The Angular Distribution of Neutrons Emerging from a Plane Surface

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The Wiener-Hopf expression for the angular distribution in Milne's standard case is transformed into a form suitable for numerical evaluation. The results of the evaluation carried out by the Mathematical Tables Project are given.

1. INTRODUCTION

THE angular distribution of the neutrons emerging from a body of purely scattering material into a vacuum will in general depend upon the law of scattering and the source distribution in the body and, to a certain extent, also upon the size and shape of the body.

An important standard case, of interest in connection with fast as well as with slow neutron problems, is the following:

The body is infinite in two directions and is bounded by a plane. It contains no sources, and no neutrons enter the plane from the outside. A constant neutron current flows in the outward direction perpendicular to the plane. The law of scattering is isotropic without energy loss.

The problem of the determination of the angular distribution of the emerging neutrons for this case has been solved by Wiener and Hopf.[1] A simplified derivation of their expression has been given by Placzek and Seidel.[2] The present paper deals with a transformation of this expression into a more practical form and its numerical evaluation.[3]

2. THEORY

Let μ be the cosine of the angle between the direction of motion of the neutron and the outward normal, and $\varphi(\mu)d\mu$ the probability for the direction cosine of an emerging neutron to lie between μ and $\mu+d\mu$, so that

$$\int_0^1 \varphi(\mu)d\mu=1. \tag{1}$$

The function $\varphi(\mu)$ is given by[4] Eq. (46) of Placzek and Seidel.

$$\varphi(\mu)=\tfrac{1}{2}(1+\mu)\times\exp\left[\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\log[\sin^2x/(1-x\cot x)]}{1-(1-\mu^2)\sin^2x}dx\right]. \tag{2}$$

We integrate by parts, noting that

$$\mu\int dx/[1-(1-\mu^2)\sin^2x]=\operatorname{art}(\mu\tan x)$$

and

$$\frac{d}{dx}\log\left[\frac{\sin^2x}{1-x\cot x}\right]=3\cot x-\frac{x}{1-x\cot x},$$

and obtain

$$\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\log[\sin^2x/(1-x\cot x)]}{1-(1-\mu^2)\sin^2x}dx=\frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{x}{1-x\cot x}-3\cot x\right\}\operatorname{art}(\mu\tan x)dx, \tag{3}$$

since the integrated part vanishes.

The second part of this integral can be evaluated in closed form. Putting $\tan x=y$ and using

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[1] N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse (1931), p. 696; see also E. Hopf, Cambridge Tracts No. 31 (1934).

[2] G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947).

[3] After the present work was done, results of a numerical evaluation of the original Wiener-Hopf expression have been published by S. Chandrasekhar (Astrophys. J. 99, 180 1944). The use of this function for numerical integrations, of which the following paper by Mark gives a significant example, requires the more accurate table given in the present paper.

[4] In G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947), the *inward* direction is counted positive, while for the purposes of the present paper it has been more convenient to define the *outward* direction as the positive one. Also, the function $\psi(0,\mu)$ in Placzek and Seidel is normalized to unit *current*, while we use here normalization to unit *density* (Eq. (1)). In order to express $\varphi(\mu)$ by $\psi(0,\mu)$ we have, therefore, to put

$$\varphi(\mu)=\psi(0,-\mu)/\sqrt{3}.$$

the fact that the integral from 0 to ∞ of the product of two even functions is equal to $1/2\pi$ times the integral from 0 to ∞ of the product of their Fourier transforms, we obtain:

$$\pi^{-1}\int_0^{\pi/2}\cot x\,\mathrm{art}(\mu\tan x)dx$$

$$=\pi^{-1}\int_0^{\infty}[\mathrm{art}(\mu y)/y(1+y^2)]dy$$

$$=\tfrac{1}{2}\int_0^{\infty}\mathrm{E}(z/\mu)e^{-z}dz=\tfrac{1}{2}\log(1+\mu),\quad (4)$$

where

$$\mathrm{E}(u)=\int_u^{\infty}v^{-1}e^{-v}dv$$

is the exponential integral.

