The Milne Problem for a Large Plane Slab with Constant Source and Anisotropic Scattering*

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Formulas are derived for the asymptotic neutron density and the neutron current emerging from an infinite plane slab whose thickness is large compared to the scattering mean free path. It is supposed that the slab sustains a uniform source of neutrons. The neutrons suffer capture in addition to scattering. The scattering is assumed to take place in accordance with the law: $(1/4\pi)(1+3f_1\mu)$ where f_1 is a constant, and μ is the cosine of the angle of scattering. Expressions for the asymptotic neutron density in the slab and the emerging current, in the limiting case of isotropic scattering, are also given.

1. INTRODUCTION

WE consider the following problem: an infinite plane slab of material of half-thickness d, bounded by vacuum on both sides, contains a source of neutrons uniformly distributed throughout the slab. The neutrons are scattered anisotropically[1] without change of energy and also suffer capture. The half-thickness, d, is assumed large compared to the scattering mean free path. We wish to obtain expressions for (a) the asymptotic neutron density inside the slab (i.e., the neutron density at distances large compared to a scattering mean free path from either face), and (b) the neutron current leaving either face.

The transport equation governing the distribution of neutrons in the slab in the case of linear scattering and with constant neutron production is:

$$\mu\frac{\partial\psi(z,\mu)}{\partial z}+\psi(z,\mu)=\frac{1}{2\sigma}[\psi_0(z)+3f_1\mu\psi_1(z)]+\frac{q_0}{2}. \quad (1)$$

In Eq. (1), the origin of the z-axis is taken on one face of the slab, μ is the cosine of the angle between the direction of the neutron velocity and the positive z-axis, $\psi(z,\mu)d\mu$ is the number of neutrons per unit volume at the point z with direction cosine between μ and $\mu+d\mu$, σ is the ratio of the scattering mean free path to the total mean free path, q_0 is the number of neutrons produced per unit volume (assumed constant), f_1 is a measure of the deviation of the scattering function from isotropy, and, finally, $\psi_0(z)$ and $\psi_1(z)$ are the zero and first moments of the neutron distribution function, namely:

$$\psi_0(z)=\int_{-1}^{1}d\mu\psi(z,\mu), \quad (1a)$$

$$\psi_1(z)=\int_{-1}^{1}d\mu\mu\psi(z,\mu). \quad (1b)$$

The quantity $\psi_0(z)$ is the neutron density and the negative of $\psi_1(z)$ is the neutron current. The total mean free path has been taken as the unit of length and the neutron velocity set equal to unity. Equation (1) is to be solved subject to the boundary conditions:

$$\psi(0,\mu)=0 \quad \text{for} \quad \mu>0, \quad (A)$$

$$\partial\psi_0(z)/\partial z=0 \quad \text{at} \quad z=d. \quad (B)$$

Condition (A) follows because the vacuum does not return any neutrons, condition (B) because of the symmetry of the problem. We shall use the asymptotic part of the solution for $\psi_0(z)$ in (B) since the half-thickness d has been assumed large compared to the scattering mean path.

2. GENERAL SOLUTION

We solve Eq. (1) by the standard Wiener-Hopf[2] procedure. Taking the Laplace transform

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[1] We work with linear scattering, i.e., the scattering function is assumed to be expressible in terms of the zero and first harmonics; the generalization to a higher number of harmonics is also possible.

[2] Cf. N. Wiener and E. Hopf, Berliner Ber. Math. Phys. Klasse 696 (1931); see also E. Hopf, "Mathematical Problems of Radiative Equilibrium," Cambridge Tract No. 31 (1934); O. Halpern, R. Lueneburg, and O. Clark, Phys. Rev. 53, 173 (1938); E. A. Schuchard and E. A. Uehling, Phys. Rev. 58, 611 (1940).

of both sides of Eq. (1), we get:

$$(1+s\mu)\phi(s,\mu) = \frac{1}{2\sigma}[\phi_0(s)+3f_1\mu\phi_1(s)]+\frac{q_0}{2s}+\mu\psi(0,\mu), \quad (2)$$

where

$$\phi(s,\mu)=\int_0^\infty dz e^{-sz}\psi(z,\mu),$$

$$\phi_0(s)=\int_{-1}^1 d\mu\phi(s,\mu),$$

$$\phi_1(s)=\int_{-1}^1 d\mu\mu\phi(s,\mu).$$

Integrating both sides of (2) over $d\mu$ from -1 to 1, we find:

$$s\phi_1(s)=\frac{1-\sigma}{\sigma}\phi_0(s)+\frac{q_0}{s}+\int_{-1}^0 d\mu\mu\psi(0,\mu) \quad (3)$$

