PHYSICAL REVIEW VOLUME 72, NUMBER 7 OCTOBER 1, 1947

On the Lorentz Transformation of Charge and Current Densities

BORIS PODOLSKY
Graduate School, University of Cincinnati, Cincinnati, Ohio
(Received June 16, 1947)

The components of $\rho\mathbf{v}/c$ and $i\rho$ must transform as components of a four-vector, so that if measured in one coordinate system they are known in all coordinate systems. On the other hand, any operational definition of $\rho(\mathbf{r}, t)$ must take account of the positions of all particles at the same time t, that of the s-th particle being $\mathbf{r}_s(t)$. Upon performing the Lorentz transformation these will be $\mathbf{r}_s'(t_s')$, and the transformed time t_s' will be different for each particle. Another observer, in measuring ρ', would use $\mathbf{r}_s'(t')$, t' being the same for all particles. As particles are in motion $\mathbf{r}_s'(t_s')\neq\mathbf{r}_s'(t')$, and there appears to be no necessary relation between $\rho(\mathbf{r}, t)$ and $\rho'(\mathbf{r}', t')$, operationally defined in each coordinate system. It turns out, however, that if in each coordinate system the charge density is defined by $\rho(\mathbf{r}, t)=\sum_s e_s\delta(\mathbf{r}-\mathbf{r}_s(t))$, then relativistic equations of transformation hold.

1. INTRODUCTION

In classical electrodynamics one introduces the *charge density* and the three-dimensional *current density vector* $\mathbf{j}=\mathbf{u}/c$, where $\mathbf{u}$ is the local velocity of the charges, both ρ and $\mathbf{j}$ being functions of position vector $\mathbf{r}$ and time t. It is then shown that the general theory requires that the four quantities

$$s_\alpha=(j_x, j_y, j_z, i\rho),\quad \alpha=1, 2, 3, 4 \tag{1}$$

must form a four-dimensional vector, i.e., that the four components of s_α must transform as the four quantities

$$x_\alpha=(x, y, z, ict), \tag{2}$$

respectively.

The question now arises as to how one must define ρ and $\mathbf{j}$ in order that s_α may indeed transform in this way.[1] The problem may be stated more precisely as follows: How can each observer write down ρ and $\mathbf{j}$ for his own coordinate system, in such a way that they would then be connected by the Lorentz transformation with the corresponding quantities for other observers?

I intend to show that, if a charge distribution consists of a system of point charges e_s ($s=1, 2, \cdots n$; n = number of charges present), located at points $\mathbf{r}_s(t)$, and having velocities $\mathbf{u}_s(t)=d\mathbf{r}_s(t)/dt$, then we may take

$$\mathbf{j}=\mathbf{j}(\mathbf{r}, t)=\sum_s(e_s/c)\mathbf{u}_s(t)\delta(\mathbf{r}-\mathbf{r}_s(t)) \tag{3}$$

and

$$\rho=\rho(\mathbf{r}, t)=\sum_s e_s\delta(\mathbf{r}-\mathbf{r}_s(t)) \tag{4}$$

as satisfactory definitions of the current and charge densities, respectively.

[1] The difficulty involved in formulating this definition was called to my attention by Professor Carl Eckart.

2. THEORY

We shall use the vector form of the Lorentz transformation equations:

$$\mathbf{r}'=\mathbf{r}+(\beta-1)v^{-2}(\mathbf{v}\cdot\mathbf{r})\mathbf{v}-\beta t\mathbf{v}, \tag{5}$$

$$t'=\beta(t-\mathbf{v}\cdot\mathbf{r}/c^2), \tag{6}$$

where $\beta=(1-v^2/c^2)^{-\frac{1}{2}}$, $\mathbf{v}$ being the velocity of the primed coordinate system relative to the unprimed. We must therefore show that if

$$\mathbf{j}'(\mathbf{r}', t')=\sum_s(e_s/c)\mathbf{u}_s'(t')\delta(\mathbf{r}'-\mathbf{r}_s'(t')) \tag{7}$$

