of φ and the asymptotic expression (6) is best used for interpolation.

It is seen from the tables that $\varphi(\mu)$ is very close to a straight line except for small μ. Fermi's simple linear approximation[5]

$$\varphi(\mu)=(1+\sqrt{3}\mu)/(1+\sqrt{3}/2) \tag{7}$$

has an error of 7.2 percent at $\mu=0$, but for $\mu>0.1$ its error is below one percent throughout. This has to be kept in mind when discussing the more complicated approximations.[6–9]

[5] E. Fermi, Ricerca Scient. **7** [2] 13 (1936).

[6] A. Unsoeld, *Physik der Sternatmosphaeren* (Julius Springer, Berlin, 1939).

[7] J. LeCaine, Phys. Rev. **72**, 564 (1947), Eq. (12).

[8] G. Placzek, Montreal Report MT 16, 1944; reissued by National Research Council of Canada, Chalk River 1947. See also Eq. (13) in LeCaine, l.c.

[9] S. Chandrasekhar, Astrophys. J. **99**, 180 (1944); **101**, 348 (1945).

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The Neutron Density Near a Plane Surface

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The exact solution of Milne's integral equation is expressed as a real integral with non-oscillating integrand. This expression has been derived from the Wiener-Hopf solution for the Laplace transform of the density. The integrand involves the angular distribution of neutrons emerging from the surface, and the tabulation of this function by the Mathematical Tables Project given by Placzek has been used in the numerical evaluation of the integral. The values of the first three moments of the difference between the density and the asymptotic density, and an expansion of the density for points near the boundary are also given. Various authors have proposed or obtained approximations to the solution of this problem, and some of these approximations are referred to and compared with the exact solution.

1. INTRODUCTION

THE purpose of this paper is to determine the neutron density in Milne's problem as described by Placzek and Seidel.[1] We shall adopt the notation and definitions of their paper and make frequent references to its results.

The neutron density, $\psi_0(z)$, satisfies Milne's integral equation

$$\psi_0(z)=\tfrac{1}{2}\int_0^\infty \psi_0(z')E(|z-z'|)dz',$$

with $E(x)=-Ei(-x)$. An expression for the Laplace transform of the solution of this equation has been obtained by Wiener and Hopf.[2,3] A simplified derivation of their result is given in PS.

The angular distribution of the emerging neutrons is, except for a factor, equal to the Laplace transform of the density (PS, Eq. (19)), so that the problem of determining the emergent angular distribution is simply that of evaluating the expression for this Laplace transform; and an extensive and accurate tabulation of this is now available.[4] However, considerable difficulties have been encountered in attempts to invert the Laplace transform of the density in order to obtain the density itself, and an exact yet manageable integral for $\psi_0(z)$ does not seem to have been given heretofore. In this paper it is

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** This paper, except for minor modifications, corrections, and improvements in some of the numerical work, was issued as a report of the Theoretical Division of the Montreal Laboratory on April 15, 1944.

[1] G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947). Hereafter this paper will be referred to as PS.

[2] N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse (1931), p. 696.

[3] E. Hopf, *Mathematical Problems of Radiative Equilibrium*, Cambridge tracts No. 31, 1934.

[4] G. Placzek, Phys. Rev. **72**, 556 (1947).

shown that

$$\psi_0(z)=3\left\{z+z_0-\tfrac{1}{4}\int_0^1 (e^{-z/\mu}d\mu)/\psi(0,\ -\mu)[(1-\mu\ \mathrm{arth}\mu)^2+\pi^2\mu^2/4]\right\}, \qquad (1)$$

where $z_0=0.710446$ (PS, Eq. (40)), and $\psi(0,\ -\mu)$, the angular distribution of emerging neutrons, is related to the function $\varphi(\mu)$ tabulated in the preceding paper[4] by

$$\varphi(\mu)=\psi(0,\ -\mu)/\sqrt{3}.$$

$\psi_0(z)$ has been evaluated by numerical integration of (1) with the help of the tabulation of $\varphi(\mu)$.

In addition to the tabulation of the exact neutron density for this standard problem, a few terms of an analytical expansion of the density function, valid for small z, have been included (Section 5). The first neglected term is $O(z^3\ \log^3 z)$, and the terms given represent the density with an error of less than 0.1 percent over the range $0\leq z<0.1$, where the mean free path is taken as the unit length.

