Refraction of Plane Shock Waves*

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(Received March 6, 1947)

It is assumed that when a plane shock wave is incident on an interface between two gases of different densities ρ and ρ_1, and different ratios of specific heats γ and γ_1, respectively, a three shock configuration results, involving an incident shock, a reflected shock, and a transmitted shock. It is further assumed that in the various angular domains the pressure is constant. The Rankine-Hugoniot equations are used to formulate the following conditions: (a) the pressure across the interface is continuous and (b) the deflection of the flow caused by the incident and reflected waves is equal to that caused by the transmitted wave. Rational polynomial equations of the twelfth degree are obtained, the roots of which determine the position and strength of the reflected and transmitted waves as functions of the strength, angle of incidence of the incident wave, and three parameters characterizing the pair of gases involved. The solutions of these equations are studied as multiple branched functions of the five parameters. It is shown that one branch behaves similarly to the acoustic case, and it is suggested that this branch is the only physically realizable one. Relations are obtained between the strength of the incident shock, its angle of incidence, and the three parameters characterizing the pair of gases which determine the ranges of these parameters where real physically realizable solutions may exist. One of these relations shows that the configuration is impossible for angles of incidence corresponding to the angle of total reflection. The cases for which numerical computations were made are listed, and the method of computation is briefly described. These computations were planned and supervised by Mrs. Adele Goldstein and were carried out on the Eniac which was made available through the cooperation of the Army Ordnance department.

1. INTRODUCTION

THE purpose of this paper is to give a discussion of some results on the phenomena associated with the reflection and refraction of a plane shock wave incident upon an interface between two gaseous media. We shall assume that the neighborhood of the line of intersection of the incident wave, I, and the surface separating the two media is divided into angular regions by a reflected shock wave, R, and a transmitted shock wave, M, as illustrated in Fig. 1. We shall also assume that all quantities of interest are constant in each of these angular regions. Thus we implicitly assume that the phenomenon of reflection is stationary as seen by one traveling with the line of intersection of the incident wave and the interface.

The computation of all relevant quantities may be reduced to that of determining the angles DOR and DOM of Fig. 1, that is, the angle between the reflected wave and the interface, the angle of reflection, and the angle between the transmitted wave and interface, the angle of refraction, respectively. These angles in turn may be determined by use of the fact that the Rankine-Hugoniot equations must hold across any shock and that the appropriate boundary conditions must be satisfied. These are:

(A) In the domain ROM there is no discontinuity in pressure. Using the notation of Fig. 1 we have $p''=p_1'$.

(B) The total deflection of the flow through the incident and reflected wave, $\delta+\delta'$, must equal the deflection through the transmitted wave, δ_1. Thus $\delta+\delta'=\delta_1$. (See Section 2 for the notation.)

When these conditions are imposed, as is done

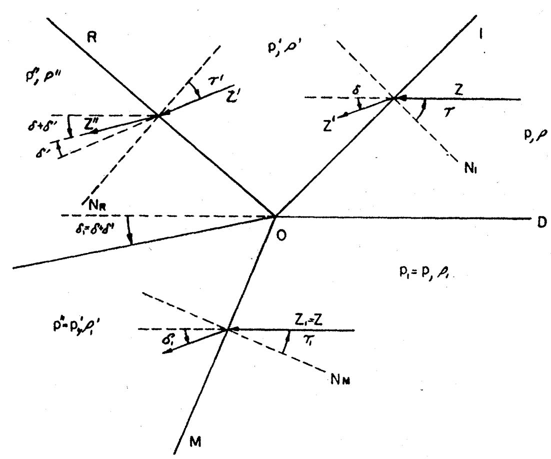

FIG. 1. The assumed shock configuration. For notation see Section 2.

* This work was started while the author was employed at Princeton University on work sponsored by the Office of Naval Research, contract N6ori-105 Task II.

below, we are led to an algebraic equation for the tangent of the angle τ', and the angle between the normal to the reflected wave and the flow incident upon that wave. Solving this equation is equivalent to solving a polynomial of twelfth degree, the coefficients of which are functions of the parameters characterizing the incident wave, essentially the angle of incidence and some measure of the strength of the incident wave. They are also functions of the characteristics of the two gases. These will be taken as the ratio of the velocities of sound in the two media, and the ratios of the specific heats for each medium. Thus each equation which has to be solved has coefficients which are functions of five parameters.

In general there will be many real solutions of these equations. Some of these may be dismissed on the ground that they violate some physical requirement such as the requirement that the flow incident upon the reflected shock wave be supersonic. However, after this is done we are still left with multiple solutions. We shall consider the solutions as a multiple branched function of the parameters involved and select those solutions which lie on branches which behave as $\eta\rightarrow 1$ as do sound waves.

In particular we shall not consider in detail solutions with positive values of τ', that is, configurations in which the reflected wave lies ahead of the normal to the flow behind the incident wave, for in the sonic case such solutions do not seem to occur.

2. NOTATION

The notation used is given in part by Fig. 1. The lines OI, OR, and OM are the incident, reflected, and transmitted shocks, and the lines N_I, N_R, and N_M are the normals to these shocks. The vectors $\mathbf{Z}$, $\mathbf{Z}'$, and $\mathbf{Z}_1=\mathbf{Z}$ are the flows incident on these shocks and z and z' are the magnitudes of these vectors. The line OD represents the interface between the two media; that above OD and to the right of OI is the medium into which the incident shock is traveling. It is called the unprimed medium and has a pressure density, sound velocity, and ratio of specific heats given by p, ρ, c, and γ, respectively. The medium below OD is the medium on which the incident shock impinges, and its characteristics are labeled by the same letters as those in the upper medium but are distinguished by a subscript 1. The pressure between the incident shock and the reflected shock is denoted by p', and other quantities are similarly labeled. The pressure behind the reflected shock is denoted by p'' and that behind the transmitted shock by p_1', with a similar convention for other quantities.

