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The Vector Meson Field and Projective Relativity

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In the projective theory of relativity, if, instead of the projective metric $G_{\alpha\beta}$ of index $2N$, one uses the restricted metric $\gamma_{\alpha\beta}(\equiv G_{\alpha\beta}/G_{00})$ of index zero, one is led to a formal unification of the gravitational and electromagnetic fields of the general theory of relativity. The present paper discusses the case of the general metric $G_{\alpha\beta}$. It is shown that a natural four-dimensional, gauge invariant variational principle, involving the curvature scalar of $G_{\alpha\beta}$, yields field equations unifying the gravitational and vector meson fields, the range of the meson force being determined by the inverse length $(12)^{\frac{1}{2}}N$. Reasons are given for supposing that an extension to conformal geometry could prove of interest.

I. INTRODUCTION

THE projective theory of relativity[2] was proposed in order to give a four-dimensional significance to the five-dimensional theory, initiated by Kaluza[3] and developed by Klein,[4] which led to a formal unification of the gravitational and electromagnetic fields of the general theory of relativity. The projective formalism developed by Veblen,[5] was used by Veblen and Hoffmann in P. R. in 1930. In 1932 an equivalent formalism in terms of homogeneous coordinates in space-time was given by van Danzig.[6] The use of homogeneous coordinates was further discussed by Pauli,[7] whose formulation of the projective theory brought out similarities between it and the 1931 theory of Einstein and Mayer.[8] The relationship between the Einstein-Mayer theory and the projective theory had also been discussed by other writers.[9]

In the present paper, the original formalism of the projective theory is adhered to. Though less symmetric, it seems to the writer more direct than the formalism used by van Danzig, Schouten, and Pauli, especially in regard to the question of the index (called the excess in the van Danzig treatment), which is fundamental for the present discussion. Indeed, the reciprocal of the index has here the significance that, to within a numerical factor, it is the range of the meson force.

Since the ideas and basic formulas of the projective theory of relativity have been fully described in the literature,[10] they will not be repeated in detail here. It is convenient, however, to list the following items:

Latin indices have the range 1, 2, 3, 4, and Greek indices the range 0, 1, 2, 3, 4.

In four dimensions, a projective tensor $T_{\beta\cdots}{}^{\alpha\cdots}$ has 5^h components of the form $\exp(Nx^0)\,\bar{T}^{\alpha\cdots}_{\beta\cdots}(x^a)$, where N is a constant called the index, and x^0 is a gauge variable which undergoes transformations of the type

$$x'^0 = x^0 + \log[\rho(x^a)]. \qquad (1)$$

Under a transformation of coordinates and gauge variable, $T_{\beta\cdots}{}^{\alpha\cdots}$ transforms as

$$T'_{\beta\cdots}{}^{\alpha\cdots}(x') = \frac{\partial x'^\alpha}{\partial x^\gamma}\cdots\frac{\partial x^\delta}{\partial x'^\beta}\cdots T_{\delta\cdots}{}^{\gamma\cdots}(x(x')). \qquad (2)$$

The fundamental symmetric projective tensor

[1] On leave from Queens College, Flushing, New York.
[2] O. Veblen and B. Hoffmann, Phys. Rev. **36**, 810 (1930); we shall refer to this paper as P. R.
[3] Th. Kaluza, Sitz. Preuss. Akad. Wiss. **54**, 966 (1921).
[4] O. Klein, Zeits. f. Physik **37**, 895 (1926); *ibid.* **46**, 188 (1927).
[5] O. Veblen, J. Lond. Math. Soc. **42**, 140 (1929); Quart. J. Math. (Oxford) **1**, 60 (1930).
[6] D. van Danzig, Math. Ann. **106**, 400 (1932); J. A. Schouten and D. van Danzig, Zeits. f. Physik **78**, 639 (1932). Van Danzig pointed out that his homogeneous coordinates X^α were related to the variables x^α of Veblen's formalism by the transformation

$$x^0 \rightarrow X^0 = e^{x^0}, \quad x^a \rightarrow X^a = e^{x^0}x^a, \quad (a=1, 2, 3, 4).$$

This amounted to a transition from an affine representation in which portions of space-time are pictured as cross sections of systems of parallel lines in a five-dimensional space to an affine representation in which the lines meet in a point.
[7] W. Pauli, Ann. d. Physik **18**, 305 (1933).
[8] A. Einstein and W. Mayer, Sitz. Preuss. Akad. Wiss. **25**, 541 (1931).
[9] J. A. Schouten and D. van Danzig, Proc. Amsterdam **34**, 1398 (1931); O. Veblen, *Projektive Relativitätstheorie* (Julius Springer, Berlin, 1933), Chap. VIII; B. Hoffmann, Phys. Rev. **43**, 615 (1933).
[10] See for instance P. R., or the book "*Projektive Relativitätstheorie*" by O. Veblen, cited in reference 9, where the geometrical implications are developed in considerable detail.