There are also included (Section 4) the values of the zero, first, and second moments of the difference between the asymptotic density and the density itself: $3(z+z_0)-\psi_0(z)$.

In Section 3 note is made of the obvious fact that one may approach this semi-infinite medium problem through the formulae given by Placzek and Volkoff[5] which apply to an infinite medium.

2. AN EXPRESSION FOR $\psi_0(z)$

In PS the Laplace transform of the density,

$$\phi_0(s)=\int_0^\infty \psi_0(z)e^{-sz}dz, \qquad (2)$$

is obtained in the explicit form (PS, Eq. (36))

$$\phi_0(s)=\sqrt{3}[(s+1)\tau_-(s)/s^2], \qquad (3)$$

where $\tau_-(s)$ is defined in PS Eqs. (26) and (28). To obtain $\psi_0(z)$ we may write the inversion formula

$$\psi_0(z)=(1/2\pi i)\int_{c-i\infty}^{c+i\infty} \phi_0(s)e^{sz}dz,\quad 0<c; \qquad (4)$$

but to substitute in this from Eq. (3) leads to an immediate impasse because of the complexity of the function $\tau_-(s)$. We therefore consider substituting in (4) from PS (18),

$$\phi_0(s)=\frac{\displaystyle\int_{-1}^{0}[\mu\psi(0,\ \mu)/(1+\mu s)]d\mu}{\displaystyle 1-\tfrac{1}{2}\int_{-1}^{1} d\mu/(1+\mu s)}. \qquad (5)$$

On setting

$$\int_{-1}^{0}[\mu\psi(0,\ \mu)/(1+\mu s)]d\mu=g(s)$$

and

$$1-\tfrac{1}{2}\int_{-1}^{1} d\mu/(1+\mu s)=K(s), \qquad (6)$$

we have $\phi_0(s)=g(s)/K(s)$; and we now consider the properties of these two functions.

$$K(s)=1-\tfrac{1}{2}\int_{-1}^{1} d\mu/(1+\mu s)$$
$$=-s^2\int_0^1 [\mu^2/(1-\mu^2 s^2)]d\mu. \qquad (7)$$

From this second form of $K(s)$ it is obvious that $K(s)$ has a double zero at $s=0$; and it may also be easily seen that $K(s)$ has no other zeros in the s-plane cut from $-\infty$ to -1 and from 1 to ∞. A proof of this is given by Placzek and Volkoff[6] (or one can show that the imaginary part of the second integral in (7) is unequal to zero unless s lies on one of the axes in the s-plane). $K(s)$ has branch points at $s=\pm 1$, and is regular in the cut plane. We use that branch of $K(s)$ which is real for $-1<s<1$.

For $g(s)$, we note that this function has a branch point at $s=+1$, and is regular in the plane cut from $+1$ to $+\infty$. We use that branch of $g(s)$ which is real for real $s<1$. We have

$$g(0)=\int_{-1}^{0}\mu\psi(0,\ \mu)d\mu,$$

[5] G. Placzek and G. M. Volkoff, *Notes on Diffusion of Neutrons without Change in Energy*, M.T. 4, obtainable from Plans and Publications Branch, National Research Council of Canada, Ottawa, Canada.

[6] Appendix B of reference 5.

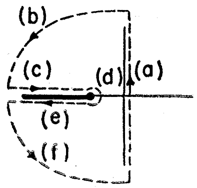

FIG. 1. Contour for the integration of Eq. (9).

which, as in PS, we take to be -1, thus normalizing the current.

We may now state that

$$\phi_0(s)=g(s)/K(s) \tag{8}$$

is regular in the s-plane cut from $-\infty$ to -1, except for a pole of the second order at $s=0$. (The regularity of $\phi_0(s)$ in the positive half-plane follows from the definition (2), and the fact that, as in reference 3, we are seeking solutions $\psi_0(z)$ of $O[\exp(az)]$, $a<1$.)