From (4), (3), and (2) we obtain now

$$\boxed{\varphi(\mu)=\frac{1}{2(1+\mu)^{\frac{1}{2}}}\exp\left[\frac{1}{\pi}\int_0^{\pi/2}\frac{x\,\mathrm{art}(\mu\tan x)}{1-x\cot x}dx\right]}\quad (5)$$

From (5) it can be seen in a simpler way than from (2) that, for very small values of μ, $\varphi(\mu)$ is asymptotically given by

$$\varphi(\mu)=\tfrac{1}{2}(1-\tfrac{1}{2}\mu\log\mu),\quad \mu\ll 1.\quad (6)$$

Thus, the derivative of $\varphi(\mu)$ becomes logarithmically infinite for $\mu=0$. This is directly connected with the well-known fact that the spatial derivative of the neutron density has a logarithmic infinity at the surface.

The integral in (5) has been evaluated numerically by the Mathematical Tables Project. For 10 values of the argument accurate values of the integral were obtained by numerical integration, and from these the intermediate values were found by interpolation. Table I gives the results of the numerical integration at intervals of 0.1, while Table II gives the interpolated values at intervals of 0.01. The second table is in a suitable form for use in connection with the numerical integration of expressions containing the function φ, except that even for this close spacing linear interpolation is quite insufficient for small values of μ. In this region the difference

TABLE I. Values of $\varphi(\mu)$ at intervals of 0.1.

μ	$\varphi(\mu)$
0	0.5000000
0.1	0.6236751
0.2	0.7251757
0.3	0.8212611
0.4	0.9146378
0.5	1.0063894
0.6	1.0970665
0.7	1.1869875
0.8	1.2763522
0.9	1.3652938
1.0	1.4539053

TABLE II. Values of $\varphi(\mu)$ at intervals of 0.01.

μ	$\varphi(\mu)$	Δ	$-\Delta_2$	μ	$\varphi(\mu)$	Δ
0.00	0.50000	1713	312	0.50	1.00639	911
1	0.51713	1401	105	1	1.01550	910
2	0.53114	1296	64	2	1.02460	909
3	0.54410	1232	45	3	1.03369	908
4	0.55642	1187	34	4	1.04277	907
0.05	0.56829	1153	27	0.55	1.05184	906
6	0.57982	1126	22	6	1.06090	905
7	0.59108	1104	18	7	1.06995	905
8	0.60212	1086	16	8	1.07900	904
9	0.61298	1070	14	9	1.08804	903
0.10	0.62368	1056	12	0.60	1.09707	902
1	0.63424	1044	10	1	1.10609	901
2	0.64468	1034	10	2	1.11510	901
3	0.65502	1024	8	3	1.12411	900
4	0.66526	1016	8	4	1.13311	900
0.15	0.67542	1008	7	0.65	1.14211	899
6	0.68550	1001	7	6	1.15110	898
7	0.69551	994	5	7	1.16008	897
8	0.70545	989	5	8	1.16905	897
9	0.71534	984	6	9	1.17802	897
0.20	0.72518	978		0.70	1.18699	896
1	0.73496	974		1	1.19595	895
2	0.74470	969		2	1.20490	895
3	0.75439	966		3	1.21385	894
4	0.76405	961		4	1.22279	894
0.25	0.77366	958		0.75	1.23173	893
6	0.78324	955		6	1.24066	893
7	0.79279	952		7	1.24959	893
8	0.80231	949		8	1.25852	892
9	0.81180	946		9	1.26744	891
0.30	0.82126	944		0.80	1.27636	891
1	0.83070	941		1	1.28526	891
2	0.84011	938		2	1.29417	890
3	0.84949	937		3	1.30307	890
4	0.85886	934		4	1.31197	890
0.35	0.86820	933		0.85	1.32087	889
6	0.87753	930		6	1.32976	889
7	0.88683	929		7	1.33865	888
8	0.89612	927		8	1.34753	889
9	0.90539	925		9	1.35642	887
0.40	0.91464	923		0.90	1.36529	888
1	0.92387	923		1	1.37417	887
2	0.93310	920		2	1.38304	887
3	0.94230	920		3	1.39191	886
4	0.95150	917		4	1.40077	887
0.45	0.96067	917		0.95	1.40964	886
6	0.96984	916		6	1.41850	885
7	0.97900	914		7	1.42735	886
8	0.98814	913		8	1.43621	885
9	0.99727	912		9	1.44506	885
0.50	1.00639	911		1.00	1.45391	

