Milne's Problem in Transport Theory

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A modified derivation of the Wiener-Hopf solution of Milne's problem is given in a form suitable for application to problems in the theory of neutron diffusion.

I. INTRODUCTION

WE consider the following problem: A half-space $z>0$ bounded by the plane $z=0$ is filled by a non-capturing medium, which scatters neutrons isotropically without changing their velocity. No sources are present in the medium and no neutrons enter the plane $z=0$ from outside. A current density of magnitude j and direction $-z$ exists in the medium. We wish to determine the stationary neutron distribution in the medium and, in particular, the angular distribution of the neutrons leaving the plane $z=0$.

This problem, which represents an important standard case in the study of neutron diffusion, is completely identical with a problem known in astrophysical literature as "Milne's case." It has been extensively discussed in connection with the determination of the law of darkening at the sun's surface.[1] Its solution, explicit as far as the angular distribution of the emerging radiation is concerned, has been obtained by Wiener and Hopf.[2] Their method can also be used with advantage for the solution of other neutron diffusion problems, but their presentation is somewhat encumbered by generalizations in directions different from those of interest to us. We shall, therefore, in the following, give a somewhat modified derivation of Wiener and Hopf's results, in such a form as will most easily lend itself to generalizations to be discussed in later papers.

II. THE TRANSPORT EQUATION AND INTEGRAL RELATIONS FOLLOWING FROM IT

We denote by μ the cosine of the angle between the direction of the neutron and the positive z direction and by $\psi(z,\mu)d\mu$ the number of neutrons per unit volume at the point z with direction cosine between μ and $\mu+d\mu$. We call

$$\psi_0(z)=\int_{-1}^{1}\psi(z,\mu)d\mu$$

the neutron density.

Choosing as unit of length the mean free path and as unit of velocity the neutron velocity, we have the transport equation

$$\mu(\partial\psi/\partial z)+\psi=\tfrac{1}{2}\psi_0, \tag{1}$$

with the boundary condition

$$\psi(0,\mu)=0 \quad \text{for} \quad \mu>0, \tag{2}$$

since no neutrons enter the medium from outside.

Two important relations can be obtained at once from (1) and (2). Integrating (1) over the variable μ we obtain, with the notation

$$j=-\int_{-1}^{1}\mu\psi(z,\mu)d\mu, \tag{3}$$

$$\partial j/\partial z=0. \tag{4}$$

Equation (4) shows that the current density j is constant. We shall put

$$j=1 \tag{5}$$

and thus have $\psi(z,\mu)$ normalized for unit current density.

We now multiply (1) by μ and integrate over

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[1] For literature, see E. Milne, *Handbuch der Astrophysik*, Vol. 3, p. 1 and E. Hopf, "Mathematical problems of radiative equilibrium," *Cambridge Tracts No. 31* (1934).
[2] N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse 696 (1931); see also E. Hopf, *Cambridge Tracts No. 31*, (1934); R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (American Mathematical Society Collection, New York, 1934), Vol. XIX, Chap. IV; E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, New York, 1937), Chap. XI.

the variable μ. Putting

$$K(z)=\int_{-1}^{1}\mu^2\psi(z,\mu)d\mu \tag{6}$$

we have

$$\partial K/\partial z=1, \tag{7}$$

and hence

$$K(z)=z+z_0, \tag{8}$$

where the constant z_0 is defined by (see Eq. (2))

$$z_0=K(0)=\int_{-1}^{0}\mu^2\psi(0,\mu)d\mu. \tag{9}$$

III. MILNE'S INTEGRAL EQUATION

On substitution of the transformation

$$\psi(z,\mu)=\chi(z,\mu)e^{-z/\mu} \tag{10}$$

into Eq. (1), integration of (1) over z with the boundary condition (2), and resubstitution from (10), we obtain

$$\psi(z,\mu)=\begin{cases}(2\mu)^{-1}\int_0^z\psi_0(z')\exp[(z'-z)/\mu]dz' & \text{if } \mu>0 \quad (11a)\\ -(2\mu)^{-1}\int_z^\infty\psi_0(z')\exp[(z'-z)/\mu]dz' & \text{if } \mu<0. \quad (11b)\end{cases}$$