The angle between the normal to a shock and the flow incident upon it is denoted by τ with an appropriate superscript or subscript. It is positive when the direction from the normal to the flow is counter-clockwise.

The angle between the emergent flow and the prolongation of the incident flow is denoted by δ with appropriate sub- or superscripts. These angles are taken to be positive if the direction from the latter to the former is counter-clockwise.

In addition we shall use the following symbols defined in terms of the symbols used in Fig. 1.

$$\eta=\rho'/\rho,\quad \eta'=\rho''/\rho',\quad \eta_1=\rho_1'/\rho_1,$$

$$x=\tan\tau,\quad x'=\tan\tau',\quad x_1=\tan\tau_1,$$

$$\Delta=\tan\delta,\quad \Delta'=\tan\delta',\quad \Delta_1=\tan\delta_1,$$

$$D=\tan(\delta+\delta')=(\Delta+\Delta')/(1-\Delta\Delta'),$$

$$B^2=(1+\eta^2x^2)(1+x'^2)^{-1},\quad \Gamma=(c/c_1)^2,$$

$$x_m'=\left[\frac{A(1+\eta^2x^2)}{(1+x^2)\Gamma\eta+A-1+\frac{1}{2}(\gamma-1)(\eta-1)}-1\right]^{\frac{1}{2}},$$

$$x_M'=\left[\frac{1+\eta^2x^2}{1+\frac{1}{2}(\gamma+1)(\eta-1)}-1\right]^{\frac{1}{2}},$$

$$z_M=1+x_M'^2,$$

$$A=\gamma(\gamma_1+1)/\gamma_1(\gamma+1),\quad a=\gamma_1/\gamma,$$

$$P=a\Gamma=(\gamma_1/\gamma)(c/c_1)^2=\rho_1/\rho,$$

$$1+x_T^2=(2\eta P)^{-1}[2\eta a+(1-a)(\eta-1)].$$

3. THE RANKINE-HUGONIOT EQUATIONS

We shall use the Rankine-Hugoniot equations in the form

$$\frac{p'-p}{p}=\frac{2\gamma}{\gamma+1}[(z/c)^2\cos^2\tau-1]$$

$$=\frac{2\gamma}{\gamma+1}\left(\frac{(z/c)^2}{1+x^2}-1\right), \tag{3.1}$$

$$\eta=\frac{\rho'}{\rho}=\frac{(\gamma+1)(p'/p)+\gamma-1}{(\gamma-1)(p'/p)+\gamma+1};$$

$$\frac{p'}{p}=\frac{(\gamma+1)\eta-(\gamma-1)}{(\gamma+1)-(\gamma-1)\eta}, \tag{3.2}$$

$$(z'/c')^2\cos^2(\tau+\delta)=(2\gamma)^{-1}[\gamma-1+(\gamma+1)(p/p')], \tag{3.3}$$

$$\tan(\tau+\delta)=\eta\tan\tau. \tag{3.4}$$

In these equations, γ is the ratio of specific heats for the gas under consideration, p'/p is the ratio of the pressure behind the shock to that ahead of it, η is the ratio of the density behind the shock to that ahead of it, z and z' are the magnitudes of the flow vectors ahead and behind the shock, respectively, c and c' are the sound velocities in these media, the angle τ is the angle between the normal to the shock and the incident flow vector, and δ is the angle of deflection of the flow. Both of these angles are illustrated in Fig. 1 and the sign convention used there is held throughout.

Equations analogous to Eqs. (3.1) to (3.4) hold across any of the shocks, I, R, and M when appropriate changes in the symbols are introduced.

Equation (3.3) may also be written as

$$(z'/c')^2(1+\eta^2x^2)^{-1}=(2\gamma)^{-1}[\gamma-1+(\gamma+1)(p/p')], \tag{3.3'}$$

and (3.4) as

$$\Delta=\tan\delta=(\eta-1)(1+\eta x^2)^{-1}x. \tag{3.4'}$$

4. THE PRESSURE CONDITION

The requirement (A), namely, $p''=p_1'$, when formulated in terms of the Rankine-Hugoniot equations leads to an expression for $x_1=\tan\tau$, as a function of η, x, $x'=\tan\tau'$, $(c/c_1)^2$, γ and γ_1. This expression is obtained as follows:

Applying (3.1) to the flow incident on the reflected wave we have

$$(p''-p')/p'=2\gamma(\gamma+1)^{-1}[(z'/c')^2(1+x'^2)^{-1}-1].$$

Substituting for $(z'/c')^2$ from (3.3′) we obtain

$$\frac{p''}{p'}=B^2\left[\frac{(\gamma-1)p'+(\gamma+1)p}{(\gamma+1)p'}\right]-\frac{\gamma-1}{\gamma+1}. \tag{4.1}$$

