Self-Energy and Interaction Energy in Podolsky's Generalized Electrodynamics

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Using a perturbation method, explicit expressions are derived for the electromagnetic self-energy and electromagnetic interaction energy of charged particles in Podolsky's generalized quantum electrodynamics. No infinities arise in the derivation, which is justified to the first order of approximation (i.e., to terms quadratic in the charges). Two noteworthy results follow from the calculations. (a) The self-energy is finite and negative. (b) The interaction energy contains no singularities.

I. INTRODUCTION

IN a series of papers [1–4] a generalized electrodynamics has been developed in which the Lagrangian contains derivatives of the field quantities. This has led to a theory in which an extraordinary field appears in combination with the usual Maxwell-Lorentz field. This extraordinary field is associated with a neutral particle of mass $m=\hbar/ac$, where a is a natural unit of length occurring in the theory. The quantization rules have been derived and the auxiliary conditions eliminated by methods which are generalizations of those used in the earlier basic work in relativistic quantum electrodynamics.

In GE III the auxiliary conditions are used to eliminate from the wave functional the ordinary scalar potential, the longitudinal component of the ordinary vector potential, the extraordinary scalar potential, and part of the longitudinal component of the extraordinary vector potential. The consequence of the transformations, which are exact, is the explicit appearance in the wave equation of terms which can be interpreted as the electrostatic self-energy and the electrostatic interaction. The present problem is to investigate the effect of the remainder of the ordinary and extraordinary vector potentials to obtain what can be interpreted as the "transverse" self-energy and the spin and retardation interactions.

II. CALCULATION OF THE PERTURBATION ENERGY

As a starting point we use some equations obtained in GE IV. In this paper a generalization of the functional formalism of Fock,[5] leads to the following set of ordinary wave equations in **k**-space for a system of charged particles [GE IV Eqs. (2.25), (2.26), and (2.27), respectively].

$$(H-i\hbar\partial/\partial t)\psi_{00}=\sum_{j=1}^{3}\int d\mathbf{k}[G_0^*(\mathbf{k},j)\psi_{10}(\mathbf{k},j)+\tilde{G}_0^*(\mathbf{k},j)\psi_{01}(\mathbf{k},j)], \tag{2.1}$$

$$(H+\hbar ck-i\hbar\partial/\partial t)\psi_{10}(\mathbf{k},j)=G_0(\mathbf{k},j)\psi_{00}, \tag{2.2}$$

$$(H+\hbar c\tilde{k}-i\hbar\partial/\partial t)\psi_{01}(\mathbf{k},j)=-\tilde{G}_0(\mathbf{k},j)\psi_{00}, \tag{2.3}$$

where[6]

$$H=\sum_{s=1}^{n}(\boldsymbol{\alpha}_s\cdot c\mathbf{p}_s+m_sc^2\beta_s)+(1/8\pi a)\sum_s\epsilon_s^2+\sum_{s,u}{}'(\epsilon_s\epsilon_u/8\pi R)[1-\exp(-R/a)], \tag{2.4}$$

[1] B. Podolsky, Phys. Rev. 62, 68 (1942), to be referred to as GE I.
[2] B. Podolsky and C. Kikuchi, Phys. Rev. 65, 228 (1944), to be referred to as GE II.
[3] B. Podolsky and C. Kikuchi, Phys. Rev. 67, 184 (1945), to be referred to as GE III.
[4] D. J. Montgomery, Phys. Rev. 69, 117 (1946), to be referred to as GE IV.
[5] V. Fock, Physik. Zeits. Sowjetunion 6, 425 (1934).
[6] Note the typographical error in GE IV Eq. (2.16) which has a factor $\frac{1}{4}$ instead of a factor $\frac{1}{8}$ before the interaction term.

$$G_0^*(\mathbf{k}, j) = (1/2\pi)^{\frac{3}{2}}(c\hbar/2k)^{\frac{1}{2}} \sum_s \epsilon_s\beta_j(1/k)\boldsymbol{\alpha}_s\cdot\mathbf{k}\times\mathbf{e}_j \exp(i\mathbf{k}\cdot\mathbf{r}_s), \tag{2.5}$$

$$\tilde{G}_0^*(\mathbf{k}, j) = (1/2\pi)^{\frac{3}{2}}(c\hbar/2\tilde{k})^{\frac{1}{2}} \sum_s \epsilon_s\tilde{\beta}_j(1/a\tilde{k})(\boldsymbol{\alpha}_s\cdot a\mathbf{k}\times\mathbf{e}_j+\boldsymbol{\alpha}_s\cdot\mathbf{e}_j) \exp(i\mathbf{k}\cdot\mathbf{r}_s), \tag{2.6}$$

$$\mathbf{R}=\mathbf{r}_s-\mathbf{r}_u, \quad \tilde{k}=(1+a^2k^2)^{\frac{1}{2}}/a \quad \text{and} \quad \beta_j^2=\tilde{\beta}_j^2=1.$$