Now substituting from (8) in (4) we obtain

$$\psi_0(z)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}[g(s)e^{sz}/K(s)]ds, \tag{9}$$

and to handle this integral we close the contour as in Fig. 1. It is shown in PS that for small values of s

$$\phi_0(s)=g(s)/K(s)=3s^{-2}+3z_0s^{-1}+\cdots \tag{10}$$

where z_0 is the number 0.710446. From Eqs. (10) and (9) we obtain

$$\psi_0(z)=(2\pi i)^{-1}\int_{(a)}[g(s)e^{sz}/K(s)]ds$$
$$=3(z+z_0)-(2\pi i)^{-1}\left[\int_{(b)}+\int_{(c)}+\int_{(d)}+\int_{(e)}+\int_{(f)}[g(s)e^{sz}/K(s)]ds\right]. \tag{11}$$

In Eq. (11) we note first that

$$\int_{(d)}=0,$$

since the branch point at -1 is a zero of the integrand. We next consider

$$\int_{(b)}+\int_{(f)}.$$

From (6) it follows that $g(s)/K(s)=O(1/|s|)$ as $|s|\to\infty$ so that there is no contribution to these integrals from the small parts of paths (b) and (f) to the right of the imaginary axis. For the rest, we set $s=Re^{i\theta}(R\to\infty)$, so that $e^{sz}=e^{zR\cos\theta}\times e^{izR\sin\theta}$; and since, for the integrals considered, θ is in the 2nd and 3rd quadrants, $\cos\theta$ is negative and hence, for R infinite,

$$\int_{(b)}+\int_{(f)}=0.$$

Thus (11) reduces to

$$\psi_0(z)=3(z+z_0)-(2\pi i)^{-1}\left[\int_{(c)}[g(s)e^{sz}/K(s)]+\int_{(e)}[g(s)e^{sz}/K(s)]ds\right]. \tag{12}$$

Along path (c), where $-\infty<s<-1$,

$$K(s)=1+(2s)^{-1}\log(s-1/s+1)-(i\pi/2s)$$
$$=1-s^{-1}\operatorname{cth}^{-1}s-i\pi/2s, \tag{13}$$

whereas, along path (e),

$$K(s)=1-s^{-1}\operatorname{cth}^{-1}s+i\pi/2s. \tag{14}$$

Using (13) and (14) in (12) and then setting $s=-t$ gives

$$\psi_0(z)=3(z+z_0)+\tfrac{1}{2}\int_1^\infty[g(-t)e^{-tz}dt]/t[(1-t^{-1}\operatorname{cth}^{-1}t)^2+\pi^2/4t^2]. \tag{15}$$

To put (15) in a convenient form we use the result (not explicitly stated in PS but immediately available from PS Eqs. (31), (32), and (35)

$$g(-t)=-\sqrt{3}\,\tau_+(-t)/(1+t),$$

and the further fact, which may be verified from Eqs. (28) and (29) of PS, that

$$\tau_+(-t)=1/\tau_-(t).$$

Combining these with Eqs. (36) and (19) of PS we obtain

$$g(-t)=-3/[2t\psi(0,-1/t)]. \tag{16}$$

Substituting (16) in (15) and then setting $t=1/\mu$ gives relation (1) for $\psi_0(z)$.

The integral in (1) has been evaluated. The results are included in Section 6 where tables of the function $\rho(z)=\frac{1}{3}\psi_0(z)$ and also of the quantity $z+z_0-\rho(z)$, which is the deviation from the asymptotic form, are given, and the method used for the integration is described.

The results of this evaluation of $\psi_0(z)$ may also be used to give the neutron density when the scattering is not isotropic but is represented by a linear expression in the cosine of the scattering angle. For, if b represent the average cosine of the scattering angle, the neutron density for such linear scattering differs from the neutron density for isotropic scattering only by a term linear in z[7]; in fact,

$$\psi_0^{(b)}(z)=\psi_0^{(is)}(z)-3bz.$$

3. AN ALTERNATIVE DERIVATIVE OF EQ. (15)

The Milne problem for a half-space may also be interpreted as a problem in an infinite medium. For this one would imagine an infinite non-capturing medium with a uniform current from infinity in a direction parallel to the z axis and with an anisotropic plane sink at $z=0$ which absorbs $\mu\psi(0,\mu)d\mu$ neutrons for negative μ between μ and $\mu+d\mu$.

