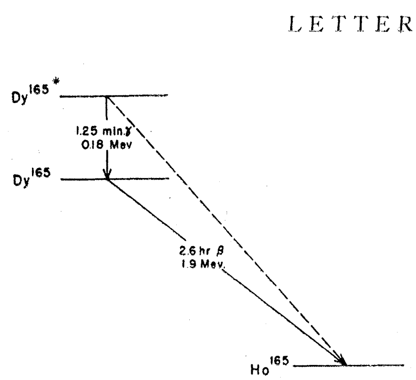

FIG. 2. Energy-level diagram for Dy^{165}.

components. The 2.6-hour activity was previously shown by a different technique[3] to be due to Dy^{165}. The present result shows that both the 1.25-minute and the 2.6-hour activities are produced by neutron irradiation of Dy^{164}, and since the ratio of 1.25-minute to 2.6-hour activities was the same with and without cadmium shielding, the short-lived activity must also have been induced by the slow neutrons in the pile. Thus the 1.25-minute activity must have been formed by an (n, γ) reaction on Dy^{164}, and thus be isomeric with the 2.6-hour activity.

The energy-level diagram for these activities is shown in Fig. 2. Decay curves of activated dysprosium, corrected for the variation of counter efficiency with energy, show that the cross sections for formation of the 1.25-minute and 2.6-hour activities are approximately equal. Thus it appears probable that only the higher excited state of Dy^{165} is formed directly in the (n, γ) reaction on Dy^{164}. If this is the case, the growth of the 2.6-hour activity after a very short bombardment should be observed. Flammersfeld looked for but did not observe this growth. A possible explanation for his negative result is that the higher excited Dy^{164} nucleus emits a beta-particle and transforms directly to Ho^{165}. The emission of one of these betas to every 100 gammas would explain Flammersfeld's results.

* This document is based on work performed under Contract No. W-31-109-eng-38 for the Atomic Energy Project at the Argonne National Laboratory.
[1] A. Flammersfeld, Zeits. f. Naturforschung 1, 190 (1946).
[2] A. J. Dempster, Proc. Am. Phil. Soc. 75, 755 (1935).
[3] Inghram, Hayden, and Hess, Phys. Rev. 71, 270 (1947).

On the Normalization of Characteristic Differentials in Continuous Spectra

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FROM the point of view of explicit determinations, the normalization of characteristic differentials in continuous spectra is one of the most delicate analytical questions in wave mechanics. In many cases, the asymptotic rule to be deduced below can be applied with little effort. What are actually needed are the spectral *densities*, but it will become clear at the end of the deduction that the asymptotic *differentiation* in question is legitimate.

The rule to be obtained is of the Stokes-Kelvin type, but is more explicit in nature. Its proof involves neither the complex machinery of the "steepest descent" nor the principle of stationary phases (Kramers). Correspondingly, nothing like the solution of the wave equation in terms of a definite integral is assumed.

If the energy level is fixed within the "elliptic," or "stable," range of the potential, both the one-dimensional and the radial Schrödinger equations are of the form

$$\psi''+f^2\psi=0, \qquad (1)$$

where the primes denote differentiations with respect to the independent variable, say r, and $f=f(r)$ is a given positive function of r. Suppose that f has first and second derivatives which, as $r\rightarrow\infty$, satisfy the following restriction:

$$\int^{\infty}|(f'/f^{\frac{3}{2}})'/f^{\frac{1}{2}}|dr<\infty. \quad (f>0). \qquad (2)$$

For instance, if $(f'/f^{\frac{3}{2}})'$ does not change its sign from a certain r onward (which is certain to be the case if f is composed of a finite number of exponentials and powers of r), and if

$$|f'|/f^2<\text{const.} \quad \text{and} \quad \int^{\infty} f'^2/f^3dr<\infty, \qquad (3)$$

then a partial integration shows that (2) must be satisfied. Clearly, (3) is fulfilled by $f=e^r$ and, if $\alpha>-1$, by $f=r^{\alpha}$. Thus, if f is regular enough, the "frequency" of (1) can be arbitrarily high (as $r\rightarrow\infty$), and it can be quite low (with $f(r)\rightarrow 0$), though not arbitrarily low.

In any case, the general solution of (1) is supplied by the real and imaginary parts of a constant multiple of a particular solution having the asymptotic form

$$\psi(r)\sim\{f(r)\}^{-\frac{1}{2}}\exp\left\{i\int^{r}f(q)dq\right\}, \qquad (4)$$

where $r\rightarrow\infty$. In fact, this holds if only condition (2) is stipulated. In (4), the choice of the lower limit of integration is immaterial, for, if the lower limit is changed, the solution ψ is replaced by a constant multiple of ψ.

Since the real and imaginary parts of (4) represent two linearly independent solutions of (1), it is clear that all solutions of (1) remain bounded if and only if $f(r)$ keeps away from 0, as $r\rightarrow\infty$; and that *all* solutions of (1) are capable of the quantum-mechanical normalization $\int|\psi(r)|^2dr=1$ if and only if

$$\int^{\infty}\{f(r)\}^{-1}dr<\infty$$

holds (the latter contingency represents the physically *degenerate* phenomenon of a "limiting circle"). In the example of Bessel's functions, which belong to $f=r^{\alpha}$ in (1), these two cases are seen to be characterized by $\alpha\geqq 0$ and $\alpha>1$, respectively, whereas the requirement of (2) was just $\alpha>-1$.