In the derivation of Eqs. (2.1), (2.2), and (2.3) the series expansion of the wave functional was broken off after three terms, which corresponds to limiting the investigation to processes involving one ordinary quantum and one extraordinary quantum. As in GE IV we treat the term on the right of (2.1) as a perturbation of the unperturbed Hamiltonian. At this point we depart from the procedure of GE IV because we seek an explicit expression for the perturbation energy rather than the corresponding matrix element.

In order to eliminate ψ_{10} and ψ_{01} from (2.1) we assume that as a zero-order approximation

$$(H-i\hbar\partial/\partial t)\psi_{10}(\mathbf{k}, j)=0 \quad \text{and} \quad (H-i\hbar\partial/\partial t)\psi_{01}(\mathbf{k}, j)=0. \tag{2.7}$$

From (2.2) and (2.3) we have thus

$$\psi_{10}(\mathbf{k}, j)=G_0(\mathbf{k}, j)\psi_{00}/\hbar ck \quad \text{and} \quad \psi_{01}(\mathbf{k}, j)=-\tilde{G}_0(\mathbf{k}, j)\psi_{00}/\hbar ck. \tag{2.8}$$

Introducing these terms into the right side of (2.1) gives as a first-order approximation the perturbation energy

$$U=-\sum_j \int d\mathbf{k}[G_0^*(\mathbf{k}, j)G_0(\mathbf{k}, j)/\hbar ck-\tilde{G}_0^*(\mathbf{k}, j)\tilde{G}_0(\mathbf{k}, j)/\hbar c\tilde{k}]. \tag{2.9}$$

This expression may be simplified by use of the following identities:

$$\begin{aligned}&\textstyle\sum_j \beta_j(1/k)\boldsymbol{\alpha}_s\cdot\mathbf{k}\times\mathbf{e}_j \exp(i\mathbf{k}\cdot\mathbf{r}_s)\beta_j(1/k)\boldsymbol{\alpha}_u\cdot\mathbf{k}\times\mathbf{e}_j \exp(-i\mathbf{k}\cdot\mathbf{r}_u)\\ &\quad=\textstyle\sum_j(1/k^2)(\boldsymbol{\alpha}_s\times\mathbf{k}\cdot\mathbf{e}_j)(\boldsymbol{\alpha}_u\times\mathbf{k}\cdot\mathbf{e}_j) \exp(i\mathbf{k}\cdot\mathbf{R})=(\boldsymbol{\alpha}_s\times\mathbf{k}/k)\cdot(\boldsymbol{\alpha}_u\times\mathbf{k}/k) \exp(i\mathbf{k}\cdot\mathbf{R})\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}&\textstyle\sum_j \beta_j(1/a\tilde{k})(\boldsymbol{\alpha}_s\cdot a\mathbf{k}\times\mathbf{e}_j+\boldsymbol{\alpha}_s\cdot\mathbf{e}_j) \exp(i\mathbf{k}\cdot\mathbf{r}_s)\beta_j(1/a\tilde{k})(\boldsymbol{\alpha}_u\cdot a\mathbf{k}\times\mathbf{e}_j+\boldsymbol{\alpha}_u\cdot\mathbf{e}_j) \exp(-i\mathbf{k}\cdot\mathbf{r}_u)\\ &\quad=\textstyle\sum_j(1/a^2\tilde{k}^2)(\boldsymbol{\alpha}_s\times a\mathbf{k}+\boldsymbol{\alpha}_s)\cdot\mathbf{e}_j(\boldsymbol{\alpha}_u\times a\mathbf{k}+\boldsymbol{\alpha}_u)\cdot\mathbf{e}_j \exp(i\mathbf{k}\cdot\mathbf{R})\\ &\quad=(1/a^2\tilde{k}^2)[a^2(\boldsymbol{\alpha}_s\times\mathbf{k})\cdot(\boldsymbol{\alpha}_u\times\mathbf{k})+\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u+a(\boldsymbol{\alpha}_s\times\mathbf{k}\cdot\boldsymbol{\alpha}_u+\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u\times\mathbf{k})] \exp(i\mathbf{k}\cdot\mathbf{R})\\ &\quad=[(\boldsymbol{\alpha}_s\times\mathbf{k}/k)\cdot(\boldsymbol{\alpha}_u\times\mathbf{k}/k)+(1/a^2\tilde{k}^2)(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)] \exp(i\mathbf{k}\cdot\mathbf{R}).\end{aligned} \tag{2.11}$$

