^2}\right].$$

Writing $(Mc^2/E)^2=1-(v/c)^2$ and using (20), we obtain

$$\frac{\Delta E}{E}=\frac{2\pi}{K\omega^2}\frac{d\omega}{dt}, \tag{23}$$

where $K$ is defined in Eq. (5). To obtain the phase equation we express $\omega$ with the aid of Eq. (7) and write $\Delta E=eV\sin\varphi$:

$$\frac{2\pi}{\omega^2}\frac{d^2\varphi}{dt^2}+\frac{KeV}{E}\sin\varphi=-\frac{2\pi}{\omega^2}\frac{d\omega_s}{dt}. \tag{24}$$

If the magnetic field is parabolic so that $K$ is constant, the phase equation may (with the aid of a little algebra) be shown to be exactly the same as Eq. (A–17).

If $K$ is not a constant, it may, in general, be expressed as a power series in $r$ with the aid of the expression for $H$ as a function of $r$. This yields an expression of the form, $K=K_0+K_1r+K_2r^2/2+\ldots$; similar expansions can then be obtained for $E$ and $\omega$. These latter, however, will normally vary so slowly with $r$ near the origin that the neglect of this variation over the first phase oscillation is permissable. However, if $K$ varies appreciably over this distance, it will be necessary to eliminate $r$ in terms of $\omega=\omega_s+d\varphi/dt$, with the aid of Eqs. (20) and (22). This will introduce terms involving $(\omega_s+d\varphi/dt)\sin\varphi$ into the phase equation and then a first integral of this equation can no longer be obtained. If $K$ does not change too much during the first phase oscillation, a rough value of the efficiency may be obtained simply by averaging $K$ over the range of radii covered during the first phase oscillation. Since the efficiency does not depend critically on K, such a procedure will probably be adequate for a rough estimate even if K changes by as much as a factor of 2 or 3 during the first phase oscillation.

In order to know whether $K$ varies much during the first phase oscillation and to correct for such variation by averaging $K$, it is necessary to estimate the range of radii covered during the first phase oscillation. To obtain this estimate, we first assume that $H$ varies parabolically and see what radius the particle would reach with such a variation. In the first phase oscillation the non-relativistic approximation for the energy can be used;

$$E=Mc^2+\tfrac{1}{2}Mv^2=Mc^2+\tfrac{1}{2}M\omega^2r^2 \simeq Mc^2(1+\omega_0^2r^2/2c^2), \quad (25)$$

where $\omega_0$ is the angular rotation frequency at the center. With the expansion (4) for $H$, we get, up to second order in $r$, with $\omega=\omega_s+d\varphi/dt$

$$\frac{d\varphi}{dt}=\omega_0\left[1+\left(h+\frac{\omega_0^2}{c^2}\right)\frac{r^2}{2}\right]-\omega_s. \quad (26)$$

Now the range of radii covered will depend on the initial value of $d\varphi/dt$; as a typical case we shall take $(d\varphi/dt)_0=0$, or $\omega_0=\omega_s$. Furthermore, for small $\varphi_s$, in which we are most interested, $\omega_s$ will not change much during a phase oscillation and hence a rough estimate may be obtained by setting $\omega_s=\text{constant}=\omega_0$.

We eliminate $dt$ in (26) by differentiating (25) and setting $(2\pi/\omega_s)dE/dt=M\omega_0^2r\,dr/dt=eV\sin\varphi$. We obtain

$$\frac{eV}{2\pi M\omega_0^2}\frac{d\cos\varphi}{dr}=\left(h+\frac{\omega_0^2}{c^2}\right)\frac{r^3}{2}.$$

Integration over $r$ with the boundary condition, $\varphi_0=\pi/2$, yields

$$\cos\varphi=\frac{2\pi M\omega_0^2}{eV}\left(h+\frac{\omega_0^2}{c^2}\right)\frac{r^4}{8}.$$

The maximum value of $r$ occurs where $\cos\varphi=1$, whence it is

$$r_{\max}=\left[\frac{4eV}{\pi M\omega_0^2(h+\omega_0^2/c^2)}\right]^{\frac{1}{4}}. \quad (27)$$

## APPENDIX III. CALCULATION OF EFFICIENCY

As explained in Section 2, our objective is to solve for the range of $\Delta\omega_s$ corresponding to phase stable motion which never returns to the origin. The range of $\Delta\omega_s$ corresponding to phase stability is already given in Eq. (10). Let us refer to this range as

$$(\Delta\omega_s/2\omega_s)_1=(\dot{\varphi}_0)_1.$$

Now it has been pointed out that there will be a critical value of $\dot{\varphi}_0$ for which the ion barely returns to the origin. Let us call this value $(\dot{\varphi}_0)_2$. It is convenient to express $(\dot{\varphi}_0)_2$ in terms of $(\varphi_0)_1$ through a dimensionless parameter $\lambda$ defined as follows

$$(\dot{\varphi}_0)_2=\lambda(\dot{\varphi}_0)_1 =\lambda[(eV/\pi Mc^2)(\omega_0^2+hc^2)F_1(\varphi_0,\varphi_s)]^{\frac{1}{2}}, \quad (28)$$

where it will be noted that $\lambda$ may take on values ranging from $-1$ to $+1$. For $\lambda=-1$, the range of $(\dot{\varphi}_0)$ for phase stability and that for particles failing to return to the origin just coincide. This will occur at the intersection of curves I and III in Fig. 1. As $\lambda\to+1$, only particles with positive $(\dot{\varphi}_0)$ near the limit of phase stability will miss the origin.