Integration of Eqs. (11) over μ yields

$$\psi_0(z)=\tfrac{1}{2}\int_0^\infty\psi_0(z')\mathrm{E}(|z-z'|)dz' \tag{12}$$

with

$$\mathrm{E}(x)=-\mathrm{Ei}(-x)=\int_x^\infty(e^{-u}/u)du. \tag{13}$$

Equation (12) is known as Milne's integral equation. Its solution determines $\psi(z,\mu)$ by Eqs. (11). In particular, we have for the angular distribution of the neutrons emerging from the medium (Eq. (11b) for $z=0$)

$$\psi(0,\mu)=-(2\mu)^{-1}\int_0^\infty\psi_0(z')\exp(z'/\mu)dz', \qquad \mu<0. \tag{14}$$

Thus, $\psi(0,\mu)$ can be represented by the Laplace transform of $\psi_0(z)$. Equations (12) and (11) are an equivalent formulation of the problem stated by Eqs. (1) and (2), and it is customary to start the treatment of the problem from Eq. (12). In view of the fact that in the case of other problems with more complicated scattering laws the reduction of the transport equation to a homogeneous integral equation for the neutron density is in general not possible, here also we shall start directly from the transport Eq. (1) with condition (2) rather than from Milne's integral equation.

IV. AN INTEGRAL EQUATION FOR THE LAPLACE TRANSFORM OF THE NEUTRON DENSITY; ASYMPTOTIC FORM OF $\psi_0(z)$

We introduce the Laplace transforms of $\psi(z,\mu)$ and $\psi_0(z)$ by

$$\phi(s,\mu)=\int_0^\infty\psi(z,\mu)e^{-sz}dz, \tag{15a}$$

$$\phi_0(s)=\int_{-1}^{1}\phi(s,\mu)d\mu=\int_0^\infty\psi_0(z)e^{-sz}dz, \tag{15b}$$

where s is a complex variable $\Re(s)>0$. We multiply (1) by e^{-sz} and integrate over z from 0 to ∞. Noting that, by partial integration,

$$\int_0^\infty e^{-sz}\partial\psi/\partial z dz = |e^{-sz}\psi(z,\mu)|_0^\infty+s\int_0^\infty e^{-sz}\psi(z,\mu)dz = -\psi(0,\mu)+s\phi(s,\mu),$$

we obtain

$$\phi(s,\mu)=[\tfrac{1}{2}\phi_0(s)+\mu\psi(0,\mu)]/(1+s\mu). \tag{16}$$

By integration of (16) over μ we have

$$\phi_0(s)\left\{1-\tfrac{1}{2}\int_{-1}^{1}(1+s\mu)^{-1}d\mu\right\}=\int_{-1}^{0}\mu\psi(0,\mu)(1+s\mu)^{-1}d\mu$$

and, since[3]

$$\tfrac{1}{2}\int_{-1}^{1}(1+s\mu)^{-1}d\mu=\frac{1}{2s}\log[(1+s)/(1-s)]=s^{-1}\operatorname{arth}s, \tag{17}$$

[3] Here and in the following we use the notation arth for $\tanh^{-1}$ and art for $\tan^{-1}$.

$$\phi_0(s)=\frac{\int_{-1}^{0}(1+s\mu)^{-1}\mu\psi(0,\mu)d\mu}{1-s^{-1}\operatorname{arth}s}. \tag{18}$$

Comparison of (15b) and (14) yields

$$\mu\psi(0,\mu)=-\tfrac{1}{2}\phi_0(-1/\mu). \tag{19}$$

Hence we may write (18)

$$\phi_0(s)\{1-s^{-1}\operatorname{arth}s\}=-\tfrac{1}{2}\int_{-1}^{0}\frac{\phi_0(-1/\mu)}{1+s\mu}d\mu. \tag{20}$$

Before attempting to solve this integral equation for $\phi_0(s)$, we note that the asymptotic form of $\psi_0(z)$ for large z is determined by the behavior of $\phi_0(s)$ at $s=0$, which can be immediately ascertained by expanding (18) in a Laurent series about $s=0$. Noting that

$$\operatorname{arth}s=s+\tfrac{1}{3}s^3+\cdots \tag{21}$$

and using the relations (3), (5), and (9), we obtain

$$\phi_0(s)=3s^{-2}+3z_0s^{-1}+\cdots \tag{22}$$

and hence the asymptotic form[4] of $\psi_0(z)$ is given by

$$\psi_0(z)\sim 3(z+z_0)\quad\text{as}\quad z\to\infty. \tag{23}$$

TABLE I. Functional properties of $\phi_0(s)$, $s^{-1}\operatorname{arth}s$, $g(s)$.