Where

$$B^2=(1+\eta^2x^2)(1+x'^2)^{-1}. \tag{4.2}$$

Hence

$$p''/p=1+(B^2-1)[1+(\gamma-1)p/(\gamma+1)p']. \tag{4.3}$$

Similarly,

$$(p_1'-p)/p=2\gamma_1(\gamma_1+1)^{-1}[(z/c)^2(1+x_1^2)^{-1}-1].$$

Substituting in this equation for z/c we obtain

$$\frac{p_1'}{p}=\frac{2\gamma_1}{\gamma_1+1}\times\left\{\Gamma\frac{1+x^2}{1+x_1^2}\left[\frac{(\gamma+1)(p'-p)}{2\gamma p}+1\right]-1\right\}. \tag{4.4}$$

Equating (4.4) and (4.3) gives the equation for x_1, namely,

$$1+x_1^2=\frac{(1+x^2)\Gamma\eta(1+x'^2)}{A(\eta^2x^2-x'^2)+\frac{1}{2}[(\gamma+1)-(\gamma-1)\eta](1+x'^2)} \tag{4.5}$$

$$=\frac{(1+x^2)\Gamma\eta}{A(B^2-1)+\frac{1}{2}[(\gamma+1)-(\gamma-1)\eta]}.$$

Consideration of Eq. (4.5) leads to relations between the variables x, η, A, and Γ which must hold in order for a real solution to exist. It follows from (4.5) that $1+x_1^2$ is a monotonic increasing function of x'^2. If

$$1-A\geqslant\tfrac{1}{2}(\gamma-1)(\eta-1),$$

the largest value of $1+x_1^2$ is finite and occurs when x' is infinite. If

$$1-A<\tfrac{1}{2}(\gamma-1)(\eta-1)$$

$1+x_1^2$ becomes infinite at x' given by

$$1+x'^2=\frac{A(1+\eta^2x^2)}{A-1+\frac{1}{2}(\gamma-1)(\eta-1)}.$$

We shall see later that for physical reasons x' must be restricted to the range given by

$$1+x^{2\prime}\leqslant 1+x'_M{}^2=\frac{1+\eta^2x^2}{1+\frac{1}{2}(\gamma+1)(\eta-1)}.$$

The value of $1+x_1^2$ corresponding to this value of $1+x'^2$ is finite and positive in both cases and given by

$$1+x_1^2=\frac{(1+x^2)\Gamma\eta}{\eta+\frac{1}{2}(A-1)(\gamma+1)(\eta-1)}.$$

Hence x_1 is not real in the range we are interested in unless

$$(1+x^2)\Gamma\geqslant 1+\tfrac{1}{2}(1-a)(\eta-1)/a\eta. \tag{4.6}$$

The inequality (4.6) reduces to the well-known condition for the existence of a transmitted wave in the acoustic approximation obtained by setting $\eta=1$, namely, $(1+x^2)\Gamma\geqslant 1$. The value of x for which the equality holds defines the angle of total reflection. However, if the equality holds in (4.6), the only possible allowable value of x'^2 is $x'_M{}^2$ and for this value of x'^2, $x_1=0$. This means that the deflection by the transmitted wave is zero but the deflection by the incident and (sonic) reflected wave is different from zero. Hence we cannot have an allowable solution if the equality in (4.6) holds. That is, there can be no analog of total reflection in the case of shocks.

Of course, if $\Gamma>1$ and $a>1$, (4.6) is satisfied for any real values of x and η. In case $\Gamma>1$ and $a<1$ the equality in (4.6) will lead to a real curve in the x, η plane for some values of x and η if

$$2/(\gamma+1)>(\Gamma-1)a/(1-a).$$

In case $\Gamma<1$ and $a<1$ the equality in (4.6) leads to a real curve in the x, η plane for all values of x and η. In case $\Gamma<1$ and $a>1$ this curve becomes partly imaginary if the inequality given above holds.

Since x_1^2 must be real and positive, it follows from (4.5) that

$$1+x'^2\geqslant 1+x'_m{}^2=\frac{A(1+\eta^2x^2)}{(1+x^2)\Gamma\eta+A-1+\frac{1}{2}(\gamma-1)(\eta-1)}. \tag{4.7}$$

That is, if x, η, Γ, and A are such that the right-hand side of (4.7) is greater than one, then allowable values of x'^2 are bounded from below. If this right-hand side is less than one, this lower bound is zero.

It may readily be verified that (4.6) is equivalent to the condition

$$1+x'_M{}^2\geqslant 1+x'_m{}^2.$$

There are choices of the parameters x,η, Γ, and A such that $1+x'_m{}^2\geqslant 1$ and $1+x'_m{}^2\leqslant 1$. In such a case (4.6) is satisfied, and x and η satisfy the condition:

$$x^2>\tfrac{1}{2}(\gamma+1)(\eta-1)/\eta^2.$$

5. DEFLECTION BY THE UPPER PATH

In order to calculate the deflection of the flow through the shocks I and R, the upper path, we calculate $\tan\delta$ and $\tan\delta'$. These are given by (3.4′), namely,

$$\Delta=\tan\delta=(\eta-1)(1+\eta x^2)^{-1}x, \tag{5.1}$$

$$\Delta'=\tan\delta'=(\eta'-1)(1+\eta'x'^2)^{-1}x', \tag{5.2}$$

and hence

$$D=\tan(\delta+\delta')=(\Delta+\Delta')/(1-\Delta\Delta')^{-1}. \tag{5.3}$$

Our next concern is to calculate η' as a function of η, x, x', A, and Γ. From Eq. (3.2) we have

$$\eta'=\frac{(\gamma+1)(p''/p')+\gamma-1}{(\gamma-1)(p''/p')+\gamma+1}.$$

Substituting in this equation from (4.1) we obtain after some algebraic manipulation

$$\eta'=\frac{(\gamma+1)B^2}{(\gamma-1)(B^2-1)+(\gamma+1)\eta}=\frac{(\gamma+1)(1+\eta^2x^2)}{(\gamma-1)(\eta^2x^2-x'^2)+(\gamma+1)\eta(1+x'^2)}. \tag{5.4}$$