$G_{\alpha\beta}$ is of index $2N$. Because

$$\partial x^0/\partial x'^a = 0, \tag{3}$$

the component G_{00} is a projective scalar. It is denoted by Φ^2, so that Φ is a projective scalar of index N.

The zero index projective tensor, $\gamma_{\alpha\beta}$, is defined by

$$\gamma_{\alpha\beta} = G_{\alpha\beta}/G_{00} = G_{\alpha\beta}/\Phi^2. \tag{4}$$

Because of (3), the components $\gamma_{\alpha 0}$ of $\gamma_{\alpha\beta}$ form a projective vector. It is denoted by φ_α. The components φ_a of φ_α acquire an added gradient under a change of the gauge variable (1) and are identified in P. R. with the electromagnetic four-potential. We may remark here that they do not have this significance in the present paper.

The zero index projective tensor, $g_{\alpha\beta}$, defined by

$$g_{\alpha\beta} = \gamma_{\alpha\beta} - \varphi_\alpha\varphi_\beta \tag{5}$$

has

$$g_{\alpha 0} = 0. \tag{6}$$

It follows from (3) that g_{ab} is an affine tensor. It is identified with the Riemannian metric of Einstein's general theory of relativity. The antisymmetric tensor φ_{ab} (which turns out to be affine) is defined by

$$\varphi_{ba} = \frac{1}{2}\left(\frac{\partial \varphi_a}{\partial x^b} - \frac{\partial \varphi_b}{\partial x^a}\right), \tag{7}$$

with

$$\varphi^a{}_b = g^{ac}\varphi_{cb} \quad \text{and} \quad \varphi^{ab} = g^{ac}g^{bd}\varphi_{cd}. \tag{8}$$

The curvature tensor formed from $\gamma_{\alpha\beta}$ is denoted by $B_{\beta\gamma\delta}{}^\alpha$. Also $B_{\beta\delta} = B_{\beta\alpha\delta}{}^\alpha$ and $B = \gamma^{\beta\delta}B_{\beta\delta}$. The corresponding tensors formed from g_{ab} are denoted by $R_{bcd}{}^a$, R_{bd}, R.

In P. R. the field equations based on the projective tensor of zero index are obtained from the variational principle.

$$\delta\int B(\gamma)^{\frac{1}{2}}dx^1dx^2dx^3dx^4 = 0, \tag{9}$$

by variation of $\gamma_{\alpha\beta}$, the variations $\delta\gamma_{\alpha\beta}$ vanishing at the boundary of the region of integration. Since B is of zero index, and $\gamma = g$ (see P. R. p. 815), the variational integral in (9) is an affine scalar.

The field equations resulting from (9) may be expressed in the form

$$T_{\alpha\beta} \equiv \Gamma_{\alpha\beta} - \varphi_\alpha\varphi_\beta\Gamma = 0, \tag{10}$$

where

$$\Gamma_{\alpha\beta} \equiv B_{\alpha\beta} - \tfrac{1}{2}\gamma_{\alpha\beta}B, \quad \Gamma \equiv \gamma^{\alpha\beta}\Gamma_{\alpha\beta}. \tag{11}$$

Since $T_{\alpha\beta}$ is a projective tensor of index zero, T^{ab}, $T_0{}^a$, and T_{00} are affine tensors. Thus the fifteen equations (10) separate naturally into the affine sets

$$T^{ab} \equiv (R^{ab} - \tfrac{1}{2}g^{ab}R) + 2(g^{cd}\varphi^a{}_c\varphi^b{}_d + \tfrac{1}{4}g^{ab}\varphi^c{}_d\varphi^d{}_c) = 0, \tag{12}$$

$$T_0{}^a \equiv \varphi^{ab}{}_{,b} = 0, \tag{13}$$

$$T_{00} \equiv R = 0. \tag{14}$$

Here the comma denotes the affine covariant derivative with respect to g_{ab}. Equation (14) is an algebraic consequence of (12); it is obtained by multiplying (12) by g_{ab} and summing over a and b. Thus there are only fourteen algebraically independent field equations. This is important, because with $\gamma_{00} = 1$ there are only fourteen independent quantities $\gamma_{\alpha\beta}$, and fifteen independent field equations would give an over-determination of the field.

The field equations (12) and (13) are precisely those of the general theory of relativity for the case of a field of gravitation and electromagnetism in the absence of matter. Equation (12) equates the gravitational stress-energy-momentum tensor to that of the electromagnetic part of the field, while Eq. (13), together with the definition of φ_{ab}, gives the Maxwell equations in their general relativity form.

In this way the projective theory of relativity provided a unification of the gravitational and electromagnetic fields in the general theory of relativity.