where we have made use of boundary condition (A). On substitution for $\phi_1(s)$ into (2), division by $(1+s\mu)$ and integration over $d\mu$ from -1 to 1, Eq. (2) is transformed into:

$$\phi_0(s)\left[1-\frac{\tanh^{-1}s}{\sigma s}+\frac{3f_1(\sigma-1)}{\sigma^2s^2}\left(1-\frac{\tanh^{-1}s}{s}\right)\right] = \frac{\sigma q_0}{s}\left[\frac{3f_1}{\sigma^2s^2}\left(1-\frac{\tanh^{-1}s}{s}\right)+\frac{\tanh^{-1}s}{\sigma s}\right] + g_+(s)+\frac{3f_1g_+(0)}{\sigma s^2}\left(1-\frac{\tanh^{-1}s}{s}\right). \quad (4)$$

In Eq. (4), we have written:

$$g_+(s)=\int_{-1}^0 d\mu\frac{\mu\psi(0,\mu)}{(1+s\mu)} \quad (5)$$

so that

$$g_+(0)=\int_{-1}^0 d\mu\mu\psi(0,\mu);$$

$g_+(0)$ represents the negative of the neutron current flowing into vacuum.

Equation (4) can be rewritten in the form:

$$[s\phi_0(s)+\sigma q_0]K(s)=3f_1/\sigma s^2H(s) \times[sg_+(0)+\sigma q_0]+[sg_+(s)+\sigma q_0], \quad (6)$$

where

$$K(s)=1-\frac{\tanh^{-1}s}{\sigma s}+\frac{3f_1(\sigma-1)}{\sigma^2s^2}\left(1-\frac{\tanh^{-1}s}{s}\right),$$

$$H(s)=(s-\tanh^{-1}s)/s.$$

Expressing $H(s)$ in terms of $K(s)$:

$$H(s)=\frac{\sigma^2s^2K(s)-\sigma(\sigma-1)s^2}{\sigma s^2+3(\sigma-1)f_1},$$

we may transform Eq. (6) into:

$$\Phi(s)K(s)=G(s), \quad (7)$$

where

$$\Phi(s)=[s\phi_0(s)+\sigma q_0][\sigma s^2+3(\sigma-1)f_1] -3f_1\sigma[\sigma g_+(0)+\sigma q_0], \quad (7a)$$

$$G(s)=[sg_+(s)+\sigma q_0][\sigma s^2+3(\sigma-1)f_1] -3f_1(\sigma-1)[sg_+(0)+\sigma q_0]. \quad (7b)$$

In Eq. (7), $\Phi(s)$ is analytic in the plane $Re(s)>\nu$ (where ν is the positive root of $K(s)$), $G(s)$ is analytic in the half-plane $Re(s)<1$, and $K(s)$ is analytic in the strip $|Re(s)|<1$.

Just as in the case of no capture,[2] it can be shown that in the strip $|Re(s)|<1$, $K(s)$ has two zeros, namely, $\pm\nu$, and approaches unity as $|s|\to\infty$. We may, therefore, adopt the usual device of defining a function:

$$\tau(s)=\left(\frac{s^2-1}{s^2-\nu^2}\right)K(s). \quad (8)$$

The function $\log\tau(s)$ is analytic and single-valued in the strip $|Re(s)|<1$, provided a particular determination of the logarithm is chosen,[3] and approaches zero as $|s|\to\infty$ in the strip. The usual decomposition then follows:

$$\tau(s)=\tau_+(s)/\tau_-(s), \quad (8a)$$

where

$$\tau_+(s)=\exp\left[\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} du\frac{\log\tau(u)}{(u-s)}\right],$$

$$\tau_-(s)=\exp\left[\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty} du\frac{\log\tau(u)}{(u-s)}\right],$$

and $|Re(s)|<\beta$ with $\nu<\beta<1$. Introducing (8) and (8a) into (7) yields:

$$\Phi(s)\frac{(s^2-\nu^2)}{(s+1)}\cdot\frac{1}{\tau_-(s)}=\frac{(s-1)}{\tau_+(s)}G(s). \quad (9)$$

The left-hand side of Eq. (9) is analytic in the half-plane $Re(s)>\nu$ and the right-hand side is analytic in the half-plane $Re(s)<\beta$. Since there is

[3] The determination $\log 1=0$ is chosen.

a region of overlap, each side is the analytic continuation of the other. Examination of the behavior of the two sides of (9) as $|s|\to\infty$ in the plane shows that they approach infinity as $|s|^3$. By an extension of Liouville's theorem, it follows that each side may be equated to a polynomial of order three. We therefore write:

$$\Phi(s)\frac{(s^2-\nu^2)}{(s+1)\tau_-(s)}=C_0+C_1s+C_2s^2+C_3s^3$$
$$=(s-1)G(s)/\tau_+(s) \qquad (10)$$

where C_0, C_1, C_2, and C_3 are constants.