Function	Domain of regularity*	Proof
$\phi_0(s)$	$\Re(s)>0$	Follows from definition (15b) and asymptotic form (23)
$s^{-1}\operatorname{arth}s$	$-1<\Re(s)<1$	Follows from definition (17)
$g(s)$	$\Re(s)<1$	Follows from definition (24)**

* One can actually show that the functions are analytic in domains wider than those indicated above, but we will not need to make use of this fact.

** From (24) it is seen that the only possible singularities of $g(s)$ can occur for those values of s for which the denominator of the integrand vanishes.

[4] That $\psi_0(z)$ has this asymptotic form follows also directly from (1) by noting that $\psi(z,\mu)$ must be almost isotropic for large z. It is therefore legitimate to write $\psi(z,\mu)$ for large z in the form

$$\psi(z,\mu)=\tfrac{1}{2}\psi_0(z)-\tfrac{3}{2}j\mu=\tfrac{1}{2}\psi_0(z)-\tfrac{3}{2}\mu. \tag{A}$$

Equation (A) can be considered as an expansion of $\psi(z,\mu)$ in Legendre polynomials neglecting all higher terms (diffusion approximation). Introduction of (A) into (1) yields

$$\psi_0(z)=3(z+\text{const.}).$$

V. SOLUTION OF THE INTEGRAL EQUATION FOR $\phi_0(s)$ BY THE METHOD OF WIENER AND HOPF

Setting

$$g(s)=\int_{-1}^{0}(1+s\mu)^{-1}\mu\psi(\mu,0)d\mu, \tag{24}$$

we may write (18) in the form

$$\phi_0(s)\{1-s^{-1}\operatorname{arth}s\}=g(s). \tag{25}$$

This integral equation can be solved by considering the domain in which the functions occurring in the equation are analytic. We shall form a certain function containing $\phi_0(s)$ and $(s^{-1}\operatorname{arth}s)$ which will have the property of being analytic and bounded in the whole complex s-plane. According to Liouville's theorem in the theory of complex variables, such a function must be a constant. From this, therefore, it will be possible to determine $\phi_0(s)$ in terms of the known function $(s^{-1}\operatorname{arth}s)$.

The functions occurring in Eq. (25) can immediately be shown to be analytic in the domains given in Table I.

In order to solve (25) we shall try to re-write it in such a manner that the left-hand side is analytic in a half-plane and the right-hand side in another half-plane overlapping the first so that both half-planes fill out the whole plane. Then the two sides of the equation can be considered as the analytical continuations of each other and therefore will represent the same function. This function will be analytic on the whole plane. If, in addition, it turns out to be bounded in the whole plane, it must be equal to a constant as mentioned above.

As it stands, Eq. (25) is not in the desired form, since the function $1-s^{-1}\operatorname{arth}s$ is analytic in a strip rather than a half-plane, namely, the strip $-1<\Re(s)<1$. Such a function can, however, be written as a quotient of two functions, each analytic in a half-plane, provided that the function (a) has no zeros in the strip and (b) that it tends to unity as $|s|\to\infty$ in the strip. The function $1-s^{-1}\operatorname{arth}s$ has, according to (21), a double zero at the origin and, as may easily be shown, no other zeros in the strip. We can satisfy condition (a) by replacing the function by $s^{-2}(1-s^{-1}\operatorname{arth}s)$. This function, however, will not satisfy condition (b). Hence, we form the function

$$\tau(s) = s^{-2}(s^2-1)(1-s\ \mathrm{arth}s) \tag{26}$$

which satisfies both conditions. In terms of $\tau(s)$, Eq. (25) reads

$$[s^2/(s^2-1)]\phi_0(s)\tau(s) = g(s). \tag{27}$$

In order to decompose $\tau(s)$ in the desired manner we consider $\log\tau(s)$ which is single-valued in the strip $-1<\mathcal{R}(s)<1$, provided we choose a particular determination of the logarithm. We shall choose it so that $\log 1=0$. Since $\tau(s)$ satisfies condition (b), $\log\tau(s)\to 0$ as $|s|\to\infty$ in the strip. Consequently, $\log\tau(s)$ can be represented by the Cauchy integral formula in the form