II. VARIATIONAL PRINCIPLE BASED ON $G_{\alpha\beta}$

An attempt was made in P. R. to use field equations based on the general projective metric $G_{\alpha\beta}$ of index $2N$. The field equations there discussed were of the type (10), with the curvature tensor formed from $G_{\alpha\beta}$ instead of $\gamma_{\alpha\beta}$. Like their prototype, these equations were related by one algebraic identity, but they were not derived from a variational principle. Their form was complicated and though a relativistic Schrödinger equation for the quantity $\Phi^{5/3}$ could be found in the field equations, the possible physical significance of the field equations was obscure.

Let $P_{\beta\gamma\delta}{}^{\alpha}$ be the projective curvature tensor formed from $G_{\alpha\beta}$, and let its contractions be $P_{\beta\delta}$ and $\bar{P}$, where

$$\bar{P} = G^{\beta\delta} P_{\beta\delta}. \tag{15}$$

Using the five-dimensional representation, it would be natural to consider field equations resulting from the variational principle

$$\delta \int \bar{P}(G)^{\frac{1}{2}} dx^0 dx^1 dx^2 dx^3 dx^4 = 0, \tag{16}$$

under variations of the $G_{\alpha\beta}$. Because of the integration over x^0, the integral in (16) is gauge invariant. When the field equations are separated into their affine sets, fourteen are identical with those given explicitly in P. R. on page 820, while the fifteenth can be expressed in the form of a relativistic Schrödinger equation for the quantity Φ, with an addition to the usual mass term. This fifteenth equation cannot be regarded as giving a scalar field in addition to the gravitational and electromagnetic fields, however, since, as in P. R., the quantity Φ enters the field equations only in the combination $\theta_a \equiv (\varphi_a - N^{-1}\partial \log\Phi/\partial x^a)$. The physical significance of the equations remains obscure.

Now the projective theory of relativity is a four-dimensional theory. It is therefore of interest to consider a variational principle based on a quadruple integral over space and time. In this case there will be no integration with respect to the gauge variable x^0, so that if we wish the variational integral to be gauge invariant we must have it a scalar of zero index, and thus affine.

Since $G_{\alpha\beta}$ is of index $2N$, $G^{\alpha\beta}$ is of index $(-2N)$. The Christoffel symbols formed from $G_{\alpha\beta}$ are of index zero. So is the projective curvature tensor, $P_{\beta\gamma\delta}{}^{\alpha}$, formed from them, and also the generalized Ricci tensor, $P_{\beta\delta}$, obtained by contraction. This means that the projective curvature scalar $\bar{P}$ is of index $(-2N)$. Also, because $G_{\alpha\beta}$ is of index $2N$, its determinant, G, is of index $10N$.

Hence the integrand, $\bar{P}(G)^{\frac{1}{2}}$, in (16) is of index $3N$.

The simplest way to convert it into a quantity of zero index is to use the projective scalar Φ of index N which is contained in $G_{\alpha\beta}$. By using Φ we do not bring into the variational integral any quantities other than components of $G_{\alpha\beta}$ and their first and second derivatives.

Therefore, we propose to consider the following gauge invariant, four-dimensional variational principle:

$$\delta \int \{\bar{P}(G)^{\frac{1}{2}}/\Phi^3\} dx^1 dx^2 dx^3 dx^4 = 0. \tag{17}$$

We shall show that, whether we vary $G_{\alpha\beta}$ or $\gamma_{\alpha\beta}$, the resulting field equations, though not identical, are physically equivalent.

If, following the usage of Veblen, we write

$$P = \gamma^{\alpha\beta} P_{\alpha\beta} \tag{18}$$

so that P is of zero index, we have, by (15),

$$\bar{P} = \Phi^{-2} P. \tag{19}$$

Also, by (4),

$$G = \Phi^{10}\gamma = \Phi^{10} g. \tag{20}$$

Hence the variational principle (17) can be written in the alternative form

$$\delta \int P(g)^{\frac{1}{2}} dx^1 dx^2 dx^3 dx^4 = 0. \tag{21}$$

Later a slight modification of (17) and (21) will be introduced.