and

$$\rho'(\mathbf{r}', t')=\sum_s e_s\delta(\mathbf{r}'-\mathbf{r}_s'(t')) \tag{8}$$

are the current and charge densities defined in the primed coordinate system, then, in an analogous manner to Eqs. (5) and (6),

$$\mathbf{j}'(\mathbf{r}', t')=\mathbf{j}(\mathbf{r}, t)+(\beta-1)v^{-2}\mathbf{v}\cdot\mathbf{j}(\mathbf{r}, t)\mathbf{v}-\beta\mathbf{v}\rho(\mathbf{r}, t)/c \tag{9}$$

and

$$\rho(\mathbf{r}', t')=\beta[\rho(\mathbf{r}, t)-\mathbf{v}\cdot\mathbf{j}(\mathbf{r}, t)/c]. \tag{10}$$

More explicitly, we have to show that the equations

$$\mathbf{j}'(\mathbf{r}', t')=\sum_s(e_s/c)\{\mathbf{u}_s(t)+(\beta-1)v^{-2}\mathbf{v}\cdot\mathbf{u}_s(t)\mathbf{v}-\beta\mathbf{v}\}\delta(\mathbf{r}-\mathbf{r}_s(t)) \tag{11}$$

and

$$\rho'(\mathbf{r}', t')=\sum_s e_s\beta\{1-\mathbf{v}\cdot\mathbf{u}_s(t)/c^2\}\delta(\mathbf{r}-\mathbf{r}_s(t)), \tag{12}$$

obtained by substitution into Eqs. (9) and (10) of the values of $\mathbf{j}$ and ρ from Eqs. (3) and (4), reduce to Eqs. (7) and (8).

In these equations, in which $\mathbf{r}$ and t are arbitrary position vector and time, and $\mathbf{r}'$ and t' the corresponding quantities connected with $\mathbf{r}$ and t by Eqs. (5) and (6), $\mathbf{r}_s(t)$ must be supposed to be some known or measurable functions of t, corresponding to the particle motion in the unprimed coordinate system, while $\mathbf{r}_s'(t')$ must give the same motion as seen in the primed system. A difficulty arises because of the fact that if, with t fixed, we apply Eqs. (5) and (6) to t and the corresponding $\mathbf{r}_s$, these will go over into t_s' and the corresponding $\mathbf{r}_s'$, with t_s' in general not equal to t'. In fact we obtain:

$$\mathbf{r}_s'(t_s') = \mathbf{r}_s(t) + (\beta-1)v^{-2}\mathbf{v}\cdot\mathbf{r}_s(t)\mathbf{v} - \beta t\mathbf{v} \quad (13)$$

and

$$t_s' = \beta(t - \mathbf{v}\cdot\mathbf{r}_s(t)/c^2), \quad (14)$$

so that the transformed time t_s' varies from particle to particle and is equal to t' only when $\mathbf{r}_s(t) = \mathbf{r}$.

It appears, therefore, that we cannot express $\mathbf{r}_s'(t')$ explicitly in terms of $\mathbf{r}_s(t)$, $\mathbf{r}$, and t. Fortunately this also turns out to be unnecessary. To see this, we note that the only events $(\mathbf{r}, t)$ for which the right-hand members of Eqs. (11) and (12) do not vanish are those for which

$$\mathbf{r} = \mathbf{r}_s(t),$$

and that then

$$t_s' = t', \quad \text{and} \quad \mathbf{r}_s'(t_s') = \mathbf{r}_s'(t') = \mathbf{r}', \quad (15)$$

as can readily be seen by comparison of Eqs. (13) and (14) with Eqs. (5) and (6). These events are, therefore, to be found among the events $(\mathbf{r}', t')$ for which the right-hand members of Eqs. (7) and (8) do not vanish. The converse is also easily shown.