Using the one-dimensional form of the formulae given by Placzek and Volkoff (reference 5, Section 5), taking $\mu'\psi(0,\mu')\delta(\mu-\mu')\delta(z)$, $\mu<0$, for the "source" term, and taking τ as Fourier transform variable, one gets

$$\phi_0(\tau)\{1-\tau^{-1}\operatorname{art}\tau\}=\int_{-1}^{0}\mu\psi(0,\mu)(1-i\tau\mu)^{-1}d\mu. \quad (17)$$

Now taking the Fourier inverse of this gives

$$\psi_0(z)=\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}\phi_0(\tau)e^{-i\tau z}d\tau$$

$$=\frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic}\frac{\int_{-1}^{0}\mu\psi(0,\mu)(1-i\tau\mu)^{-1}d\mu}{1-\tau^{-1}\operatorname{art}\tau}e^{-i\tau z}d\tau, \quad (c>0).$$

By the substitution $s=-i\tau$ this is reduced to (9) above; or the results of Placzek and Volkoff's Appendix B, may be applied to obtain (15) directly.

4. NOTE ON THE MOMENTS

$$\int_0^\infty z^n\{3(z+z_0)-\psi_0(z)\}dz=M_n$$

We give the values of M_0, M_1, M_2. These are obtained from the relation

$$\int_0^\infty e^{-sz}[3(z+z_0)-\psi_0(z)]dz=3s^{-2}+3z_0s^{-1}-\phi_0(s)$$

by expanding each side in powers of s and equating coefficients. Using (3) we have $\phi_0(s)=\sqrt{3}s^{-2}\times(s+1)[\tau_-(0)+s\tau_-'(0)+\frac{1}{2}s^2\tau_-''(0)+\cdots]$. If we define $c_n=(-1)^n\tau_-^{(n)}(0)/\tau_-(0)$, and recall (PS, Section 6) that $\tau_-(0)=\sqrt{3}$, then it follows from the statements above that

$$M_0=3(c_1-\tfrac{1}{2}c_2),\ M_1=\tfrac{3}{2}(c_2-\tfrac{1}{3}c_3),\ M_2=(c_3-\tfrac{1}{4}c_4).$$

To evaluate these we use the following relations, the first of which is given in Section 6 of PS and the rest of which may be obtained by a similar method:

$c_1=1-z_0=0.289554$; $c_2=c_1^2+2/5=0.483842$;

$c_3=c_1^3+16c_1/5+k$, with[8]

$$k=\int_0^1\frac{\mu d\mu}{(1-\mu\operatorname{arth}\mu)^2+\frac{1}{4}\pi^2\mu^2}=0.344708,$$

so that $c_3=1.29556$; and $c_4=612/175+4c_3c_1-3c_1^4-12c_1^2/5=4.77538$. The moments are now seen to have the values:

$$M_0=3(3-5z_0^2)/10=0.14290,\quad M_1=0.07798,\ M_2=0.10172.$$

Hence, at once, for the difference function,

$$z_{Av}=0.546,\qquad (z^2)_{Av}=0.712.$$

One may now, by expanding each side of (5) in powers of s, obtain the values of the current moments of the angular distribution:

$$\psi^{(n)}(0)=\int_{-1}^{0}\mu^n\psi(0,\mu)d\mu$$

for $n=3$, 4, and 5. From PS Eq. (14) we see also

[7] C. Mark, *Milne's Problem for Anisotropic Scattering* Eq. (42). MT-26, obtainable from Plans and Publications Branch, National Research Council of Canada, Ottawa.

[8] This results from a numerical integration by Bengt Carlson.

that

$$\psi^{(n)}(0)=\tfrac{1}{2}(-1)^n\int_0^\infty E_{n+1}(z)\psi_0(z)dz,$$

where

$$E_n(z)=\int_1^\infty u^{-n}e^{-zu}du.$$

The results are (in addition to $\psi^{(1)}(0)=-1$, $\psi^{(2)}(0)=z_0$), $\psi^{(3)}(0)=-(3+5z_0^2)/10=-0.55237$, $\psi^{(4)}(0)=0.45226$, $\psi^{(5)}(0)=-0.38304$.