On substituting (5.4) in (5.2) and (5.3) we obtain

$$\Delta'=\frac{[2(B^2-1)-(\eta-1)(\gamma+1)]x'}{(\gamma+1)(1+\eta x^2)\eta-2(B^2-1)}=\frac{(x'_M{}^2-x'^2)x'}{(1+x'^2)[1+\frac{1}{2}(\gamma+1)(1+x'_M{}^2)]-(1+x'_M{}^2)} \tag{5.5}$$

where $1+x'_M{}^2$ is defined by (5.7) below.

$$\Delta'=\frac{[2(\eta^2x^2-x'^2)-(\eta-1)(\gamma+1)(1+x'^2)]x'}{(\gamma+1)\eta(1+\eta x^2)(1+x'^2)-2(\eta^2x^2-x'^2)}, \tag{5.5'}$$

and

$$D=\frac{(\eta x-x')(\gamma+1)(1+x'^2)(\eta-1)(1+x^2)+2(\eta^2x^2-x'^2)(x'(1+\eta x^2)-(\eta-1)x)}{(\gamma+1)(1+x'^2)[\eta(1+\eta x^2)^2+(\eta-1)^2xx']-2(\eta^2x^2-x'^2)(1+\eta x^2+(\eta-1)xx')}. \tag{5.6}$$

Since, by assumption, the reflected wave must be a shock wave, we must have $\eta'\geqslant 1$. Hence we must have

$$1+x'^2\leqslant 1+x'_M{}^2=\frac{1+\eta^2x^2}{1+\frac{1}{2}(\gamma+1)(\eta-1)}. \tag{5.7}$$

Thus $|x_M'|$ is an upper limit for $|x'|$. In case the right-hand side of Eq. (5.7) is less than one, no reflected shock is possible. The equation $x_M'=0$ is equivalent to

$$1+\eta^2x^2=1+\tfrac{1}{2}(\gamma+1)(\eta-1).$$

This equation determines an angle of incidence for each shock strength η, such that the reflected wave is sonic, and for x, less than that given by this equation; the reflected wave does not exist because the flow behind the incident wave is subsonic.

6. THE FUNCTION $D(x')$

For the purpose of the discussion to follow it is convenient to summarize some of the properties of the function $D(x')$ for fixed values of γ, η, and x, that is, for an incident shock of given strength passing through a fixed medium at a fixed angle of incidence. We shall assume that

$$z_M=1+x'_M{}^2=\frac{1+\eta^2x^2}{1+\frac{1}{2}(\gamma+1)(\eta-1)}\geqslant 1.$$

$D(x')$ is positive if x' is negative and $x'^2>x'_M{}^2$. As x' increases $D(x')$ decreases monotonically and at $x'=-|x_M'|$ it is equal to Δ. As x' increases further $D(x')$ passes through a minimum for a negative x' such that

$$\begin{aligned}1+x'^2=1+(x'^2)_{\min}=&\frac{z_M}{1+\frac{1}{2}(\gamma+1)z_M}\\ &\times\{1-\tfrac{1}{4}(\gamma+1)z_M+[(\tfrac{1}{4}(\gamma+1))^2z_M{}^2\\ &+\tfrac{1}{2}(\gamma^2-1)z_M+\gamma+1]^{\frac{1}{2}}\}\geqslant z_M.\end{aligned} \tag{6.1}$$

This is the only minimum $D(x')$ possesses. There is also a maximum at the positive value of x' satisfying (6.1). This maximum is greater than Δ since $\Delta=D(x')$ at $x'=0$, as well as x', such that $x'^2=x'_M{}^2$.

After passing through the maximum $D(x')$ decreases monotonically with increasing x' and crosses the x' axis at $x'=\eta x$.

The minimum of $D(x')$ will be negative if and only if x, η, and γ are such that

$$x^2>\frac{2(\eta-1)[(\gamma-1)(1+\eta x^2)+2](1+\eta x^2)}{z_M(1+\eta^2x^2)}. \tag{6.2}$$

If this inequality holds there are two negative roots of the equation $D(x')=0$, given by

$$x'=z_M\left\{\frac{-x}{1+\eta x^2}\pm\left[\frac{x^2}{(1+\eta x^2)^2}-\frac{2(\eta-1)[(\gamma-1)(1+\eta x^2)+2]}{(1+\eta^2x^2)(1+\eta x^2)z_M}\right]^{\frac{1}{2}}\right\}.$$

If the inequality in (6.2) becomes an equality, the minimum of $D(x')$ is zero. The angle of incidence defined by the resulting equation is the so-called extreme angle beyond which regular reflection of shocks from a rigid wall is impossible.[1]

7. DEFLECTION BY THE LOWER PATH

We next calculate η_1 in order to compute

$$\Delta_1=(\eta_1-1)(1+\eta_1x_1{}^2)^{-1}x. \tag{7.1}$$

We have

$$\eta_1=\frac{(\gamma_1+1)(p_1'/p)+\gamma_1-1}{(\gamma_1+1)+(\gamma_1-1)(p_1'/p)}=\frac{(\gamma_1+1)(p''/p)+\gamma_1-1}{\gamma_1+1+(\gamma_1-1)(p''/p)}.$$