Thus the events for which the δ-functions in Eqs. (7) and (8) do not vanish are *the same events* for which the δ-functions in Eqs. (11) and (12) do not vanish.

It is therefore possible to establish a relation between $\delta(\mathbf{r}'-\mathbf{r}_s'(t'))$ and $\delta(\mathbf{r}-\mathbf{r}_s(t))$. This is done in the Appendix and gives:

$$\delta(\mathbf{r}'-\mathbf{r}_s'(t')) = \beta[1-\mathbf{v}\cdot\mathbf{u}_s(t)/c^2]\delta(\mathbf{r}-\mathbf{r}_s(t)). \quad (16)$$

With this relation Eq. (11) obviously reduces to Eq. (7), while Eq. (12) becomes:

$$\mathbf{j}'(\mathbf{r}', t') = \sum_s (e_s/\beta c)\{\mathbf{u}_s(t) + (\beta-1)v^{-2}\mathbf{v}\cdot\mathbf{u}_s(t) - \beta\mathbf{v}\} \times \frac{\delta(\mathbf{r}'-\mathbf{r}_s'(t))}{1-\mathbf{v}\cdot\mathbf{u}_s(t)/c^2}. \quad (17)$$

This may be further simplified, however, by introducing $\mathbf{u}_s'(t')$. Thus, differentiation of Eqs. (13) and (14) gives:

$$d\mathbf{r}_s'(t_s') = d\mathbf{r}_s(t) + (\beta-1)v^{-2}\mathbf{v}\cdot d\mathbf{r}_s(t)\mathbf{v} - \beta\mathbf{v}dt \quad (18)$$

and

$$dt_s' = \beta[dt - \mathbf{v}\cdot d\mathbf{r}_s(t)/c^2]. \quad (19)$$

Hence

$$\mathbf{u}_s'(t_s') = d\mathbf{r}_s'(t_s')/dt_s' = \frac{\mathbf{u}_s(t) + (\beta-1)v^{-2}\mathbf{v}\cdot\mathbf{u}_s(t)\mathbf{v} - \mathbf{v}\beta}{\beta[1-\mathbf{v}\cdot\mathbf{u}_s(t)/c^2]}. \quad (20)$$

Substituting this relation into Eq. (17) and making use of Eq. (15), we finally obtain Eq. (8).

APPENDIX

Let $F(\mathbf{r}')$ be any function of $\mathbf{r}'$, and let $\mathbf{r}-\mathbf{r}_s(t)$, regarded as a function of $\mathbf{r}'$ and t', be $\mathbf{R}$. Thus

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_s(t) = \mathbf{R}(\mathbf{r}', t'). \quad (21)$$

Then

$$\iiint_{-\infty}^{\infty} F(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}_s(t))dx'dy'dz'$$

$$= \iiint_{-\infty}^{\infty} F(\mathbf{r}')\delta(\mathbf{R})dx'dy'dz'$$

$$= \iiint_{-\infty}^{\infty} F(\mathbf{r}')\frac{\partial(x', y', z')}{\partial(R_x, R_y, R_z)}\delta(\mathbf{R})dR_xdR_ydR_z$$

$$= \left[F(\mathbf{r}')\frac{\partial(x', y', z')}{\partial(R_x, R_y, R_z)}\right]_{\mathbf{R}=0}$$

$$= \left[F(\mathbf{r}')\frac{\partial(x', y', z')}{\partial(R_x, R_y, R_z)}\right]_{\mathbf{r}'=\mathbf{r}_s'(t')},$$

by Eq. (15). But, the last expression is also equal to

$$\iiint_{-\infty}^{\infty} F(\mathbf{r}')\frac{\partial(x',y',z')}{\partial(R_x,R_y,R_z)}\delta(\mathbf{r}'-\mathbf{r}_s'(t'))dx'dy'dz'.$$