5. NOTE ON AN EXPANSION OF $\psi_0(z)$ FOR SMALL VALUES OF z

Using Eq. (46) of PS and Eq. (5) of Placzek,[4] one can write

$$\log(s\phi_0(s)/\sqrt{3})=-\tfrac{1}{2}\log(1+s^{-1})+\pi^{-1}\int_0^{\pi/2}\frac{y\,\mathrm{art}(s^{-1}\tan y)}{1-y\,\mathrm{ctn}y}dy. \quad (18)$$

It may be shown that the integral in (18) can, for large values of s, be expressed in the form

$$\frac{1}{2s}\log s+\frac{2a+1}{2s}+\frac{4-\pi^2}{16s^2}+O(s^{-3}\log s),$$

where

$$a=\frac{1}{\pi}\int_0^{\pi/2}\tan y\left\{\frac{y}{1-y\,\mathrm{ctn}y}-\frac{\pi}{2}\right\}dy=1.0674.$$

From this asymptotic form of $\phi_0(s)$ for $s\to\infty$ we can deduce a from for $\psi_0(z)$ for small z. The expansion obtained is

$$\psi_0(z)=\sqrt{3}\{1-\tfrac{1}{2}z\log z+1.2788z+\tfrac{1}{16}z^2(\log z)^2-0.3822z^2\log z+0.7068z^2+O(z^3(\log z)^3)\}. \quad (19)$$

This expansion, as may be seen from the following tables, represents the density quite well for values of $z<0.1$ It could then be used to evaluate integrals containing $\psi_0(z)$ analytically over just that part of the range where $\psi_0(z)$ would give trouble in a numerical integration.

TABLE I. Evaluation of $q(z)=z+z_0-\rho(z)$, $\Delta=[q(z)-q_{\mathrm{app}}(z)]\times10^4$.

z	Correct	Lecaine[12]	Δ	Placzek[4]	Δ	Eddington[10]	Δ	Expansion (Section 5)	Δ
0	0.1331	0.1314	17	0.1331	0	0.1271	60	0.1331	0
0.01	0.1222	0.1213	9	0.1222	0	0.1193	29	0.1222	0
0.02	0.1150	0.1145	5	0.1151	−1	0.1076	74	0.1150	0
0.03	0.1092	0.1089	3	0.1094	−2	0.1016	76	0.1091	1
0.05	0.0997	0.0996	1	0.1001	−4	0.0921	76	0.0995	2
0.1	0.0825	0.0826	−1	0.0835	−10	0.0757	68	0.0817	8
0.2	0.0609	0.0609	0	0.0623	−14	0.0566	43	0.0582	27
0.3	0.0471	0.0470	1	0.0482	−11	0.0452	19		
0.4	0.0373	0.0371	2	0.0379	−6	0.0374	−1		
0.5	0.0301	0.0298	3	0.0301	0	0.0317	−16		
0.6	0.0246	0.0243	3	0.0240	6	0.0273	−27		
0.7	0.0203	0.0200	3	0.0191	12	0.0238	−35		
0.8	0.0169	0.0165	4	0.0153	16	0.0210	−41		
0.9	0.0141	0.0138	3	0.0122	19	0.0186	−45		
1.0	0.0119	0.0116	3	0.0097	22	0.0166	−47		
1.2	0.0085	0.0082	3	0.0061	24	0.0134	−49		
1.5	0.0053	0.0051	2	0.0030	23	0 0100	−47		
2.0	0.0025	0.0024	1	0.0008	17	0.0066	−41		
2.5	0.00125	0.00117	0.8	0.0001	11	0.0047	−35		
3.0	0.00064	0.00059	0.5	−0.0003	9	0.0036	−30		
3.5	0.00033	0.00030	0.3	−0.0001	4	0.0029	−26		
4.0	0.00018	0.00016	0.2						
5.0	0.000048	0.000046	0.02						

6. NUMERICAL RESULTS; DISCUSSION

If we define the function $q(z)=z+z_0-\rho(z)$, where $\rho(z)=\frac{1}{3}\psi_0(z)$, and set

$$1/h(\mu)=\varphi(\mu)[(1-\mu\,\mathrm{arth}\mu)^2+\pi^2\mu^2/4], \quad (20)$$

then from (1)

$$4\sqrt{3}q(z)=\int_0^1 h(\mu)e^{-z/\mu}d\mu. \quad (21)$$

Table I contains the results of the evaluation of $q(z)$. The last figure is considered to be reliable. The method used was as follows: The function $2+\mu\log\mu-h(\mu)$, which vanishes at $\mu=0$ and has a finite slope there, turned out to be almost straight from $\mu=0$ to $\mu=0.9$, so that it could be well approximated in that range by a polynomial $p(\mu)$ of the form $a\mu+b\mu^2+c\mu^3+d\mu^4$. Coefficients for $p(\mu)$ were chosen so that the function $k(\mu)=2+\mu\log\mu-h(\mu)-p(\mu)$, which vanishes at $\mu=0$, would also vanish at $\mu=0.1$, 0.2, and 0.9, and have $\int_0^1k(\mu)d\mu=0(\int_0^1h(\mu)d\mu$, known from the value of $q(0)$). It then turned out that $k(\mu)$ also vanished between $\mu=0.5$ and $\mu=0.6$, and that from $\mu=0$ to 0.9, $k(\mu)$ is quite smooth and rather small (having a maximum ~0.028 near $\mu=0.8$). Relation (21) was then rewritten in the form