The problem is now to evaluate the C's. From (10) it is evident that $C_0=-G(0)/\tau_+(0)$, which is zero, since $G(0)=0$ (cf. (7b)) and $\tau_+(0)$ is finite. C_1 is also zero as can be seen by writing

$$C_1s+C_2s^2+C_3s^3=(s-1)G(s)/\tau_+(s),$$

differentiating both sides with respect to s and setting $s=0$. Hence:

$$G(s)=s^2(C_2+C_3s)\tau_+(s)/(s-1). \qquad (11)$$

Inserting (7b) for $G(s)$ and rearranging terms, we find:

$$[sq_+(s)+\sigma q_0]=\frac{s^2(C_2+C_3s)\tau_+(s)+3f_1(\sigma-1)[sg_+(0)+\sigma q_0](s-1)}{(s-1)[\sigma s^2+3(\sigma-1)f_1]}. \qquad (12)$$

From (12) it would follow that $[sg_+(s)+\sigma q_0]$ has poles at $s=\pm\zeta$ where[4] $\zeta=i[3f_1(1-1/\sigma)]^{\frac{1}{2}}$; since this is impossible, it follows that the numerator of the right-hand side of (12) must vanish for $s=\zeta$ and $s=-\zeta$, i.e.,

$$(C_2+C_3\zeta)\tau_+(\zeta)$$
$$-\sigma[\zeta g_+(0)+\sigma q_0](\zeta-1)=0, \qquad (13a)$$
$$(C_2-C_3\zeta)\tau_+(-\zeta)$$
$$-\sigma[-\zeta g_+(0)+\sigma q_0](-\zeta-1)=0. \qquad (13b)$$

Equations (13a) and (13b) yield values for C_2 and C_3, namely:

$$C_2=\frac{\zeta^2 g_+(0)(\alpha+\beta/\zeta)-\sigma q_0(\alpha+\beta\zeta)}{(\alpha^2-\beta^2)}, \qquad (14a)$$

$$C_3=\frac{-g_+(0)(\alpha+\beta\zeta)+\sigma q_0(\alpha+\beta/\zeta)}{(\alpha^2-\beta^2)}, \qquad (14b)$$

with

$$\alpha=(2\sigma)^{-1}[\tau_+(\zeta)+\tau_+(-\zeta)],$$
$$\beta=(2\sigma)^{-1}[\tau_+(\zeta)-\tau_+(-\zeta)]. \qquad (14c)$$

Equations (14a) and (14b) express C_2 and C_3 in terms of the unknown constant $g_+(0)$; that is as it should be since we still must take into account boundary condition (B). To determine $g_+(0)$, we find the asymptotic solution for $\psi_0(z)$ and then impose the boundary condition (B).

To find the asymptotic solution for $\psi_0(z)$, we write down the expression for $\phi_0(s)$ given by (10) and (7a) and find the contribution to the Laplace inverse from the poles.[5] For $\phi_0(s)$, we have:

$$\phi_0(s)=-\frac{\sigma q_0}{s}+\frac{3f_1[sg_+(0)+\sigma q_0]}{s(s^2-\zeta^2)}$$
$$+\frac{\tau_-(s)s(s+1)(C_2+C_3s)}{\sigma(s^2-\zeta^2)(s^2-\nu^2)}. \qquad (15)$$

The contribution to $\psi_0(z)$ (the Laplace inverse of $\phi_0(s)$) from the poles is:

$$\psi_0(z)_{\text{asymp}}=\frac{\sigma q_0}{(\sigma-1)}$$
$$+\frac{C_2}{2\sigma(\nu^2-\zeta^2)}[\tau_-(\nu)(1+\nu)e^{\nu z}$$
$$+(1-\nu)e^{-\nu z}\tau_-(-\nu)]$$
$$+\frac{C_3\nu}{2\sigma(\nu^2-\zeta^2)}[(1+\nu)e^{\nu z}\tau_-(\nu)$$
$$-(1-\nu)e^{-\nu z}\tau_-(-\nu)]. \qquad (16)$$

We may rewrite (16) in the form:

$$\psi_0(z)_{\text{asymp}}=\frac{\sigma q_0}{\sigma-1}+C_2A(\nu,\zeta)\cosh[\nu(z+z_0)]$$
$$+C_3\nu A(\nu,\zeta)\sinh[\nu(z+z_0)], \qquad (17)$$

[4] If $f_1<0$, ζ is real; however, $|\zeta|<1$ so that the pole still cannot exist.