III. LEMMAS CONCERNED WITH THE VARIATION

Because of the Φ^{-3} factor in (17), the method of Palatini[11] is not applicable to the present variation. Obtaining the field equations in their general, projective tensor form is thus difficult. But since we are interested mainly in the affine subsets of the projective field equations, the problem may be avoided by using the following lemmas:

Lemma I

After making the usual Green's theorem transformations of the calculus of variations and discarding terms because the variations of $G_{\alpha\beta}$ vanish at the boundary, let

$$\delta \int K dx^1 dx^2 dx^3 dx^4 = \int T^{\alpha\beta} \delta G_{\alpha\beta} dx^1 dx^2 dx^3 dx^4, \tag{22}$$

[11] A. Palatini, Rend. del Circolo Matematico di Palermo **XLIII**, 205 (1919).

under variations of $G_{\alpha\beta}$, so that the field equations resulting from

$$\delta\int K dx^1 dx^2 dx^3 dx^4 = 0 \qquad (23)$$

are, on multiplying by G_{00} to make them of zero index,

$$G_{00}T^{\alpha\beta} = 0. \qquad (24)$$

Let indices of $T^{\alpha\beta}$ be lowered by $\gamma_{\alpha\beta}$. Then the affine equations

$$G_{00}T^{ab} = 0, \qquad (25)$$

$$G_{00}T_0{}^a = 0, \qquad (26)$$

$$G_{00}\gamma_{\alpha\beta}T^{\alpha\beta} \equiv G_{00}T = 0,\text{[12]} \qquad (27)$$

are the Euler conditions for (22) under variations of g_{ab}, φ_a, and $\log G_{00}$, respectively.

Proof of Lemma I

By (4) and (5), we have

$$G_{\alpha\beta} = G_{00}\gamma_{\alpha\beta} = G_{00}(g_{\alpha\beta} + \varphi_\alpha\varphi_\beta). \qquad (28)$$

So

$$\delta G_{\alpha\beta} = G_{00}(\delta g_{\alpha\beta} + \varphi_\alpha\delta\varphi_\beta + \varphi_\beta\delta\varphi_\alpha) + \gamma_{\alpha\beta}\delta G_{00}. \qquad (29)$$

Also, by (6) and the fact that $\varphi_0 = \gamma_{00} = 1$, we have

$$\delta g_{\alpha 0} = 0, \quad \delta\varphi_0 = 0. \qquad (30)$$

Therefore, by (22), and using the fact that $T^{\alpha\beta}$ must be symmetric,

$$\delta\int K dx^1 dx^2 dx^3 dx^4 = \int T^{\alpha\beta}\delta G_{\alpha\beta} dx^1 dx^2 dx^3 dx^4$$

$$= \int T^{\alpha\beta}\{G_{00}(\delta g_{\alpha\beta} + 2\varphi_\alpha\delta\varphi_\beta) + \gamma_{\alpha\beta}\delta G_{00}\} dx^1 dx^2 dx^3 dx^4$$

$$= \int (G_{00}T^{ab}\delta g_{ab} + 2G_{00}T^{\alpha b}\varphi_\alpha\delta\varphi_b + G_{00}T^{\alpha\beta}\gamma_{\alpha\beta}\delta\log G_{00}) dx^1 dx^2 dx^3 dx^4$$

$$= \int (G_{00}T^{ab}\delta g_{ab} + 2G_{00}T_0{}^b\delta\varphi_b + G_{00}T\delta\log G_{00}) dx^1 dx^2 dx^3 dx^4,$$

since

$$T^{\alpha b}\varphi_\alpha = T^{\alpha b}\gamma_{\alpha 0} = T_0{}^b.$$

Thus variation of g_{ab} alone yields (25), variation of φ_b alone yields (26), and variation of $\log G_{00}$ alone yields (27).

[12] Note that this equation involves T, rather than T_{00}.

Lemma II

After making the usual Green's theorem transformations of the calculus of variations and discarding terms because the variations of $\gamma_{\alpha\beta}$ vanish at the boundary, let

$$\delta\int K dx^1 dx^2 dx^3 dx^4 = \int \bar{T}^{\alpha\beta}\delta\gamma_{\alpha\beta} dx^1 dx^2 dx^3 dx^4, \qquad (31)$$

under variations of $\gamma_{\alpha\beta}$. Then the field equations resulting from

$$\delta\int K dx^1 dx^2 dx^3 dx^4 = 0 \qquad (32)$$

will be

$$\bar{T}^{\alpha\beta} - \delta_0{}^\alpha\delta_0{}^\beta\bar{T} = 0. \qquad (33)$$

where

$$\bar{T} \equiv \bar{T}^{\alpha\beta}\gamma_{\alpha\beta}. \qquad (34)$$

Also, the affine equations

$$\bar{T}^{ab} = 0, \qquad (35)$$

$$\bar{T}_0{}^a = 0, \qquad (36)$$

will be the Euler conditions for (32) under variations of g_{ab}, φ_a, respectively, while the fifteenth affine equation,

$$\bar{T}_{00} - \bar{T} = 0, \qquad (37)$$

of the set (33) will be an algebraic consequence of (35).