Substituting from (4.3) we obtain

$$\eta_1=\gamma_1(\gamma+1)\frac{2A(B^2-1)+[\gamma+1-(\gamma-1)\eta]}{2\gamma(\gamma_1-1)(B^2-1)+\gamma_1(\gamma+1)[\gamma+1-(\gamma-1)\eta]}, \tag{7.2}$$

[1] cf. J. Von Neumann, *Oblique Reflection of Shocks* (Explosives Research Report No. 12, U. S. Navy Department, Bureau of Ordnance, October 1943).

and

$$\Delta_1=\frac{2\gamma(B^2-1)x_1}{(\gamma+1)(1+x^2)\gamma_1\Gamma\eta-2(B^2-1)\gamma} \tag{7.3}$$

$$\Delta_1=\frac{2\gamma(\eta^2x^2-x'^2)x_1}{(\gamma+1)(1+x^2)(1+x'^2)\gamma_1\Gamma\eta-2(\eta^2x^2-x'^2)\gamma}. \tag{7.4}$$

We note that the condition that the transmitted wave be a shock wave is $\eta_1\geqslant 1$ which is automatically satisfied since $\gamma_1\geqslant 1$.

The deflection condition (B) is then satisfied by finding an x' such that

$$D=\Delta_1 \tag{7.5}$$

where D is given by (5.6) and Δ_1 by (7.4). The x' must be such that

$$1+x'_m{}^2\leqslant 1+x'^2\leqslant 1+x'_M{}^2. \tag{7.6}$$

In case $x'_m{}^2$ is negative the left-hand inequality is replaced by $1+x'^2\geqslant 1$. One solution of (7.5) is given by $x'=\eta x$ for then $D=\Delta_1=0$. However, this value of x' violates (7.6) and does not correspond to a physically possible position of the reflected shock.

8. THE FUNCTION $\Delta_1(x')$

The problem has been reduced to solving (7.5) where the quantities involved are defined by (7.4), (5.6), and (4.5); in addition we are interested only in those solutions satisfying (7.6). We have already seen that no physically real solutions can exist unless

$$(1+x^2)\Gamma>1+(1-a)(\eta-1)/2a\eta$$

and

$$1+\eta^2x^2\geqslant 1+\tfrac{1}{2}(\gamma+1)(\eta-1). \tag{8.1}$$

We now discuss some properties of the function $\Delta_1(x')$. Since the behavior of x_1, as a function of x' has already been discussed, it is convenient to write Δ_1 as a function of x_1 and the parameters x, η, γ, γ_1, and Γ. It follows from (4.5) and (7.3) that

$$\Delta_1=\frac{[z_{1M}-(1+x_1^2)]x_1}{(1+x_1^2)[1+\frac{1}{2}(\gamma_1+1)z_{1M}]-z_{1M}}, \tag{8.2}$$

where

$$z_{1M}=\frac{2\eta(1+x^2)\Gamma}{2\eta-(\gamma+1)(\eta-1)}>1, \tag{8.3}$$

as a consequence of the first of (8.1).

Equation (8.2) may be obtained from the relation between Δ' and x' by replacing x' by x_1, z_M by z_{1M}, and γ by γ_1. The character of Δ_1 as a function of x_1 may be readily obtained, therefore, from that of Δ' as a function of x'. In any case it may be seen readily that it is an odd function of x_1 and as x' increases from minus infinity Δ_1 decreases, becoming zero at the value of x_1 given by $1+x_1^2=z_{1M}$. As x_1 increases further, Δ_1, passes through a negative minimum and then increases to zero at $x_1=0$. The value of x_1 for which the minimum occurs may be obtained by making the replacements mentioned above in Eq. (6.1).

In discussing Δ_1 as a function of x' we distinguish between four cases:

$$1+x'_m{}^2\geqslant 1,\quad 1-A<\tfrac{1}{2}(\gamma-1)(\eta-1), \tag{8.4}$$

for which

$$\infty\geqslant 1+x_1^2\geqslant 1$$

as

$$\frac{2A(1+\eta^2x^2)}{2(A-1)+(\gamma-1)(\eta-1)}\geqslant 1+x'^2\geqslant 1+x'_m{}^2,$$

$$1+x'_m{}^2\geqslant 1,\quad 1-A\geqslant\tfrac{1}{2}(\gamma-1)(\eta-1), \tag{8.5}$$

for which

$$N\geqslant 1+x_1^2\geqslant 1\quad\text{as}\quad \infty\geqslant 1+x'^2>1+x'_m{}^2,$$

$$1+x'_m{}^2<1,\quad 1-A\geqslant\tfrac{1}{2}(\gamma-1)(\eta-1), \tag{8.6}$$

for which

$$N\geqslant 1+x_1^2\geqslant M\quad\text{as}\quad \infty\geqslant 1+x'^2\geqslant 1,$$

and

$$1+x'_m{}^2<1,\quad 1-A<\tfrac{1}{2}(\gamma-1)(\eta-1), \tag{8.7}$$

for which

$$\infty\geqslant 1+x_1^2\geqslant M$$

as

$$\frac{2A(1+\eta^2x^2)}{2(A-1)+(\gamma-1)(\eta-1)}\geqslant 1+x'^2\geqslant 1.$$

The quantities N and M are functions of the parameters x, η, Γ, γ, and γ_1, which are determined by setting $1+x'^2$ equal to infinity and unity, respectively, in Eq. (4.5). They are positive and greater than unity for all allowed values of the parameters.