[5] The branch-point contribution yields the non-asymptotic part of the solution.

where

$$A(\nu,\zeta)=[(1-\nu^2)\tau_-(\nu)\tau_-(-\nu)]^{\frac{1}{2}}/\sigma(\nu^2-\zeta^2), \quad (17a)$$

$$z_0=\frac{1}{2\nu}\log\left\{\frac{(1+\nu)\tau_-(\nu)}{(1-\nu)\tau_-(-\nu)}\right\}. \quad (17b)$$

The boundary condition (B) applied to (17) now yields:

$$C_2+C_3\nu\coth[\nu(d+z_0)]=0. \quad (18)$$

Using the definitions (14a) and (14b) for C_2 and C_3, Eq. (18) permits us to solve for $g_+(0)$ with the result:

$$g_+(0)=\sigma q_0\frac{\{(\alpha+\beta/\zeta)-\nu^{-1}(\alpha+\beta\zeta)\tanh[\nu(d+z_0)]\}}{\{(\alpha+\beta\zeta)-\nu^{-1}\zeta^2(\alpha+\beta/\zeta)\tanh[\nu(d+z_0)]\}}. \quad (19)$$

Equations (17), (17a), (17b), and (19) constitute the asymptotic solution for the neutron density in the slab; the negative of (19) is the current leaving the $z=0$ face of the slab. The quantities $\tau_-(\nu)$ and $\tau_-(-\nu)$ can be evaluated by converting their contour integral representations into real integrals.

3. LIMITING CASE OF ISOTROPIC SCATTERING

It is interesting to mention the limiting case of isotropic scattering; allowing f_1 and therefore ζ to approach zero, the asymptotic solution for $\psi_0^{(i)}(z)$ becomes:

$$\psi_0^{(i)}(z)_{\text{asymp}}=(\sigma/(\sigma-1))q_0+A(\nu_0)\cosh[\nu_0(d-z)], \quad (20)$$

where

$$A(\nu_0)=\frac{-\sigma^2q_0}{\cosh[\nu_0(d+z_0^{(i)})]}\cdot\left\{\frac{2(1-\nu_0^2)}{[1-\sigma(1-\nu_0^2)](\sigma-1)}\right\}^{\frac{1}{2}}. \quad (20a)$$

The emerging current $g_+^{(i)}(0)$ becomes:

$$g_+^{(i)}(0)=-\sigma q_0\{\nu_0^{-1}\tanh[\nu_0(d+z_0^{(i)})]-1-[\tau_+'^{(i)}(0)/\tau_+^{(i)}(0)]\}. \quad (21)$$

In Eqs. (20) and (21) the quantities ν_0 and $z_0^{(i)}$ are defined by:

$$\nu_0^{-1}\tanh^{-1}\nu_0=\sigma, \quad (22a)$$

$$z_0^{(i)}=(2\nu_0)^{-1}\log\left\{\frac{(1+\nu_0)\tau_-^{(i)}(\nu_0)}{(1-\nu_0)\tau_-^{(i)}(-\nu_0)}\right\}. \quad (22b)$$

The term $[\tau_+'^{(i)}(0)/\tau_+^{(i)}(0)]$ in Eq. (21)is the logarithmic derivative of $\tau_+^{(i)}(s)$ evaluated at $s=0$. The definitions of $\tau_+^{(i)}(s)$ and $\tau_-^{(i)}(s)$ are:

$$\tau_+^{(i)}(s)=\exp\left\{\frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty}du\frac{\log\tau^{(i)}(u)}{(u-s)}\right\}, \quad (23a)$$

$$\tau_-^{(i)}(s)=\exp\left\{\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}du\frac{\log\tau^{(i)}(u)}{(u-s)}\right\}, \quad (23b)$$

where

$$\tau^{(i)}(s)=[(s^2-1)/(s^2-\nu_0^2)]\times(1-(\sigma s)^{-1}\tanh^{-1}s). \quad (23c)$$

From (23b) it is possible to show that

$$[\tau_+'^{(i)}(0)/\tau_+^{(i)}(0)]=\bar{z}_0-1$$

where:[6]

$$\bar{z}_0=\frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{3}{\sin^2x}-\frac{1}{1-x\cot x}\right\}dx=0.7104\cdots. \quad (24)$$

The simplest case of all is the case of isotropic scattering and no capture. In this case, the current becomes (q_0d) while the asymptotic neutron density $\bar{\psi}_0^{(i)}$ is:

$$\bar{\psi}_0^{(i)}(z)_{\text{asymp}}=-\tfrac{3}{2}q_0z^2+A[z_0+\bar{z}_0^{(i)}], \quad (25)$$

where $A=3q_0d$,

$$\bar{z}_0^{(i)}=\bar{z}_0+(15\bar{z}_0^2-1)/30d,$$

with $\bar{z}_0$ defined by (24).

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[6] E. Hopf, "Mathematical Problems of Radiative Equilibrium," Cambridge Tract No. 31 (1934).