Proof of Lemma II

That the field equations are (33) rather than just $\bar{T}^{\alpha\beta} = 0$, is due to the fact that $\gamma_{00} = 1$ always. For, since $\delta\gamma_{00} = 0$, we may infer from (32) and (31) only that[13]

$$\bar{T}^{\alpha\beta} = \delta_0{}^\alpha\delta_0{}^\beta W,$$

where W is some function of x^a; and contraction by means of $\gamma_{\alpha\beta}$ shows that $W = \bar{T}$.

The proof that (35), (36) result from variation of g_{ab}, φ_a, respectively, is so similar to the proof in Lemma I it need not be given in detail.

That (37) is an algebraic consequence of (35) follows from P. R. page 820, footnote 7. Thus, in the present notation,

$$\bar{T} = \gamma_{\alpha\beta}\bar{T}^{\alpha\beta} = (g_{\alpha\beta} + \varphi_\alpha\varphi_\beta)\bar{T}^{\alpha\beta}$$

$$= (g_{\alpha\beta} + \gamma_{\alpha 0}\gamma_{\beta 0})\bar{T}^{\alpha\beta} = g_{ab}\bar{T}^{ab} + \gamma_{\alpha 0}\gamma_{\beta 0}\bar{T}^{\alpha\beta}$$

$$= g_{ab}\bar{T}^{ab} + \bar{T}_{00}.$$

[13] See P. R., p. 818.

Therefore,

$$\bar{T}_{00}-\bar{T}\equiv g_{ab}\bar{T}^{ab}. \tag{38}$$

From (38), we see incidentally that (27) can be reduced to $G_{00}T_{00}=0$ by algebraic manipulation of it and (25).

Lemma III

The field equations (25) are identical with the field equations (35), and the field equations (26) are identical with the field equations (36).

Proof of Lemma III

Equations (25) and (35) are each the result of varying g_{ab} in (22) (which is the same as (32)). Equations (26) and (36) are each the result of varying φ_a.

IV. THE AFFINE FIELD EQUATIONS

By Lemmas I, II, and III, we see that varying $G_{\alpha\beta}$ yields fifteen field equations without an algebraic identity between them, while varying $\gamma_{\alpha\beta}$ yields fifteen equations, of which only fourteen are algebraically independent. Further, we see that, so far as these fourteen equations are concerned, it does not matter whether we vary $G_{\alpha\beta}$ or $\gamma_{\alpha\beta}$. It will appear in the course of the work that, though there is no algebraic relation between the fifteen field equations obtained by varying $G_{\alpha\beta}$, there is a simple differential relation which in effect reduces them to fourteen independent equations equivalent to those obtained by varying $\gamma_{\alpha\beta}$.

To perform the variations, we use the value of P as calculated by Veblen,[14] namely,

$$P=B-8N\theta^{a}{}_{,a}+12N^2(1+\theta^a\theta_a), \tag{39}$$

where the comma denotes the covariant derivative with respect to g_{ab}, and

$$\theta_\alpha\equiv\varphi_\alpha-\Phi_\alpha, \tag{40}$$

$$\Phi_\alpha\equiv N^{-1}\partial\log\Phi/\partial x^\alpha. \tag{41}$$

Since $\Phi_0=1$ and $\varphi_0=1$, we have $\theta_0=0$. Therefore, because of (3), θ_a is an affine vector. It will be recalled that φ_a is not affine, being altered by an additive gradient under a change of the gauge variable. This added gradient is just cancelled by the alteration induced in Φ_a by the change of gauge, with the result that θ_a is left unaffected.

Because of (39), the variational principle (17), in the form (21), becomes

$$\delta\int\{B-8N\theta^a{}_{,a}+12N^2(1+\theta^a\theta_a)\}(g)^{\frac{1}{2}}\times dx^1dx^2dx^3dx^4=0. \tag{42}$$

By the familiar formula for the contracted covariant derivative of a contravariant vector, we have

$$\theta^a{}_{,a}(g)^{\frac{1}{2}}=\frac{\partial}{\partial x^a}\{g^{\frac{1}{2}}\theta^a\}.$$

Therefore, since variations vanish at the boundary, Green's theorem shows that the second term of the integrand in (42) will contribute nothing to the field equations.

The part $B(g)^{\frac{1}{2}}$ of the integrand is the same as the integrand of (9), the result of varying which is already known. This leaves only the term $12N^2(1+\theta^a\theta_a)(g)^{\frac{1}{2}}$ needing special attention.

We shall use the following well-known formulas:

Since $g^{ab}g_{bc}=\delta_c{}^a$

$$g_{bc}\delta g^{ab}=-g^{ab}\delta g_{bc},$$

or

$$\delta g^{cd}=-g^{ac}g^{bd}\delta g_{ab}. \tag{43}$$

Also

$$\delta g=(\delta g_{ab})(gg^{ab}),$$

so that

$$\delta(g)^{\frac{1}{2}}=\tfrac{1}{2}(g)^{\frac{1}{2}}g^{ab}\delta g_{ab}. \tag{44}$$

Performing the needed variations, we have in turn:

$$\delta_{(g_{ab})}\int(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=\int(\tfrac{1}{2}g^{ab})(g)^{\frac{1}{2}}\delta g_{ab}dx^1dx^2dx^3dx^4. \tag{45}$$

[14] Quart. J. Math. (Oxford) 1, 60 (1930), Eq. (5.17) for the case $n=4$, and Eqs. (5.6), (5.3).