In arriving at the last expression we use the known properties of the triple scalar product and the vector equation

$$(\boldsymbol{\alpha}_s\times\mathbf{k}/k)\cdot(\boldsymbol{\alpha}_u\times\mathbf{k}/k)=\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u-(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k). \tag{2.12}$$

The perturbation energy thus becomes

$$\begin{aligned}U=-\sum_{s,u}(\epsilon_s\epsilon_u/16\pi^3)\int d\mathbf{k} \exp(i\mathbf{k}\cdot\mathbf{R})[&(\boldsymbol{\alpha}_s\times\mathbf{k}/k)\cdot(\boldsymbol{\alpha}_u\times\mathbf{k}/k)(1/k^2)-(\boldsymbol{\alpha}_s\times\mathbf{k}/k)\cdot(\boldsymbol{\alpha}_u\times\mathbf{k}/k)(1/\tilde{k}^2)\\ &-(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)(1/a^2\tilde{k}^4)].\end{aligned} \tag{2.13}$$

It can be shown[7] that the three terms have the following origins. The first term is the contribution of the transverse component of the ordinary field. It is the only term present in the usual electrodynamics, in which case it gives rise to an infinite self-energy. The second term is the contribution of the transverse component of the extraordinary field. The third term is the contribution of the longitudinal component of the extraordinary field. Its magnitude has been determined by the treatment of the auxiliary conditions in GE II and GE III.

[7] To do so requires a formalism in which all vector quantities are resolved into transverse and longitudinal components rather than the formalism of GE IV. Such a development has been carried out. It also leads to Eq. (2.13).

Using (2.12) and rearranging terms, it follows that

$$U=-\sum_{s,\,u}(\epsilon_s\epsilon_u/16\pi^3)\Big\{\int d\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{R})\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u(1/k^2-1/\tilde{k}^2)$$
$$\int d\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{R})(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)(1/k^2-1/\tilde{k}^2)$$
$$-\int d\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{R})(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)(1/a^2\tilde{k}^4)\Big\}. \quad (2.14)$$

To evaluate the integrals we use polar coordinates in **k**-space, taking **R** as the direction of the polar axis. The volume element is $k^2\sin\theta dkd\theta d\varphi$, where θ and φ are the polar and azimuth angles, respectively. As $\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u$ is independent of **k**, it is unaffected by the integrations. The factor $(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)\times(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)$ however, requires more careful attention. Introducing α_x, α_y, and α_z, the well-known Dirac matrices, and

$$k_x/k=\sin\theta\cos\varphi, \qquad k_y/k=\sin\theta\sin\varphi, \qquad k_z/k=\cos\theta$$

we have

$$(\boldsymbol{\alpha}_s\cdot\mathbf{k}/k)(\boldsymbol{\alpha}_u\cdot\mathbf{k}/k)=\alpha_{sx}\alpha_{ux}\sin^2\theta\cos^2\varphi+\alpha_{sy}\alpha_{uy}\sin^2\theta\sin^2\varphi+\alpha_{sz}\alpha_{uz}\cos^2\theta$$
$$+(\alpha_{sx}\alpha_{uy}+\alpha_{sy}\alpha_{ux})\sin^2\theta\sin\varphi\cos\varphi+(\alpha_{sy}\alpha_{uz}+\alpha_{sz}\alpha_{uy})\sin\theta\cos\theta\sin\varphi$$
$$+(\alpha_{sz}\alpha_{ux}+\alpha_{sx}\alpha_{uz})\sin\theta\cos\theta\cos\varphi. \quad (2.15)$$

Upon integration with respect to φ over the range $0-2\pi$ the cross products all vanish. Using

$$\alpha_{sx}\alpha_{ux}+\alpha_{sy}\alpha_{uy}+\alpha_{sz}\alpha_{uz}=\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u \quad \text{and} \quad \alpha_{sz}\alpha_{uz}=(\boldsymbol{\alpha}_s\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R),$$

the remaining terms give

$$U=-\sum_{s,\,u}(\epsilon_s\epsilon_u/16\pi^2)[\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u(2I_1-I_1+I_2-I_3+I_4)+(\boldsymbol{\alpha}_s\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R)(I_1-3I_2+I_3-3I_4)], \quad (2.16)$$