of φ and the asymptotic expression (6) is best used for interpolation.

It is seen from the tables that $\varphi(\mu)$ is very close to a straight line except for small μ. Fermi's simple linear approximation[5]

$$\varphi(\mu)=(1+\sqrt{3}\mu)/(1+\sqrt{3}/2) \qquad (7)$$

has an error of 7.2 percent at $\mu=0$, but for $\mu>0.1$ its error is below one percent throughout. This has to be kept in mind when discussing the more complicated approximations.[6–9]

[5] E. Fermi, Ricerca Scient. **7** [2] 13 (1936).

[6] A. Unsoeld, *Physik der Sternatmosphaeren* (Julius Springer, Berlin, 1939).

[7] J. LeCaine, Phys. Rev. **72**, 564 (1947), Eq. (12).

[8] G. Placzek, Montreal Report MT 16, 1944; reissued by National Research Council of Canada, Chalk River 1947. See also Eq. (13) in LeCaine, l.c.

[9] S. Chandrasekhar, Astrophys. J. **99**, 180 (1944); **101**, 348 (1945).

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The Neutron Density Near a Plane Surface

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The exact solution of Milne's integral equation is expressed as a real integral with non-oscillating integrand. This expression has been derived from the Wiener-Hopf solution for the Laplace transform of the density. The integrand involves the angular distribution of neutrons emerging from the surface, and the tabulation of this function by the Mathematical Tables Project given by Placzek has been used in the numerical evaluation of the integral. The values of the first three moments of the difference between the density and the asymptotic density, and an expansion of the density for points near the boundary are also given. Various authors have proposed or obtained approximations to the solution of this problem, and some of these approximations are referred to and compared with the exact solution.

1. INTRODUCTION

THE purpose of this paper is to determine the neutron density in Milne's problem as described by Placzek and Seidel.[1] We shall adopt the notation and definitions of their paper and make frequent references to its results.

The neutron density, $\psi_0(z)$, satisfies Milne's integral equation

$$\psi_0(z)=\tfrac{1}{2}\int_0^\infty \psi_0(z')E(|z-z'|)dz',$$

with $E(x)=-Ei(-x)$. An expression for the Laplace transform of the solution of this equation has been obtained by Wiener and Hopf.[2,3] A simplified derivation of their result is given in PS.

The angular distribution of the emerging neutrons is, except for a factor, equal to the Laplace transform of the density (PS, Eq. (19)), so that the problem of determining the emergent angular distribution is simply that of evaluating the expression for this Laplace transform; and an extensive and accurate tabulation of this is now available.[4] However, considerable difficulties have been encountered in attempts to invert the Laplace transform of the density in order to obtain the density itself, and an exact yet manageable integral for $\psi_0(z)$ does not seem to have been given heretofore. In this paper it is

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** This paper, except for minor modifications, corrections, and improvements in some of the numerical work, was issued as a report of the Theoretical Division of the Montreal Laboratory on April 15, 1944.

[1] G. Placzek and W. Seidel, Phys. Rev. **72**, 550 (1947). Hereafter this paper will be referred to as PS.

[2] N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse (1931), p. 696.

[3] E. Hopf, *Mathematical Problems of Radiative Equilibrium*, Cambridge tracts No. 31, 1934.

[4] G. Placzek, Phys. Rev. **72**, 556 (1947).