Also

$$\delta_{(g_{ab})}\int \theta^a\theta_a(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=\delta_{(g_{ab})}\int (\theta_a\theta_b g^{ab}(g)^{\frac{1}{2}})dx^1dx^2dx^3dx^4$$

$$=\int (\theta_a\theta_b)(\delta g^{ab}+\tfrac{1}{2}g^{ab}g^{cd}\delta g_{cd})(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$$

$$=\int (\theta_a\theta_b)(-g^{ac}g^{bd}+\tfrac{1}{2}g^{ab}g^{cd})(g)^{\frac{1}{2}}\delta g_{cd}dx^1dx^2dx^3dx^4$$

$$=\int (-\theta^c\theta^d+\tfrac{1}{2}g^{cd}\theta^a\theta_a)(g)^{\frac{1}{2}}\delta g_{cd}dx^1dx^2dx^3dx^4. \tag{46}$$

Again,

$$\delta_{(\varphi_a)}\int \theta^a\theta_a(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=\delta_{(\varphi_a)}\int g^{ab}(\varphi_a-\Phi_a)(\varphi_b-\Phi_b)(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$$

$$=\int 2g^{ab}(\varphi_b-\Phi_b)\delta\varphi_a(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$$

$$=\int 2\theta^a(g)^{\frac{1}{2}}\delta\varphi_a dx^1dx^2dx^3dx^4, \tag{47}$$

and finally, by (40) and (41),

$$\delta_{(\log G_{00})}\int \theta^a\theta_a(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=\delta_{(\log G_{00})}\int g^{ab}\left(\varphi_a-\frac{1}{2N}\frac{\partial \log G_{00}}{\partial x^a}\right)\left(\varphi_b-\frac{1}{2N}\frac{\partial \log G}{\partial x^b}\right)(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$$

$$=\int 2g^{ab}\left(\varphi_a-\frac{1}{2N}\frac{\partial \log G_{00}}{\partial x^a}\right)\left(-\frac{1}{2N}\frac{\partial(\delta \log G_{00})}{\partial x^b}\right)(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4$$

$$=\int\left(-\frac{1}{N}g^{ab}\theta_a(g)^{\frac{1}{2}}\right)\frac{\partial(\delta \log G_{00})}{\partial x^b}dx^1dx^2dx^3dx^4$$

$$=\int \frac{1}{(g)^{\frac{1}{2}}}\left(\frac{\partial}{\partial x^b}\left\{\frac{1}{N}g^{ab}\theta_a(g)^{\frac{1}{2}}\right\}\right)(g)^{\frac{1}{2}}\delta \log G_{00}dx^1dx^2dx^3dx^4$$

$$=\int\left(\frac{1}{N}\theta^a{}_{,a}\right)(g)^{\frac{1}{2}}\delta \log G_{00}dx^1dx^2dx^3dx^4. \tag{48}$$

In combining the results (45), (46), (47), and (48) with the known results for the part $B(g)^{\frac{1}{2}}$ in (42), we must allow for a minus sign that occurs in the latter:

$$\delta\int B(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=\int \Gamma_{\alpha\beta}(g)^{\frac{1}{2}}\delta\gamma^{\alpha\beta}dx^1dx^2dx^3dx^4=-\int \Gamma^{\alpha\beta}(g)^{\frac{1}{2}}\delta\gamma_{\alpha\beta}dx^1dx^2dx^3dx^4, \text{ by (43).}$$

Thus the fifteen field equations arising from variation of $G_{\alpha\beta}$ have the affine form

$$(R^{ab}-\tfrac{1}{2}g^{ab}R)'-6N^2g^{ab}+2(g^{cd}\varphi^a{}_c\varphi^b{}_d+\tfrac{1}{4}g^{ab}\varphi^c{}_d\varphi^d{}_c)-12N^2(\theta^a\theta^b-\tfrac{1}{2}g^{ab}\theta^c\theta_c)=0, \tag{49}$$

$$\varphi^{ab}{}_{,b}+24N^2\theta^a=0, \tag{50}$$

$$\theta^a{}_{,a}=0. \tag{51}$$

The field equations arising from variation of $\gamma_{\alpha\beta}$ are, as we have seen, equivalent to the affine sets (49), (50). The fifteenth equation, (51), is, however, a direct consequence of (50). For, taking the divergence of Eq. (64) we have

