which is about 0.16 percent, occurs for a value of z a little greater than 0.1.

Eddington's approximation,[10] is

$$\rho(z)=(z+17/24)\left[\frac{2+3z-\{E_2(z)-\frac{3}{2}E_3(z)\}}{2+3z-3\{E_4(z)-\frac{3}{2}E_5(z)\}}\right].$$

This seems to be the best approximation available in previously published literature, although it was done at a rather early stage. The error of 1 percent at the boundary, increases to 1.3 percent at $z=0.02$ and is less than 0.5 percent after $z=0.3$.

The expansion of Section 5 is also evaluated in Table I. This has the correct analytical form near the boundary and is a good approximation up to $z=0.05$. By combining this with Lecaine's approximation one would have a fairly simple and very accurate representation of the density over the entire range.

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Note. Since the first writing of this paper, Wick[13] and Chandrasekhar[14] have published applications to this problem of the method of expanding the angular distribution $\psi(z,\mu)$ in Legendre polynomials in μ. Chandrasekhar's highest approximation, in which three exponentials are used, still has a maximum error (near $z=0.1$) of more than 3.5 percent, while even the elementary approximation referred to above which uses only one exponential has a maximum error of about 1.6 percent. In the light of the criteria which should be applied to approximations here, and the other examples already given, it should be pointed out that the polynomial method does not seem to be well adapted to the problem we are considering. Of course, in more complicated problems where simple iteration and variation techniques are not available, the polynomial method has had many very successful applications.

[13] G. C. Wick, "Über ebene Diffusionsprobleme," Zeits. f. Physik **121**, 702 (1943).
[14] S. Chandrasekhar, Astrophys. J. **101**, 348 (1945).

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Application of a Variational Method to Milne's Problem

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An approximate solution of Milne's integral equation for the neutron density is obtained by a variational method with high accuracy in simple analytical form. The extrapolated asymptotic density at the boundary is given by this method correct to 0.4 parts in a million. The density itself has a maximum error of 0.3 percent which occurs at the boundary and of less than 0.05 percent for all distances beyond 0.05 mean free paths. A simple expression for the angular distribution of emerging neutrons is also obtained.

THE exact solution of Milne's integral equation[1]

$$\psi_0(z)=\tfrac{1}{2}\int_0^\infty \psi_0(z')E_1(|z-z'|)dz' \qquad (1)$$

with

$$E_1(x)=-Ei(-x)=\int_1^\infty (e^{-xv}/v)dv$$

has been obtained by Mark[2] and evaluated by numerical integration.

We obtain here an approximate solution of this equation in simple analytical form by employing a variational technique.

From the integral equation it is seen that

$$\psi_0(z)=z+q(z), \qquad (2)$$

where $\lim_{z\to\infty}q(z)=z_0$. From Eqs. (9) and (14)

* Report issued May 15, 1944.
[1] For literature, see references 2 and 3.

[2] C. Mark, Phys. Rev. **72**, 558 (1947).

of the paper by Placzek and Seidel,[3] we have:

$$z_0=3\int_{-1}^{0}\mu^2\psi(0,\mu)d\mu$$

$$=\tfrac{3}{2}\int_0^\infty \psi_0(z')E_3(z')dz' \qquad (3)$$

$$=\tfrac{3}{2}\left[\tfrac{1}{4}+\int_0^\infty q(z')E_3(z')dz'\right],$$

where[4]

$$E_n(x)=\int^\infty e^{-xv}v^{-n}dv. \qquad (3')$$

From (1) and (2) we obtain

$$q(z)=\tfrac{1}{2}\int_0^\infty q(z')E_1(|z-z'|)dz'+\tfrac{1}{2}E_3(z). \qquad (4)$$

We then consider the functional[5]

$$F(\bar{q})=\frac{\int_0^\infty \bar{q}(z)\left[\bar{q}(z)-\tfrac{1}{2}\int^\infty \bar{q}(z')E_1(|z-z'|)dz'dz\right]}{\left[\int_0^\infty \bar{q}(z)E_3(z)dz\right]^2}. \qquad (5)$$

When $\bar{q}=q(z)$, $F(\bar{q})$ becomes, using (4) and (3),

$$F(q)=\left[2\int_0^\infty q(z)E_3(z)dz\right]^{-1}=(4z_0/3-\tfrac{1}{2})^{-1}. \qquad (6)$$

Also $F(q)$ is a minimum of $F(\bar{q})$.[6]