$$4\sqrt{3}q(z)=\int_0^1[2-p(\mu)]e^{-z/\mu}d\mu+\int_0^1\mu\log\mu e^{-z/\mu}d\mu+\int_0^1(e^{-z}-e^{-z/\mu})k(\mu)du \quad (22)$$

$$=T_1+T_2+T_3.$$

In Eq. (22) T_1 can be expressed in terms of the functions

$$E_n(z)=\int_1^\infty u^{-n}e^{-zu}du,$$

which are tabulated;

$$T_2=-\tfrac{1}{4}+z-\tfrac{1}{4}z^2\log^2 z+(\tfrac{3}{4}-\tfrac{1}{2}\gamma)z^2\log z -(\tfrac{7}{8}-\tfrac{3}{4}\gamma+\tfrac{1}{4}\gamma^2+\pi^2/24)z^2 -\sum_3^\infty(-)^n z^n/(n-2)^2 n!;$$

and T_3 is the only term requiring numerical integration. Although $k(\mu)$ is appreciable and varies rapidly between $\mu=0.9$ and $\mu=1$, the integrand in T_3 is smooth and small, so that the maximum contribution of T_3 to the value of $q(z)$, which occurs near $z=1.5$, is only $7\cdot 10^{-5}$. As a check on the evaluation, we may use the value of M_0 given in Section 4 by which we should have

$$\int_0^\infty q(z)dz=0.04763.$$

The contribution to this coming from T_1 and T_2, which may be obtained exactly, is 0.04777, and a numerical integration of the values obtained for T_3 (which is negative) gave a contribution from this term of -0.00017.

Table II gives the evaluation of $\rho(z)$.

Several of the numerous approximations to $q(z)$ are compared with the correct function. For the approximations, the differences: $[q(z)-q_{\text{app}}(z)]\times 10^4$ are also given. In considering the approximations one must remember that the difference function in this problem is never very large and decreases rapidly, whereas $z+z_0$ increases, so that for large values of z one can tolerate very large relative errors in the difference function, and it is only for $z\ll 1$ that it must be given to good accuracy. Furthermore, even the simple straight-line approximation to $\rho(z)$ which fits at the boundary,[9] $\rho(z)=z+1\sqrt{3}$, gives at the worst an error of 9 percent (for $z\doteq 0.4$) in the value of $\rho(z)$. With this in mind, it is seen that the great complication (from an analytical point of view) of an approximation such as Eddington's[10] which still has errors up to 1.3 percent (for $z\doteq 0.02$) is quite unnecessary.

[9] E. Fermi, Ricerca Scient. **7** [2], 13 (1936).

TABLE II. Evaluation of $\rho(z)$.

z	$z+z_0$	$\rho(z)$
0	0.7104	0.5773
0.01	0.7204	0.5982
0.02	0.7304	0.6154
0.03	0.7404	0.6312
0.05	0.7604	0.6607
0.1	0.8104	0.7279
0.2	0.9104	0.8495
0.3	1.0104	0.9633
0.4	1.1104	1.0731
0.5	1.2104	1.1803
0.6	1.3104	1.2858
0.7	1.4104	1.3901
0.8	1.5104	1.4935
0.9	1.6104	1.5963
1.0	1.7104	1.6985
1.2	1.9104	1.9019
1.5	2.2104	2.2051
2.0	2.7104	2.7079
2.5	3.2104	3.2092
3.0	3.7104	3.7098
3.5	4.2104	4.2101
4.0	4.7104	4.7102

In fact, the elementary approximation $\rho(z)$ $z+z_0-a_0e^{-\alpha_0 z}$ ($a_0=0.133096$, $\alpha_0=3.6986$) given by Placzek[11] gives at the worst an error of about 2 percent (for z between 0.05 and 0.1). The approximations of Lecaine[12] (errors ≤ 0.3 percent) and Placzek[11] (errors ≤ 0.16 percent) are, however, significant improvements over the simple exponential approximation. Lecaine's approximation,[12] obtained by a variational method, is

$$\rho(z)=z+0.710446\times[1-0.342895E_2(z)+0.315870E_3(z)].$$

This has the advantage of a simple analytical form. The error of 0.3 percent at the boundary is quickly reduced, and for $z\geq 0.05$ the error is less than 0.05 percent.