$$\varphi^{ab}{}_{,ba}+24N^2\theta^a{}_{,a}=0. \tag{52}$$

But, because φ^{ab} is antisymmetric, we have

$$\varphi^{ab}{}_{,b}=\frac{1}{(g)^{\frac{1}{2}}}\frac{\partial((g)^{\frac{1}{2}}\varphi^{ab})}{\partial x^b},$$

and so

$$\varphi^{ab}{}_{,ba}=\frac{1}{(g)^{\frac{1}{2}}}\frac{\partial}{\partial x^a}\left\{\frac{\partial(g)^{\frac{1}{2}}\varphi^{ab}}{\partial x^b}\right\}=\frac{1}{(g)^{\frac{1}{2}}}\frac{\partial^2((g)^{\frac{1}{2}}\varphi^{ab})}{\partial x^a\partial x^b},$$

and this vanishes because of the antisymmetry of φ^{ab}. Therefore (52) reduces to (51).

Thus the fifteen field equations arising from variation of $G_{\alpha\beta}$ place upon the field quantities g_{ab}, θ_a exactly the same restrictions as are imposed by the field equations arising from variation of $\gamma_{\alpha\beta}$.

Since φ_{ab} is defined with a $\frac{1}{2}$ factor, let us write

$$\theta_{ab}=2\varphi_{ab}=\frac{\partial\varphi_a}{\partial x^b}-\frac{\partial\varphi_b}{\partial x^a}=\frac{\partial\theta_a}{\partial x^b}-\frac{\partial\theta_b}{\partial x^a}. \tag{53}$$

Then the field equations (49), (50) become

$$(R^{ab}-\tfrac{1}{2}g^{ab}R)-6N^2g^{ab}+\tfrac{1}{2}(g^{cd}\theta^a{}_c\theta^b{}_d+\tfrac{1}{4}g^{ab}\theta^c{}_d\theta^d{}_c)-12N^2(\theta^a\theta^b-\tfrac{1}{2}g^{ab}\theta^c\theta_c)=0, \tag{54}$$

$$\theta^{ab}{}_{,b}+12N^2\theta^a=0. \tag{55}$$

Taking g_{ab} to be of signature $(---+)$, and N^2 to be positive, we see that, except for the term $(-6N^2g^{ab})$, these are the classical (i.e., unquantized) field equations for a vector meson and gravitational field in the general theory of relativity.

Though the term $(-6N^2g^{ab})$ has the appearance of a cosmological term, it is of quite the wrong order of magnitude for such a term. For if (55) is to represent a vector meson field in the galilean case, $12N^2$ must be of the order of magnitude 10^{25} cm^{-2}, while the usual cosmological constant, being of the order of magnitude of the reciprocal of the square of the radius of the universe, is more like 10^{-50} cm^{-2}. The presence of the term $(-6N^2g^{ab})$ would mean that, for the case $\theta_a=0$, there would be a solution for g_{ab} pertaining to a space-time of constant negative curvature, the radius of curvature being of the order of magnitude of the range of nuclear forces.

It is easy to remove the term $(-6N^2g^{ab})$ from the field equations. If one replaces the variational principle (21) by

$$\delta\int(P-12N^2)(g)^{\frac{1}{2}}dx^1dx^2dx^3dx^4=0, \tag{56}$$

the resulting field equations are identical with those obtained from (21), except that the term $(-6N^2g^{ab})$ no longer appears in (54).[15] Despite the presence of the projective scalar Φ in the fundamental projective tensor $G_{\alpha\beta}$, there is no rigorous scalar meson field equation in the field equations of the present paper; the scalar Φ enters only in the combination denoted by θ_a.

[15] If one retains the $(-6N^2g_{ab})$ term and makes the usual approximation for weak gravitational fields of assuming that only g_{44} differs significantly from its galilean value, one finds, for the case $\theta_a=0$, that g_{44} satisfies the scalar meson equation. This result is of dubious significance, however, since the rigorous solution for the static spherically symmetric case is known and g_{44} does not there have the form of a meson potential, nor is the weak field approximation there justified for small values of the radial coordinate. Thus though g_{44} might appear as a scalar meson outside the nucleus the indications are that it would not so appear inside.

I have tried to find a rigorous solution for the static, spherically symmetric case, with $\theta_a\neq 0$, with or without the $(-6N^2g_{ab})$ term, but the field equations, after promising manipulation, became too complicated to solve. In an endeavor to assess the effect of the gravitational field on the singularities of the meson field, I arbitrarily replaced the vector meson field by a scalar meson field in the hope that this would yield a rigorous solution. Surprisingly, the field equations proved more recalcitrant than those pertaining to the case of the vector meson, and no rigorous solution was forthcoming.