If we let $\bar{q}=$constant, $F(\bar{q})=9/4$ and from (6)

$$z_0=17/24. \qquad (7)$$

We now assume as an approximation to q, a function $\bar{q}$ of the form

$$\bar{q}=[1-AE_2(z)-BE_3(z)]z_0. \qquad (8)$$

This form is suggested by the fact that if the method of successive approximations is applied to Eq. (4), beginning with $q=z_0$ the next approximation is

$$q \doteq z_0-\tfrac{1}{2}z_0E_2(z)+\tfrac{1}{2}E_3(z).$$

We choose A and B such that $F(\bar{q})$ assumes an extreme value. In evaluating $F(\bar{q})$ with $\bar{q}$ given by (8) it is necessary to evaluate integrals of the form

$$\int_0^\infty E_m(z)dz\int_0^\infty E_n(z')E_1(|z-z'|)dz'$$

$$=\int_0^\infty E_m(z)dz\int_0^z E_n(z')E_1(z-z')dz'$$

$$+\int_0^\infty E_m(z)dz\int_0^\infty E_n(z+z')E_1(z')dz',$$

with m, $n=2$ or 3. By substituting from Eq. (3′) and interchanging orders of integration, this can be reduced to

$$\int_1^\infty s^{-(m+1)}\log(1+s)ds\int_1^\infty v^{-n}(s+v)^{-1}dv$$

$$+\int_1^\infty v^{-(n+1)}\log(1+v)dv\int_1^\infty s^{-m}(s+v)^{-1}ds,$$

which can be evaluated for any m and n. Thus

$$\int_0^\infty E_2(z)dz\int_0^\infty E_2(z')E_1(|z-z'|)dz'=\tfrac{1}{3}+\pi^2/12-(4/3)\log 2,$$

$$\int_0^\infty E_3(z)dz\int_0^\infty E_2(z')E_1(|z-z'|)dz'=\tfrac{1}{3}\log 2-\tfrac{1}{12},$$

$$\int_0^\infty E_3(z)dz\int_0^\infty E_3(z')E_1(|z-z'|)dz'=(56/45)\log 2-79/360-\pi^2/18,$$

and

$$F(q)=\frac{\tfrac{1}{4}+A\tfrac{2}{3}(\log 2-1)-\tfrac{1}{8}B+AB\tfrac{1}{3}(1-\log 2)+A^2\{\tfrac{1}{2}-\pi^2/24\}+B^2\{\pi^2/36-(2/9)\log 2-13/144\}}{\{\tfrac{1}{3}-\tfrac{1}{8}A-B\tfrac{1}{5}(2\log 2-1)\}^2}. \qquad (9)$$

[3] G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947).
[4] For tables see G. Placzek, *The Functions* $E_n(x)$, MT-1, obtainable from Plans and Publications Branch, National Research Council of Canada, Ottawa, Canada.
[5] R. E. Marshak, Phys. Rev. **71**, 688 (1947).
[6] B. Davison, Phys. Rev. **71**, 694 (1947).

Now, solving the equations

$$\partial F/\partial A=0, \quad \partial F/\partial B=0,$$

we find

$$A=0.3428949, \quad B=-0.3158704,$$

and F for this value of A and B is 2.235831. From (6),

$$2.235831 \doteqdot 1/(\tfrac{1}{3}4z_0-\tfrac{1}{2}),$$

whence

$$z_0 \doteqdot 0.7104457. \quad (10)$$

The function of form (8) which best approximates $q(z)$ is

$$q_1(z)=0.7104457(1-0.3428949E_2 +0.3158704E_3). \quad (11)$$

This procedure is designed to give an approximation to the value of F. Since z_0 is related to F directly, whereas q is related only through an integral, the accuracy of the approximation to z_0 will be better than that of the approximation to $q(z)$. The true value of z_0 is 0.71044609,[3] the error in the approximation (10) being thus 4×10^{-5} percent. The approximation (11) has been tabulated and compared with the exact solution by Mark.[2] The maximum error in $z+q_1(z)$ is 0.3 percent and occurs at the boundary. Beyond $z=0.05$ the error is less than 0.05 percent.

Another method of approximation has been discussed by Placzek.[7] Although that approximation is better at the boundary, the present one is an improvement for $z\geqq0.05$, and has the further advantage of a simpler analytical form.

The emergent angular distribution implied by (11), normalized to unit density is

$$\varphi(\mu)=0.501362+0.671543\mu+[0.210927\mu +0.194303\mu^2]\log(1+1/\mu), \quad 0\leqslant\mu\leqslant1. \quad (12)$$

Comparison with the table of the exact function[8] shows that the maximum error of (12) is 0.3 percent at $\mu=0$ and decreases quickly as μ increases. The analytical form of (12) is however not much simpler than the form of Placzek's approximation (reference 7, Eq. (18)):

$$\varphi(\mu)=0.52414+0.43301\mu-0.02414(1+\alpha\mu)^{-1} +\{0.30763\mu+0.43301\mu^2 -0.04916\mu(1+\alpha\mu)^{-1}\}\log(1+1/\mu), \quad (13)$$

where $\alpha=2.62032$, which gives the correct value for $\mu=0$ and represents the true function through the whole range with an error of less than 0.1 percent.

The variational method has also been used for the treatment of a generalization of the present problem to capturing media. The results will be given in a separate paper.

[7] G. Placzek, *The Neutron Density Near a Plane Surface*, I. MT-16, obtainable from Plans and Publications Branch, National Research Council of Canada, Ottawa, Canada.

[8] G. Placzek, Phys. Rev. **72**, 556 (1947).