Placzek's approximation,[11] obtained by an iteration method, is

$$\rho(z)=z+z_0+\tfrac{1}{2}[E_3(z)-z_0E_2(z)] -(a/2\alpha)\{[\log(\alpha+1/\alpha-1) +E_i((a-1)z)]e^{-\alpha z}+E_1(z)\},$$

with $a=0.11354$, $\alpha=2.62032$. This approximation is devised so as to be correct at the boundary. For some purposes the region close to the boundary is that of greatest interest. The maximum error,

[10] A. S. Eddington, *Internal Constitution of the Stars* (Cambridge University Press, Teddington, England, 1926).

[11] G. Placzek, *The Neutron Density near a Plane Surface*, I. MT-16, obtainable from Plans and Publications Branch, National Research Council of Canada, Ottawa.

[12] J. Lecaine, Phys. Rev. **72**, 564 (1947).

which is about 0.16 percent, occurs for a value of z a little greater than 0.1.

Eddington's approximation,[10] is

$$\rho(z)=(z+17/24)\left[\frac{2+3z-\{E_2(z)-\frac{3}{2}E_3(z)\}}{2+3z-3\{E_4(z)-\frac{3}{2}E_5(z)\}}\right].$$

This seems to be the best approximation available in previously published literature, although it was done at a rather early stage. The error of 1 percent at the boundary, increases to 1.3 percent at $z=0.02$ and is less than 0.5 percent after $z=0.3$.

The expansion of Section 5 is also evaluated in Table I. This has the correct analytical form near the boundary and is a good approximation up to $z=0.05$. By combining this with Lecaine's approximation one would have a fairly simple and very accurate representation of the density over the entire range.

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Note. Since the first writing of this paper, Wick[13] and Chandrasekhar[14] have published applications to this problem of the method of expanding the angular distribution $\psi(z, \mu)$ in Legendre polynomials in μ. Chandrasekhar's highest approximation, in which three exponentials are used, still has a maximum error (near $z=0.1$) of more than 3.5 percent, while even the elementary approximation referred to above which uses only one exponential has a maximum error of about 1.6 percent. In the light of the criteria which should be applied to approximations here, and the other examples already given, it should be pointed out that the polynomial method does not seem to be well adapted to the problem we are considering. Of course, in more complicated problems where simple iteration and variation techniques are not available, the polynomial method has had many very successful applications.

[13] G. C. Wick, "Über ebene Diffusionsprobleme," Zeits. f. Physik **121**, 702 (1943).
[14] S. Chandrasekhar, Astrophys. J. **101**, 348 (1945).

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Application of a Variational Method to Milne's Problem

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An approximate solution of Milne's integral equation for the neutron density is obtained by a variational method with high accuracy in simple analytical form. The extrapolated asymptotic density at the boundary is given by this method correct to 0.4 parts in a million. The density itself has a maximum error of 0.3 percent which occurs at the boundary and of less than 0.05 percent for all distances beyond 0.05 mean free paths. A simple expression for the angular distribution of emerging neutrons is also obtained.

THE exact solution of Milne's integral equation[1]

$$\psi_0(z)=\tfrac{1}{2}\int_0^\infty \psi_0(z')E_1(|z-z'|)dz' \qquad (1)$$

with

$$E_1(x)=-Ei(-x)=\int_1^\infty (e^{-xv}/v)dv$$

has been obtained by Mark[2] and evaluated by numerical integration.

We obtain here an approximate solution of this equation in simple analytical form by employing a variational technique.

From the integral equation it is seen that

$$\psi_0(z)=z+q(z), \qquad (2)$$

where $\lim_{z\to\infty}q(z)=z_0$. From Eqs. (9) and (14)

* Report issued May 15, 1944.
[1] For literature, see references 2 and 3.
[2] C. Mark, Phys. Rev. **72**, 558 (1947).