In order to prove that (1) possess a solution $\psi=\psi(r)$ satisfying (4), it is sufficient to apply the method of variation of constants, by placing $\psi=\phi/f^{\frac{1}{2}}$. An easy calculation shows that this substitution transforms (1) into

$$d^2\phi/ds^2+(1+F)\phi=0, \qquad (5)$$

where the new independent variable is

$$s=s(r)=\int^{r} f(q)dq, \quad (f>0), \tag{6}$$

and the coefficient $F=F(s)$ in (5) is the function[1]

$$F=\tfrac{3}{4}f'^2/f^4-\tfrac{1}{2}f''/f^3. \tag{7}$$

It is understood that $f=f(r)$ and its derivatives with respect to r must, in (7), be expressed as functions of s by means of the inverse of the substitution (6).

Suppose that $r=\infty$ corresponds to $s=\infty$, i.e., that the integral defined by (6) is divergent (otherwise the situation is quite trivial). Then (6) shows that

$$\int^{\infty} |F| ds<\infty \tag{8}$$

holds if and only if

$$\int^{\infty} |F| f dr<\infty \tag{9}$$

does. On the other hand, it is readily verified from (7) that (9) is equivalent to (2). Hence, (2) means that (8) is satisfied.

It is known[2] that, if F is any continuous function satisfying (8), then $\phi(s)-\phi_0(s)\rightarrow 0$ and $d(\phi-\phi_0)/ds\rightarrow 0$, as $s\rightarrow\infty$, is a pair of asymptotic requirements which establishes a one-to-one correspondence between the solutions, $\phi=\phi(s)$, of (5) and the solutions, $\phi_0=\phi_0(s)$, of the trivial approximation, $(d^2\phi_0/ds^2)+\phi_0=0$, to (5). Since this trivial differential equation admits of the solution $\phi_0(s)=e^{is}$, it follows that (5) has a solution satisfying the limit relation $\phi(s)/e^{is}\rightarrow 1$, where $s\rightarrow\infty$. Finally, it is seen from (6) that this limit relation is equivalent to (4) by virtue of the definition, $\psi=\phi/f^{\frac{1}{2}}$, of ϕ.

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[1] The transformations applied to (1) can be interpreted as an adaptation of Liouville's substitution (as to the latter, see, e.g., E. L. Ince, *Ordinary Differential Equations* (Longmans Green, and Company, London, 1927), pp. 270–271).

[2] The result quoted is due to M. Bôcher, Trans. Am. Math. Soc. 1, 40 (1900). Today it has more than one proof very primitive in nature (see, for example, H. Weyl, *The Theory of Groups and Quantum Mechanics* (Methuen and Company, Ltd., London, 1931), pp. 71–73).

The Isotopic Composition of Normal Krypton and Xenon

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RECENTLY a mass-spectrometer investigation of the isotopes of krypton and xenon resulting from the fission of uranium-235 by thermal neutrons was reported.[1] This investigation was carried out with a 180°-deflection Nier-type mass spectrometer.[2] In the course of a parallel investigation with a 90°-deflection sector-type instrument,[3] it was discovered that the measured abundance of krypton 80 in normal krypton was 10 percent greater than the value previously reported by Nier.[4] Because of this discrepancy, it was decided to make a careful investigation of the isotopic composition of normal krypton and xenon. An automatic ion-current recording unit,[5] which increases the precision of isotopic-abundance measurements, was used in this work.

Various possible systematic errors which might arise from resolution difficulties, mass discrimination, ion-current discrimination, or secondary electron emission from the collector plate were investigated. These errors in the isotopic abundances were found to be less than one percent under our operating conditions. The purified gas samples were free from contamination, and there was no evidence of isotopic fractionation either during the course of sample preparation or during the passage of the gas through the capillary leak into the mass spectrometer tube.

Results for Krypton

Abundance data obtained from twenty recorded mass spectrograms of normal krypton using the 90° mass spectrometer and from ten such spectrograms using the 180° instrument are presented in Table I, along with the earlier data of Nier.[4]

TABLE I. Isotopic composition of normal krypton.

	90° M.S.		180° M.S.		Nier*
Mass unit	Abundance (percent)	Mean deviation	Abundance (percent)	Mean deviation	Abundance (percent)
78	0.343	±0.003	0.341	±0.0003	0.35
80	2.223	±0.009	2.223	±0.002	2.01
82	11.510	±0.040	11.490	±0.010	11.53
83	11.490	±0.030	11.470	±0.020	11.53
84	57.000	±0.090	57.040	±0.040	57.10
86	17.420	±0.030	17.440	±0.030	17.47

* See reference 4.

From this table it may be seen that the abundance data for krypton 80 from the present investigation is greater by ten percent than the value reported by Nier. Considering the good agreement between our 90° and 180° data, it would appear that Nier's value must have been in error. This fact was brought to the attention of Professor Nier, who found, on returning to his original records of 1937, that he had used the wrong shunt factor in computing the background correction for krypton 80.[6] In Table II is presented the average of our 90° and 180° mass spectrometer abundance data for normal krypton, along with the corrected data of Nier.[6] Table II gives the final results on the isotopic abundance of normal krypton.

TABLE II. Isotopic composition of normal krypton.

Mass unit	Present data (average) abundance (percent)	Nier's corrected data** abundance (percent)
78	0.342	0.346
80	2.228	2.261
82	11.500	11.500
83	11.480	11.500
84	57.020	56.950
86	17.430	17.430

** See reference 6.

From this table, the good agreement between Nier's corrected values and our abundance data for krypton 82, 83, 84, and 86 is evident. However, in the case of krypton 78 and 80, a discrepancy of about one percent is noted.

Results for Xenon

Abundance data from six complete recorded mass spectrograms of normal xenon using the 180° mass spectrometer are presented in Table III, in which Nier's earlier data are also included.[4]