The graph of Δ_1 as a function of x' is symmetrical with respect to both axes when positive and negative values of x_1 are taken, since Δ_1 is an even function of x'. We shall describe it in the first quadrant for the various cases. When (8.4) holds, Δ_1 is imaginary for $x'<|x_m'|$. It is zero at $x'=|x_m'|$ and has an infinite slope at this point. As x' increases, Δ_1 increases, passes through a maximum, and then vanishes at $x'=\eta x$.

The function in the first quadrant has two branches at $x'=\eta x$. The branch which emerges from this point for increasing x' is increasing and has a vertical asymptote at the value of x' given in Eq. (8.4). The function of Δ_1 is imaginary beyond this vertical asymptote.

When (8.5) holds, the character of this second branch of Δ_1 at the point $x'=\eta x$ changes. It is still increasing, but instead of having a vertical asymptote it remains finite as x' goes to infinity.

When (8.6) holds, the function Δ_1 is real for all values of x' less than ηx and does not vanish anywhere except at $x'=\eta x$. It has a second branch which behaves as in the case described by (8.4). When (8.7) holds, the function Δ_1 behaves as in (8.6) for $x'\leqslant \eta x$ and as in (8.5) for $x'>\eta x$.

9. EXISTENCE OF SOLUTIONS FOR $x'>-\eta x$

From the facts already cited about the functions $D(x')$ and $\Delta_1(x')$, it is possible to discuss various solutions of the equation $\Delta_1=D$ as a function of the parameter x for a given combination of gases (Γ, γ, and γ_1 fixed) and for a strength given of incident shock (η fixed). The different solutions of $\Delta_1=D$ will correspond to different branches of the function $x'(x)$. Two branches may or may not intersect. We shall now show that for suitable choices of parameters Γ, γ_1, γ, and η, branches which pass through the point $x'=-|x_M'|$, exist. At this point, the reflected wave disappears. This phenomenon occurs in acoustics, and this suggests that only those solutions which lie on this branch correspond to physically possible ones.

We begin the discussion by assuming x large, so that the incident wave strikes the density discontinuity almost head on. At the point $x'=-\eta x$, Δ_1 is zero and D is positive so that Δ_1-D is positive. Moreover, it is decreasing as x' increases in the neighborhood of $x'=-\eta x$. This point is, of course, outside the range of a physically possible solution, since that range is given by

$$1+x'^2\leqslant 1+x'_M{}^2\leqslant \eta^2x^2.$$

We now compare the value of Δ_1 and D at $x'=-|x_M'|$. The latter is given by Δ since $\Delta'=0$ here. It follows from (7.3) that for $1+x'^2=1+x'_M{}^2$

$$\Delta_1=\frac{(\eta-1)x_1}{(1+x^2)P\eta-(\eta-1)},\tag{9.1}$$

where

$$P=a\Gamma=\gamma_1c^2/\gamma c_1{}^2=\rho_1/\rho,\tag{9.2}$$

and

$$1+x_1{}^2=\frac{(1+x^2)\Gamma\eta a}{a\eta+\frac{1}{2}(1-a)(\eta-1)}=\frac{1+x^2}{1+x_T{}^2}\geqslant 1,\tag{9.3}$$

as follows from the first condition expressed in (8.1), where

$$1+x_T{}^2=\frac{2\eta a+(1-a)(\eta-1)}{2\eta P}>0.\tag{9.4}$$

Writing

$$\Delta_1{}^2(x'_M{}^2)-\Delta^2=\frac{(\eta-1)^2d}{[(1+x^2)P\eta-(\eta-1)]^2(1+\eta x^2)^2},$$

it may readily be verified that

$$d=\frac{1+x^2}{1+x_T{}^2}[x^4\eta^2(1-P^2(1+x_T{}^2))+2\eta x^2 \times(1-[\tfrac{1}{2}\eta(P-1)^2+P](1+x_T{}^2))-x_T{}^2].\tag{9.5}$$

Hence $d=0$ if and only if

$$x^2=\frac{(P+\frac{1}{2}\eta(P-1)^2)(1+x_T{}^2)-1\pm\{(1+x_T{}^2)[\eta(1+x_T{}^2)(P+\frac{1}{4}\eta(P-1)^2)-(\eta-1)]\}^{\frac{1}{2}}(P-1)}{\eta(1-P^2(1+x_T{}^2))}.\tag{9.6}$$

When $a=1$ there is only one positive root to $d=0$, and (9.6) reduces to

$$x^2=\frac{1-P}{2P}+\left[\left(\frac{1-P}{2P}\right)^2+\frac{1}{P\eta^2}\right]^{\frac{1}{2}}. \qquad (9.7)$$

When $\eta=1$ there is only one possible choice of the sign in (9.6), and it reduces to

$$x^2=\frac{\Gamma-1}{P^2-\Gamma}=\frac{(c/c_1)^2-1}{(\rho_1/\rho)^2-(c/c_1)^2}. \qquad (9.8)$$

In the usual linear theories of reflection and refraction of sound waves the condition for the absence of a reflected wave is shown[2] to be given by (9.8).