$$\log\tau(s) = (2\pi i)^{-1}\int_{\beta-i\infty}^{\beta+i\infty}(u-s)^{-1}\log\tau(u)du - (2\pi i)^{-1}\int_{-\beta-i\infty}^{-\beta+i\infty}(u-s)^{-1}\log\tau(u)du$$

where $0<\beta<1$ and $-\beta<\mathcal{R}(s)<\beta$. Defining

$$\tau_-(s) = \exp\left[(2\pi i)^{-1}\int_{-\beta-i\infty}^{-\beta+i\infty}(u-s)^{-1}\log\tau(u)du\right], \tag{28}$$

$$\tau_+(s) = \exp\left[(2\pi i)^{-1}\int_{\beta-i\infty}^{\beta+i\infty}(u-s)^{-1}\log\tau(u)du\right], \tag{29}$$

we obtain

$$\log\tau(s) = \log\tau_+(s) - \log\tau_-(s)$$

or

$$\tau(s) = \tau_+(s)/\tau_-(s), \tag{30}$$

where $\tau_-(s)$ is regular and different from zero in the half-plane $\mathcal{R}(s)>-\beta$, while $\tau_+(s)$ is regular and different from zero in the half-plane $\mathcal{R}(s)<\beta$. Putting Eq. (30) into (27), one may write it in the form

$$s^2\phi_0(s)/(s+1)\tau_-(s) = (s-1)g(s)/\tau_+(s). \tag{31}$$

Here, the left-hand side is regular in $\mathcal{R}(s)>0$, while the right-hand side is regular in $\mathcal{R}(s)<\beta$. Thus, each side of (31) represents the analytic continuation of the other, so that either side is regular in the whole finite s-plane. It is necessary now to investigate the order of magnitude of each side of (31). From (28) and (29) it follows (see appendix) that in any half-plane $\mathcal{R}(s)>-\beta'>-\beta$, $\tau_-(s)$ is bounded away from zero, that is,

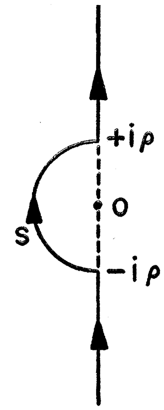

FIG. 1. Path of integration for $\tau_-(0)$.

$|\tau_-(s)|>C_1>0$ and similarly in any half-plane $\mathcal{R}(s)<\beta'<\beta$, $|\tau_+(s)|>C_2>0$, where C_1 and C_2 are constants. Furthermore, the definitions (15b) and (24) imply that in their respective half-planes of regularity $\phi_0(s)$ and $g(s)$ are $0(1/|s|)$ at infinity. These conditions imply that each side of (31) is bounded at infinity. Since each side is regular elsewhere it must be bounded in the whole plane. Hence, it fulfills the conditions of Liouville's theorem and must be identically equal to a constant C. Therefore, from (31)

$$\phi_0(s) = C(s+1)\tau_-(s)/s^2. \tag{32}$$

In order to determine C we expand $\phi_0(s)$ in a Laurent series about $s=0$

$$\phi_0(s) = C\tau_-(0)s^{-2} + C[\tau_-(0)+\tau_-'(0)]s^{-1}+\cdots. \tag{33}$$

Comparison with (22) yields

$$C = 3/\tau_-(0). \tag{34}$$

We now evaluate $\tau_-(0)$. From (28) we have

$$1/\tau_-(0) = \exp\left[-(2\pi i)^{-1}\int_{-\beta-i\infty}^{-\beta+i\infty}u^{-1}\log\tau(u)du\right].$$