This is especially curious in view of the discovery by many writers[16] that in terms of a five-dimensional, or of a homogeneous four-dimensional formalism, in the galilean case, the vector and scalar (or pseudo-scalar) meson fields fit naturally together as a single unit. The electromagnetic field, too, is absent from the present field equations, unless we set $N=0$, in which case the whole character of the field equations is changed, there being fifteen independent equations for fifteen field quantities which have the significance of gravitational and electromagnetic fields and a new scalar field. There is, however, no longer a fundamental length in the theory.[17]

It may be that a broader geometrical base is needed than the projective geometry affords. It has long been known, for instance, that the Maxwell equations may be regarded as belonging to conformal geometry.[18] An indication of the possibilities residing in the conformal geometry may be seen in the following brief calculation, which is expressed in galilean terms:

Let φ_α be a projective vector of index N, and write

$$\varphi_{\alpha\beta}=\partial\varphi_\alpha/\partial x^\beta-\partial\varphi_\beta/\partial x^\alpha. \tag{57}$$

Consider the equation

$$\varphi_{\alpha\beta,\beta}=0. \tag{58}$$

This combines the two equations

$$\varphi_{ab,b}+\varphi_{a0,0}=0, \tag{59}$$

$$\varphi_{0b,b}=0. \tag{60}$$

The second of these is a consequence of the first, for $\varphi_{ab,ba}\equiv 0$, and so $\varphi_{a0,0a}=0$, which implies (60). The first equation, (59), can be written

$$\varphi_{ab,b}+N^2(\varphi_a-N^{-1}\varphi_{0,a})=0, \tag{61}$$

which is a vector meson equation for the vector $(\varphi_a-N^{-1}\varphi_{0,a})$. It will be noted that this closely parallels the situation in the present paper, the scalar meson being absent, and the scalar φ_0 being absorbed in the vector meson part.

Now let us go over to a restricted form of the general conformal geometry developed by E. Cartan, H. Weyl, J. M. Thomas, T. Y. Thomas, O. Veblen, and others,[19] namely, the case in which $U_0{}^0$ is independent of the space-time coordinates. In this case $\partial\log U_0{}^0/\partial x^a$ vanishes, and it is possible to express the transformation matrix $U_\tau{}^\sigma$ $(\sigma,\tau=0, 1, 2, 3, 4, 5)$ in the form of a constant multiple of a Jacobian matrix for a transformation involving six variables x^σ.

Let φ_σ be a conformal vector, having an index N, but not containing the variable x^5, and write

$$\varphi_{\sigma\tau}=\partial\varphi_\sigma/\partial x^\tau-\partial\varphi_\tau/\partial x^\sigma. \tag{62}$$

Then the equation

$$\varphi_{\sigma\tau,\tau}=0 \tag{63}$$

yields Eqs. (59), (60) above, and in addition the equation

$$\varphi_{5\alpha,\alpha}=0, \tag{64}$$

or

$$\varphi_{5,bb}+N^2\varphi_5=0,$$

which is a meson equation for the scalar φ_5, the N in this equation being the same inverse length as in the vector meson equation (61) above.

Since a second-rank symmetric conformal tensor contains not only a symmetric projective tensor of the second rank, but also a projective vector and a scalar, it is possible that the gravitational, electromagnetic, vector meson, and scalar meson fields (and perhaps a further scalar field[20]) may together form a single geometric object belonging to the general conformal geometry, and that the field equations governing them may find unitary expression within that geometry.

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[16] See, for example, C. Møller, Proc. Copenhagen **18** (1941). J. K. Lubański and L. Rosenfeld, Physica **9**, 117 (1942). A. Pais, Physica **9**, 267 (1942). K. C. Wang and K. C. Cheng, Phys. Rev. **70**, 516 (1946).

[17] Setting $N=0$ does not bring us back to the field equations of P.R., formed from $\gamma_{\alpha\beta}$ because $\gamma_{00}=1$ while, when $N=0$, G_{00} is a scalar function of x^a.

[18] E. Cunningham, Proc. Lond. Math. Soc. **8**, 77 (1910). H. Bateman, Proc. Lond. Math. Soc. **8**, 223, 469 (1910); **21**, 256 (1920). H. Weyl, Sitz. Preuss. Akad. Wiss. 465 (1918). J. A. Schouten and J. Haantjes, Physica **1**, 869 (1934). Compare also, D. van Danzig, Proc. Camb. Phil. Soc. **30**, 421 (1934).

[19] See, for example, Chapter IV of the book *Differential Invariants of Generalised Spaces* (Cambridge, 1934), by T. Y. Thomas. I use here the formalism of O. Veblen, Proc. Nat. Acad. Sci. **21**, 168 (1935).

[20] Compare, for instance, E. C. G. Stueckelberg, Helv. Phys. Acta **14**, 51 (1941).