where

$$I_1=\int_0^\infty\int_0^\pi\frac{\exp(ikR\cos\theta)\sin\theta dkd\theta}{1+a^2k^2}, \quad (2.17)$$

$$I_2=\int_0^\infty\int_0^\pi\frac{\cos^2\theta\exp(ikR\cos\theta)\sin\theta dkd\theta}{1+a^2k^2}, \quad (2.18)$$

$$I_3=\int_0^\infty\int_0^\pi\frac{a^2k^2\exp(ikR\cos\theta)\sin\theta dkd\theta}{(1+a^2k^2)^2}, \quad (2.19)$$

$$I_4=\int_0^\infty\int_0^\pi\frac{a^2k^2\cos^2\theta\exp(ikR\cos\theta)\sin\theta dkd\theta}{(1+a^2k^2)^2}. \quad (2.20)$$

These integrals may be evaluated by standard methods. No special limiting processes are necessary. The results are

$$I_1=(\pi/R)[1-\exp(-R/a)], \quad (2.21)$$

$$I_2=-\pi\exp(-R/a)(1/R+2a/R^2+2a^2/R^3)+2\pi a^2/R^3, \quad (2.22)$$

$$I_3=(\pi/2a)\exp(-R/a), \quad (2.23)$$

$$I_4=(\pi/2a)\exp(-R/a)+\pi\exp(-R/a)(1/R+2a/R^2+2a^2/R^3)-2\pi a^2/R^3. \quad (2.24)$$

Introducing these expressions into (2.16) we observe that the effect of the terms which come from the residual longitudinal component of the extraordinary field is to cancel all the singularities which arise from the terms representing the effect of the transverse components of the ordinary and extraordinary fields. We obtain finally an amazingly simple expression for the perturbation of the Hamiltonian.

$$U=-\sum_{s,u}\frac{\epsilon_s\epsilon_u}{16\pi}\left\{\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u\left[\frac{1-\exp(-R/a)}{R}\right]+(\boldsymbol{\alpha}_s\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R)\left[\frac{1-\exp(-R/a)}{R}-\frac{\exp(-R/a)}{a}\right]\right\}. \quad (2.25)$$

III. SELF-ENERGY OF CHARGED PARTICLES

To obtain the self-energy we consider only those terms in the summation corresponding to $s=u$ and let R go to zero. The second term in (2.25), which incidentally can be interpreted as the effect of retardation, vanishes. For a single particle we have left only[8]

$$U=-(\epsilon^2/16\pi a)\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}=-\tfrac{3}{4}\epsilon^2/4\pi a. \quad (3.1)$$

Combining this with the electrostatic self-energy, we obtain for the total, electromagnetic self-energy of a charged particle

$$U_s=-\tfrac{1}{4}\epsilon^2/4\pi a \quad \text{(in Heaviside units)} \quad (3.2)$$

or

$$U_s=-\tfrac{1}{4}\epsilon^2/a \quad \text{(in electrostatic units)}. \quad (3.3)$$

Thus on this theory the electromagnetic self-energy of a charged particle is necessarily distinct from the mass energy.

The wave equation for an isolated charged particle is

$$(c\boldsymbol{\alpha}\cdot\mathbf{p}+mc^2\beta-\epsilon^2/4a-i\hbar\partial/\partial t)\psi=0. \quad (3.4)$$

IV. THE ELECTROMAGNETIC INTERACTION ENERGY

Omitting the self-energy terms in (2.25) and considering the electrostatic interaction, we have for the complete electromagnetic interaction energy the following equation:

$$U_i=\sum_{s,u}{}'\frac{\epsilon_s\epsilon_u}{8\pi}\left\{(1-\tfrac{1}{2}\boldsymbol{\alpha}_s\cdot\boldsymbol{\alpha}_u)\left[\frac{1-\exp(-R/a)}{R}\right]-\tfrac{1}{2}(\boldsymbol{\alpha}_s\cdot\mathbf{R}/R)(\boldsymbol{\alpha}_u\cdot\mathbf{R}/R)\left[\frac{1-\exp(-R/a)}{R}-\frac{\exp(-R/a)}{a}\right]\right\}. \quad (4.1)$$

Note the appearance of the term $(1/a)\exp(-R/a)$ which is only effective at short ranges. If we let $a\rightarrow 0$, Eq. (4.1) reduces to Breit's formula.

The remarkable simplicity of the above result, as well as its freedom from singularities, has led the writer to the opinion that the theory upon which it is based has a deep-seated validity.

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[8] This result may be obtained more directly by letting $s=u$ and $R=0$ in Eq. (2.14). The integrations can then be carried out very simply.