We deform the path of integration into the two segments $(-i\infty, -i\rho)$, $(i\rho, i\infty)$ of the imaginary axis and the semicircle S of radius ρ, as shown on Fig. 1, ρ being any positive number. The two integrals over the segments cancel since, according to (26), the integrand is odd so that

$$1/\tau_-(0) = \exp\left[-(2\pi i)^{-1}\int_S u^{-1}\log\tau(u)du\right].$$

On letting $\rho\rightarrow 0$ in the integral, we find

$$1/\tau_-(0)=\exp[\tfrac{1}{2}\log\tau(0)]=\surd\tau(0).$$

From (26) and (21) we have $\tau(0)=\frac{1}{3}$, so that from (34)

$$C=\sqrt{3} \qquad (35)$$

and thus from (32)

$$\phi_0(s)=\sqrt{3}(s+1)\tau_-(s)/s^2. \qquad (36)$$

VI. DETERMINATION OF z_0 AND $\psi_0(0)$

Again comparing (33) with (22), we obtain

$$z_0=1+\tau_-'(0)/\tau_-(0).$$

From (28) we get

$$\tau_-'(0)/\tau_-(0)=(2\pi i)^{-1}\int_{-\beta-i\infty}^{-\beta+i\infty} u^{-2}\log\tau(u)du.$$

Integration by parts yields

$$\tau_-'(0)/\tau_-(0)=(2\pi i)^{-1}\int_{-\beta-i\infty}^{-\beta+i\infty}[\tau'(u)/u\tau(u)]du.$$

Since $\tau(u)$ is even, $\tau(0)\neq 0$, and $\tau'(0)=0$, the integrand does not have a singularity at $u=0$, and we may move the path of integration to the imaginary axis. Then, since the integrand is also even, we may write

$$\tau_-'(0)/\tau_-(0)=(\pi i)^{-1}\int_0^{i\infty}[\tau'(u)/u\tau(u)]du.$$

Introducing now the explicit form of τ as given by (26) and replacing the variable of integration by $s=it$, we obtain

$$z_0=\frac{1}{\pi}\int_0^{\infty}\left\{\frac{3}{t^2}-\frac{1}{(1+t^2)(1-t^{-1}\operatorname{art}t)}\right\}dt \qquad (37)$$

or, substituting $t=\tan x$,

$$z_0=\frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{3}{\sin^2 x}-\frac{1}{1-x\cot x}\right\}dx. \qquad (38)$$

Each of the two parts of the integrand goes to infinity as $3/x^2$ as $x\rightarrow 0$. Writing

$$z_0=I_1+I_2$$

with

$$I_1=\frac{3}{\pi}\int_0^{\pi/2}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)dx=\frac{6}{\pi^2},$$

$$I_2=\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{3}{x^2}-\frac{1}{1-x\cot x}\right)dx,$$

we have

$$z_0=\frac{6}{\pi^2}+\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{3}{x^2}-\frac{1}{1-x\cot x}\right)dx. \qquad (39)$$

The integrand in (39) can be expanded in a power series in x, which converges rapidly in the whole range of integration, and then integrated term by term, with the result[5]

$$z_0=0.71044609. \qquad (40)$$

As seen from (23), the value of the neutron density at the boundary, extrapolated from the asymptotic solution, is $3z_0$. We now wish to determine the true neutron density at the boundary, $\psi_0(0)$. It is readily seen that

$$\psi_0(0)=\lim_{s\rightarrow\infty} s\phi_0(s). \qquad (41)$$

Indeed, the relation follows formally from (15b) by introducing the new variable $u=sz$. Equation (15b) then becomes

$$s\phi_0(s)=\int_0^{\infty}\psi_0(u/s)e^{-u}du.$$

Here, letting $s\rightarrow\infty$ and interchanging the limit and integral sign,[6] we obtain

$$\lim_{s\rightarrow\infty} s\phi_0(s)=\int_0^{\infty}\psi_0(0)e^{-u}du=\psi_0(0).$$

Introducing (36) into (41) we now obtain

$$\psi_0(0)=\sqrt{3}\lim_{s\rightarrow\infty}\tau_-(s). \qquad (42)$$

From (28) it is seen that $\tau_-(s)\rightarrow 1$ as $s\rightarrow\infty$ and hence[7]

$$\psi_0(0)=\sqrt{3}. \qquad (43)$$

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[5] The computation was carried out by Dr. P. R. Wallace and Mr. B. Carlson. As will be seen from later papers, very accurate knowledge of z_0 is necessary for various approximation methods.