The sign of d in (9.5) depends on the size of $P^2(1+x_T{}^2)$ relative to unity. This quantity is the analog of the square of the acoustic impedance of the two media involved for

$$P^2(1+x_T{}^2)=P\left(a+\tfrac{1}{2}(1-a)\left(1-\frac{1}{\eta}\right)\right),$$

and when $\eta=1$ we have

$$P^2(1+x_T{}^2)=Pa=\gamma_1\rho_1/\gamma\rho=(\rho_1c_1/\rho c)^2.$$

Our subsequent discussion then falls into two cases:

$$P^2(1+x_T{}^2)\geqslant 1, \quad\text{and}\quad P^2(1+x_T{}^2)<1.$$

In the first case the incident wave impinges on a "denser" medium in an almost head-on direction, and d is negative as is the difference between Δ_1, and D at $x'=\eta x$. As x' increases, one branch of Δ_1 increases to a positive maximum and D decreases to a minimum value which we can make negative by choosing x large enough. We shall assume that for this choice of parameters η, x, γ, γ_1, and Γ the point $x'=-|x_m'|$ is greater than at least one of the two real negative roots of $D=0$. This condition can be satisfied by choosing x sufficiently large. For such a combination of parameters there must be two negative values of x' such that $\Delta_1=D$, as is obvious from a consideration of the graphs of the functions involved.

As x decreases d decreases, and one negative solution x' moves toward the point $x'=-|x_M'|$. As x decreases it may happen that both negative solutions disappear while d is still negative. This would be similar to the situations encountered in the study of three shock configurations where it is found that a limiting angle of incidence exists beyond which no three shock solutions exist. If such a thing happens, the curves $D(x')$ and $\Delta_1(x')$ are tangent for some value of x and for smaller values of x never intersect. It is evident from a consideration of the graphs of the function involved that this may occur even when $D(x')=0$ has real solutions, that is, in the region of regular reflection.

It may also happen for some choices of the parameters γ, γ_1, Γ, and η that both negative solutions still exist for the values of x greater than or equal to the greatest value for which $d=0$. In case x satisfies (9.6), $x'=-|x_M'|$ is one solution, and the reflected wave disappears. For values of x slightly less than this value, one solution lies between $x'=-\eta x$ and $x'=-|x_M'|$, and this corresponds to acoustic theory, for in that theory a compression is reflected from a "denser" medium as a compression for head-on incidence. As the angle of incidence is made more glancing, a critical angle given by (9.8) is reached at which the reflected wave disappears, and beyond this angle the reflected wave is a rarefaction.

Thus one branch of the function $x'(x)$ describing the solutions of $D=\Delta_1$ for fixed η, γ, γ_1, and Γ, and varying x passes through the point $x'=-|x_M'|$ and then lies between $x'=-\eta x$ and $x'=-|x_M'|$. Since a similar phenomenon takes place in acoustic theory and since the other branch behaves in a physically implausible manner, as we shall see below, we propose to identify this branch as the physically realizable one.

Presumably the reflected shock wave should be replaced by a Prandtl-Meyer rarefaction for values of x less than the critical value for which $d=0$. We shall not discuss this point further here.

When the value of x is decreased from that taken initially, the absolute value of the minimum of $D(x')$ decreases. This means that the second negative solution mentioned above is somewhere given by a value of x' for which $D(x')$ and $\Delta_1(x')$ are small; hence x_1 is small and x' is close to $-|x_m'|$. Since the slope of the

[2] cf. Rayleigh, *Theory of Sound* (Dover Publications, New York, 1945), Vol. II, p. 81.

$\Delta_1(x')$ curve is infinite at $x'=-|x_m'|$, x' can be close to $-|x_m'|$ and $\Delta_1(x')$ can be sizable. When this solution is in the neighborhood of $x'=-|x_m'|$ the pressure behind the transmitted wave must be very large, since x_1 is close to zero. This means that this wave is almost normal to the interface and yet it must have a large velocity parallel to the interface since it must keep up with the incident wave. Such a solution does not seem physically plausible.

The foregoing discussion may be summarized as follows: If $P(1+x_T^2)\geqslant 1$, the lower medium is acoustically the more dense. Then for x larger than the largest value for which $d=0$, that is, for x larger than the largest value given by (9.6), a physically realizable three-shock configuration may exist. It cannot exist for values smaller than this critical value, and presumably the reflected shock is replaced by a Prandtl-Meyer rarefaction. For x larger than the critical value, the existence or non-existence of a solution is determined as in the problem of three-shock configurations in a single gas. This is done at present by numerical means.

10. THE CASE $P^2(1+x_1^2)<1$

In this case, it follows from arguments similar to those used above that at $x'=-\eta x$, Δ_1-D is negative, and at $x'=-|x_M'|\Delta_1-D$ is positive for large x. Hence there is a solution of $\Delta_1=D$ between these two values of x'. There must be another solution between $x'=|x_M'|$ and $x'=-|x_m'|$ as follows from a consideration of the graphs involved. However, there is no solution on the physically realizable branch. This is to be expected, since in the case $P^2(1+x_T^2)<1$ the lower medium is the "rarer," and for head-on incidence in such a case the reflected wave is a rarefaction. Presumably the assumption of a reflected shock wave is incorrect, and this wave should be replaced by a Prandtl-Meyer rarefaction.

If x is now decreased to the greatest value given by (9.6), the solution between $x'=-\eta x$ and $x'=-|x_M'|$ moves toward $x'=-|x_M'|$. When x satisfies (9.6), there is a solution at $x'=-|x_M'|$ for which no reflected wave exists. For values of x less than this critical value, solutions of the type assumed may exist.