Therefore

$$\delta(\mathbf{r}-\mathbf{r}_s(t))=\frac{\partial(x',y',z')}{\partial(R_x,R_y,R_z)}\delta(\mathbf{r}'-\mathbf{r}_s'(t'))$$
$$=\delta(\mathbf{r}'-\mathbf{r}_s'(t'))/J, \quad (22)$$

where

$$J=\partial(R_x,R_y,R_z)/\partial(x',y',z').$$

In calculating derivatives of the components of $\mathbf{R}$ we must remember that $\mathbf{r}_s(t)$ are some given functions of t, specifying the motions of the particles in the unprimed coordinate system, while

$$\mathbf{r}=\mathbf{r}'+(\beta-1)v^{-2}(\mathbf{v}\cdot\mathbf{r}')\mathbf{v}+\beta t'\mathbf{v} \quad (23)$$

and

$$t=\beta(t'+\mathbf{v}\cdot\mathbf{r}'/c^2). \quad (24)$$

Thus, we have, for example,

$$R_x=x'+(\beta-1)v^{-2}(x'v_x+y'v_y+z'v_z)v_x+\beta t'v_x-x_s(t),$$

where x' and $x_s(t)$ are the x-components of $\mathbf{r}'$ and $\mathbf{r}_s(t)$, respectively. Therefore

$$\partial R_x/\partial x'=1+(\beta-1)v^{-2}v_x^2-(dx_s(t)/dt)\partial t/\partial x'$$
$$=1+(\beta-1)v^{-2}v_x^2-\dot{x}_s(t)\beta v_x/c^2,$$

the factor $\partial t/\partial x'$ being obtained from Eq. (24), while $\dot{x}_s(t)$ is the x-component of $\mathbf{u}_s(t)$.

In this way we obtain:

$$J=\beta[1-\mathbf{v}\cdot\mathbf{u}_s(t)/c^2]. \quad (25)$$

Combining Eqs. (22) and (25) we obtain Eq. (16).

PHYSICAL REVIEW VOLUME 72, NUMBER 7 OCTOBER 1, 1947

Recurrence Formulas for Coulomb Wave Functions

JOHN L. POWELL
University of Wisconsin, Madison, Wisconsin
(Received June 13, 1947)

Recurrence relations connecting Coulomb functions for different values of the angular momentum L and the same value of the nuclear radius parameter ρ are summarized, and their derivation is outlined.

THE Coulomb wave functions F_L and G_L, as defined by Yost, Wheeler, and Breit,[1] are solutions of the differential equation

$$\frac{d^2F}{d\rho^2}+\left\{1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}\right\}F=0 \quad (1)$$

which, for large ρ, have the asymptotic forms

$$F_L\sim\sin\left(\rho-\frac{L\pi}{2}-\eta\ \ln 2\rho+\sigma_L\right),$$

$$G_L\sim\cos\left(\rho-\frac{L\pi}{2}-\eta\ \ln 2\rho+\sigma_L\right),$$

where $\sigma_L=\arg\Gamma(L+1+i\eta)$.

[1] F. L. Yost, John A. Wheeler, and G. Breit, J.Terr. Mag. **40**, 443 (1935); Phys. Rev. **49**, 174 (1936).

F_L and G_L are the real and imaginary parts, respectively, of the function

$$Y_L=F_L+iG_L=\frac{\rho^{L+1}}{i(e^{2\pi\eta}-1)(2L+1)!C_L}\times\int_D(z-i)^{L+i\eta}(z+i)^{L-i\eta}e^{z\rho}dz, \quad (2)$$

where

$$C_L^2=\frac{[1+(\eta^2/L^2)]\cdots[1+(\eta^2/1^2)]}{1^2\cdot 3^2\cdots(2L+1)^2}\frac{2\pi\eta}{(e^{2\pi\eta}-1)}, \quad (3)$$

and D is a contour in the complex z plane which starts at $(-\infty-i)$, encircles the point $-i$ once in the positive sense, and returns to the starting point. The expression (2) may be derived by