[6] This step may readily be justified rigorously.

[7] For other derivations of this result, see M. Bronstein, Zeits. f. Physik **58**, 696, **59**, 144 (1929); E. Hopf, see reference 1.

VII. THE ANGULAR DISTRIBUTION OF THE EMERGING NEUTRONS

We wish to express the angular distribution of the emerging neutrons, as given by (19), (36), and (28), by a real integral. For this purpose, we have to transform $\tau_-(s)$. Again deforming the path of integration according to Fig. 1, we have:

$$\log\tau_-(s)=\frac{1}{2\pi i}\int_{-i\infty}^{-i\rho}\frac{\log\tau(u)}{u-s}du+\frac{1}{2\pi i}\int_S\frac{\log\tau(u)}{u-s}du+\frac{1}{2\pi i}\int_{i\rho}^{i\infty}\frac{\log\tau(u)}{u-s}du.$$

The first integral may be written as

$$\int_{-i\infty}^{-i\rho}(u-s)^{-1}\log\tau(u)du=-\int_{i\rho}^{i\infty}(u+s)^{-1}\log\tau(u)du.$$

As $\rho\to 0$, the middle integral approaches zero, for $s\neq 0$. Combining the first and third integrals we obtain

$$\log\tau_-(s)=\frac{s}{\pi i}\int_{i0}^{i\infty}\frac{\log\tau(u)}{u^2-s^2}du=-\frac{s}{\pi}\int_0^\infty\frac{\log\tau(it)}{t^2+s^2}dt$$

and, on substituting from (26),

$$\log\tau_-(s)=-\frac{s}{\pi}\int_0^\infty\frac{\log\left\{\dfrac{t^2+1}{t^2}\left(1-\dfrac{\operatorname{art}t}{t}\right)\right\}}{t^2+s^2}dt. \quad (44)$$

Putting $t=\tan x$ we have finally

$$\log\tau_-(s)=\frac{s}{\pi}\int_0^{\pi/2}\frac{\log[\sin^2x/(1-x\cot x)]}{\sin^2x+s^2\cos^2x}dx \quad (45)$$

and, with (19) and (36):

$$\psi(0,\mu)=\tfrac{1}{2}\sqrt{3}(1-\mu)\tau_-(-1/\mu)=\tfrac{1}{2}\sqrt{3}(1-\mu)\times\exp\left[\frac{-\mu}{\pi}\int_0^{\pi/2}\frac{\log[\sin^2x/(1-x\cot x)]}{1-(1-\mu^2)\sin^2x}dx\right],\quad \mu<0. \quad (46)$$

The numerical evaluation of this expression is given in the following paper.[8]

APPENDIX

It remains to be shown that the functions $\tau_-(s)$ and $\tau_+(s)$, defined in (28) and (29), are such that $|\tau_-(s)|>C_1>0$ in any half-plane $\Re(s)>-\beta'>-\beta$ and $|\tau_+(s)|>C_2>0$ in any half-plane $\Re(s)<\beta'<\beta$. We shall confine ourselves to proving the assertion for $\tau_-(s)$. An entirely analogous argument holds for $\tau_+(s)$. From (28) it is evidently sufficient to prove that in any half-plane $\Re(s)>-\beta'>-\beta$ the integral

$$\int_{-\beta-i\infty}^{-\beta+i\infty}(u-s)^{-1}\log\tau(u)du$$

is bounded. Now for large values of u the function $\tau(u)$ is of the form

$$\tau(u)=1+O(1/|u^2|),$$

so that $\log\tau(u)$ is quadratically integrable. Hence, applying Schwarz's inequality to the integral,

$$\left|\int_{-\beta-i\infty}^{-\beta+i\infty}\frac{\log\tau(u)}{u-s}du\right|^2\leqslant\int_{-\beta-i\infty}^{-\beta+i\infty}|\log\tau(u)|^2du\times\int_{-\beta-i\infty}^{-\beta+i\infty}\frac{|du|}{|u-s|^2}.$$

The assertion follows at once since the denominator of the second integral stays uniformly away from zero whenever $\Re(s)>-\beta'>-\beta$.

[8] G. Placzek, Phys. Rev. 72, 556 (1947).