The efficiency can then be written from Eq. (28) as

$$\epsilon = \frac{(1-\lambda)}{|\tau d\omega_s|dt|}\left[\frac{eV}{\pi Mc^2}(\omega_0^2+hc^2)F_1(\varphi_0, \varphi_s)\right]^{\frac{1}{2}}$$

$$= \frac{2}{|\tau d\omega_s|dt|}\left[\frac{eV}{\pi Mc^2}(\omega_0^2+hc^2)\right]^{\frac{1}{2}} L(\varphi_0, \varphi_s) \quad (29)$$

where

$$L(\varphi_0, \varphi_s) = \tfrac{1}{2}(1-\lambda)[F_1(\varphi_0, \varphi_s)]^{\frac{1}{2}}. \quad (30)$$

Since $(\dot{\varphi}_0)_2$ is the initial value of $\dot{\varphi}$ corresponding to a particle just returning to the origin, we may obtain a formula for $\dot{\varphi}$ at all times for such a particle by replacing $\dot{\varphi}_0$ in Eq. (6) by $(\dot{\varphi}_0)_2$, as given in Eq. (28). We obtain, taking $\varphi_0 = \pi/2$:

$$(\dot{\varphi})^2 = \frac{eV}{\pi Mc^2}(\omega_0^2+hc^2)[\cos\varphi + \lambda^2 \cos\varphi_s + \{\varphi - (\lambda^2+1)\pi/2 + \lambda^2\varphi_s\} \sin\varphi_s] \quad (31)$$

$$= (eV/\pi Mc^2)(\omega_0^2+hc^2)P(\varphi_1, \varphi_s, \lambda).$$

Integration of this equation yields

$$(t-t_0) = [\pi Mc^2/eV(\omega_0^2+hc^2)]^{\frac{1}{2}} \times \int_{\pi/2}^{\varphi} d\varphi/[P(\varphi, \varphi_s, \lambda)]^{\frac{1}{2}}, \quad (32)$$

where the path of integration must include all values of $\varphi$ covered by the particle in its motion. If $\dot{\varphi}_0$ is negative ($\lambda$ negative), $\varphi$ will decrease directly to a minimum value, then increase, etc. If $\dot{\varphi}_0$ is positive ($\lambda$ positive), $\varphi$ will first increase to a maximum, then decrease to a minimum, again increase, etc.

We wish to obtain the time $t_1$ required for a particle to reach its minimum radius in the first phase oscillation. This may be obtained from Eq. (32) by noting that the minimum radius is reached when $\varphi$ returns to zero from the negative side. We thus set $\varphi = 0$ as the upper limit and keep in mind the range of values of $\varphi$ actually covered in the integration. The limiting particle will have $r=0$ at $t-t_0=t_1$. From Eq. (7) we see that at $r=0$, $d\varphi/dt = \omega_0 - \omega_s = \dot{\varphi}_0 - (t-t_0)d\omega_s/dt = \dot{\varphi}_0 - t_1 d\omega_s/dt$. Expressing $\dot{\varphi}_0$ with the aid of (28) we obtain for the value of $d\varphi/dt$ at $r = r_{\min}$:

$$\left(\frac{d\varphi}{dt}\right)_{r=r_{\min}} = \lambda\left[\frac{eV}{\pi Mc^2}(\omega_0^2+hc^2)F_1(\pi/2, \varphi_s)\right]^{\frac{1}{2}} - \frac{d\omega_s}{dt}\left[\frac{\pi Mc^2}{eV(\omega_0^2+hc^2)}\right]^{\frac{1}{2}} \int_{\pi/2}^{0} \frac{d\varphi}{P^{\frac{1}{2}}}.$$

From Eq. (31) we can obtain another value of $d\varphi/dt$ at $r=r_{\min}$ by setting $\varphi=0$. We must choose the positive square root of $d\varphi/dt$ because $\varphi$ will be increasing from negative values when $r=r_{\min}$:

$$\left(\frac{d\varphi}{dt}\right)_{r=r_{\min}} = \left[\frac{eV}{\pi Mc^2}(\omega_0^2+hc^2)P(0, \varphi_s, \lambda)\right]^{\frac{1}{2}}.$$

Setting the two expressions for $(d\varphi/dt)_{r=r_{\min}}$ equal, and using Eq. (3) we obtain

$$1/2 \sin\varphi_s \int_{\pi/2}^{0} \frac{d}{[P(0, \varphi_s, \lambda)]^{\frac{1}{2}}} = [P(0, \varphi_s\lambda)]^{\frac{1}{2}} - \lambda[F_1(\pi/2, \varphi_s)]^{\frac{1}{2}}, \quad (33)$$

which implicitly defines $\lambda$ as a function of $\varphi_s$. The quantity $\lambda$ was obtained from this equation by numerical solution.