However, they cannot exist for all values of x less than the greatest value satisfying (9.6). It is evident that if there are two real values of x satisfying (9.6) this type of solution again disappears at the smaller value. Even if there is only one real value of x satisfying (9.6), this type of solution cannot exist for we have seen that there is a lower bound for x given by the requirement that x_1 must be real, namely, (8.1), which is

$$1+x^2>1+x_T^2=\frac{2\eta a+(1-a)(\eta-1)}{2\eta P}. \quad (10.1)$$

There is another lower bound given by the requirement that the flow behind the incident shock be supersonic, namely,

$$1+x_M'^2=\frac{1+\eta^2x^2}{1+\frac{1}{2}(\gamma+1)(\eta-1)}\geqslant 1. \quad (10.2)$$

We have already pointed out that the solutions cease to exist before the first of these lower bounds is reached and, there is therefore, no analogue of total reflection.

The greatest lower bound for x for the existence of solutions of the type we are considering is a function of η and the parameters characterizing the two gases. Geometrically it is determined by the condition that the curve $D(x')$ have one point of contact with the curve $\Delta_1(x')$ between the limits $x'=-\eta x$ and $x'=0$ at which the two curves are tangent.

Mathematically, this is the same type of condition which determines the greatest lower bound for the existence of solutions in case $P^2(1+x_T^2)\geqslant 1$, in the three-shock configuration problem and in the regular reflection problem. In the last problem, for values of x less than this greatest lower bound a Mach reflection takes place. This suggests that the analog of a Mach reflection may take place for values of x between the greatest solution of (9.6) and (10.1), provided (10.2) is satisfied in case $P^2(1+x_T^2)<1$.

We may summarize the case $P^2(1+x_T^2)<1$ as follows: For each η, a physically realizable three-shock configuration such as assumed can exist only if x is within a certain range. The upper limit for this range is the greatest value of x satisfying (9.6). A lower limit for this range is the largest of the following three numbers, (1) the smallest solution of (9.6) if it is distinct from

TABLE I. Key to computations.

Case	γ	γ_1	γ_1/γ	$\Gamma=(c/c_1)^2$	$P=\rho_1/\rho$	$\rho c/\rho_1 c_1$	Gases
I	1.66	1.4	0.84	0.825	0.693	1.31	
II	1.4	1.66	1.19	0.12	0.143	7.63	Air-He
III	1.4	1.4	1	0.875	0.875	1.07	O_2-N_2
IV	1.4	4/3	0.95	0.60	0.57	1.36	Air-CH_4
V	4/3	1.4	1.05	0.60	0.63	1.23	CO_2-Air
VI	1.139	1.4	1.23	0.19	0.23	1.86	Freon-Air
VII	1.139	5/3	1.46	0.02	0.03	4.71	Freon-He

the largest, (2) $|x_T|$, or (3) $(\gamma+1)(\eta-1)/2\eta^2$. Neither (2) nor (3) can be the greatest lower bound for defining this range but there must exist a value of x depending on η greater than (2) or (3) for which solutions of this type cease existing. Presumably for such η and x (and also smaller x's), an analog of Mach reflection takes place.

11. NUMERICAL COMPUTATIONS[3]

Equation (7.5) was solved numerically on the Eniac, which was made available through the cooperation of the Army Ordnance Department, for seven combinations of the parameters γ, γ_1, and Γ. These are listed in Table I. Mrs. Adele Goldstine planned and supervised the computations.

The method of computation was the following: For each combination of gases, (γ, γ_1, and Γ), an integral value of η satisfying $1\leqslant\eta\leqslant 2/(\gamma-1)$ was chosen. For each value of these parameters τ was allowed to take on values varying from zero to 89 degrees. For each choice of these five parameters $|x_M'|$ was computed. D, Δ_1, and $D-\Delta_1$ were then computed for $x'=-|x_M'|$ and all other

$$x'=-|x_M'|+ndx'\leqslant|x_M'|,$$

where $n=0, 1, 2, \cdots$ and dx' was taken to be 0.05 for some computations and 0.01 for others. The values of x' and $x'+dx'$, between which $D-\Delta_1$ changed sign, were noted and the value of $x'+dx'$ was called a solution. This method did not locate double roots. However these may be inferred in many cases from graphs of x' against x for fixed η for a given pair of gases.

The detailed results of these computations will not be given here but will be summarized briefly.

For all cases for which computations were made, the lower medium was the "rarer," and hence from the preceding discussion it was to be expected that for values of x greater than that given by (9.6) no solutions on the branch we have called physically realizable were to be found. However, for these values of x a positive and negative solution was found, and the values of x' were approximately equal to $\pm|x_m'|$. This result is caused in part by the fact that the curve $\Delta_1(x')$ has an infinite slope at $x'^2=x'^2_m$ and in part to the nature of the function $D(x')$.

In many of the cases computed, the interval of x within which the physically relizable branch can exist is less than a degree, and hence it was not explored in the numerical computations. In other cases some points on these branches were determined.

12. ACKNOWLEDGMENTS

This work was greatly aided by the cooperation received from the Office of Naval Research, The Army Ordnance Department, Mrs. Adele Goldstine, and Professor Walker Bleakney of Princeton University. In addition, I am indebted to Professor J. Von Neumann for calling this problem to my attention and for many stimulating discussions regarding it.

[3] A detailed report on the numerical computations is in preparation.