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The γ-Instability of Mesons

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The following examples of γ-instability are investigated: (a) $M_0 \rightarrow \gamma_1+\gamma_2$, (b) $M_1 \rightarrow \gamma_1+\gamma_2+\gamma_3$, (c) $M_1 \rightarrow M_0+\gamma$, (d) $M_1{}^{\pm} \rightarrow M_0{}^{\pm}+\gamma$, where M_0 and M_1 mean a pseudoscalar and a (heavier) vector meson, respectively, and the superscript $\pm$ indicates the charge, if there is one. If the coupling constant and meson masses appropriate to the Schwinger mixture are used, the lifetimes are found to be: $\tau_a = 1\times 10^{-16}$ sec., $\tau_b = 2\times 10^{-11}$ sec., $\tau_c = 1\times 10^{-18}$ sec., and $\tau_d = 4\times 10^{-18}$ sec. In all cases except (b) the calculation leads to logarithmically divergent integrals.

1. INTRODUCTION

According to current meson theory both the nucleon and the electron-neutrino carry a virtual meson cloud; conversely, a free meson carries a nucleon cloud and can emit an electron-neutrino pair. The possibility of emitting an electron-neutrino pair permits the β-decay of the free meson, and the nucleon cloud, coupling even a neutral meson to the electromagnetic field, permits γ-decay. Unfortunately, the whole status of meson theory is at present quite uncertain, and our numerical results probably have at most an order of magnitude significance, not only because of uncertainties about the form and numerical constants of the theory, but also because of the dangers of working with a divergent formalism at its limits of validity. Nevertheless the investigation given here may be of interest, since it illustrates a kind of instability common to a class of theories. The following examples of γ-decay, which are probably typical of the more interesting cases, will be considered.

$$
\begin{array}{lll}
\text{(A)} & M_0 \rightarrow \gamma_1+\gamma_2 & \tau_A = 1\times 10^{-16}\ \text{sec.} \\
\text{(B)} & M_1 \rightarrow \gamma_1+\gamma_2+\gamma_3 & \tau_B = 2\times 10^{-11} \\
\text{(C)} & M_1 \rightarrow M_0+\gamma & \tau_C = 1\times 10^{-18} \\
\text{(D)} & M_1{}^{\pm} \rightarrow M_0{}^{\pm}+\gamma & \tau_D = 4\times 10^{-18}
\end{array}
\qquad (1)
$$

where M_0 and M_1 mean pseudoscalar and vector meson, respectively, and the superscript $\pm$ indicates a charge, if there is one. Processes of this nature have recently been proposed to explain the soft component of the Auger showers.[1] Process (B) is the simplest mode of decay of a vector meson, since the binary fission is forbidden by Furry's selection rule.[2] Processes like (C) and (D) are logical possibilities in a meson theory such as the Schwinger mixture in which two kinds of mesons of different rest mass are postulated. Another possibility, not described by (C) or (D), is the decay of a heavy meson into a lighter one with a change of charge, e.g., $M_1{}^{+} \rightarrow M_0+e^{+}$. In fact, Hulthén has proposed a mixture of charged pseudoscalar and neutral scalar fields with different rest masses, and one may ask whether the heavy (neutral) particle decomposes into the lighter one. However, the decay of one meson into a lighter one with the ejection of an electron should not occur; instead the heavy meson should undergo simple β-decay. On the other hand, the processes (A)–(D) are all much faster than β-decay, since they are caused by the coupling of the meson field to the nucleons whose mesonic charge is of the order of e, whereas β-decay arises from coupling of the meson field to the electron and neutrino sources whose meson charge is only of the order of $10^{-8}e$. Whenever there is competition between β- and γ-decay, the latter will win. For example, consider $M_0 \rightarrow e^{+}+e^{-}$; this reaction, whose lifetime is of the order of 10^{-8} sec., should not be expected since (A), with a lifetime of 10^{-16} sec., is much faster.

The lifetimes are given at the right of the corresponding reaction. According to these results a charged vector meson does not disappear through direct β-decay but only as the result of (D) followed by the β-decay of $M_0{}^{\pm}$. It would also follow from the above scheme that the eventual fate of a neutral vector meson is its

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[1] J. R. Oppenheimer, New York Meeting of American Physical Society, January, 1947.
[2] S. Sakata and Y. Tanikawa, Phys. Rev. **57**, 548 (1940).

transmutation into three photons, either directly by (B) or indirectly by (C) and (A). Since, however, (B) and (C) are competing decompositions and (C) is much the faster, (B) should not be expected.

2. HAMILTONIAN AND MATRIX ELEMENTS

The interaction terms in the Hamiltonian responsible for these processes are to the first order in the coupling constants

$$(4\pi)^{\frac{1}{2}}\int \Phi^*\sum\nolimits_\alpha \tau_\alpha\{g_0[U_\alpha\beta\gamma_5] + (f_0/K_0)[\gamma_5\partial_0 U_\alpha+\boldsymbol{\sigma}\mathbf{D}_\alpha U_\alpha]\}\Phi d\mathbf{x}, \quad (2a)$$

$$(4\pi)^{\frac{1}{2}}\int \Phi^*\sum\nolimits_\alpha \tau_\alpha\{g_1[U_{0\alpha}-\mathbf{U}\boldsymbol{\alpha}] + (f_1/K_1)[\mathbf{V}_{0\alpha}\boldsymbol{\gamma}+\mathbf{V}_\alpha\beta\boldsymbol{\sigma}]\}\Phi d\mathbf{x}, \quad (2b)$$

where

α and τ_α = isotopic spin index and matrix,
$(U_0, \mathbf{U})$ = wave function of M_1,
U and Φ = wave function of M_0 and nucleon, respectively,

$$\mathbf{V}=\mathbf{D}\times\mathbf{U} \quad\text{and}\quad \mathbf{V}_0=-\partial_0\mathbf{U}-\mathbf{D}U_0,$$
$$\boldsymbol{\gamma}=-i\beta\boldsymbol{\alpha}, \quad \boldsymbol{\sigma}=\gamma_5\boldsymbol{\alpha}, \quad \gamma_5=-i\alpha_1\alpha_2\alpha_3.$$

f and g have dimensions of charge, and $K=\mu c/\hbar$, $\mathbf{D}_\alpha=\boldsymbol{\nabla}-i(e_\alpha/c)\mathbf{A}$, where $\mathbf{A}$ is the electromagnetic vector potential; $\mathbf{D}=\boldsymbol{\nabla}$ except in case (D). For simplicity we put $g=0$ and consider only the second invariants; in addition, the analysis is limited to transverse mesons. The non-relativistic terms, $\boldsymbol{\sigma}\mathbf{D}U$ and $\mathbf{V}\beta\boldsymbol{\sigma}$, play no role at all in decompositions (A) and (B), since these terms are proportional to the velocity of the meson, and one can assume that the meson is initially at rest. In (C) and (D), however, either the initial or final meson must be moving.

These interaction terms give rise to transitions which have matrix elements with the following absolute values:

M_0:

$$(2\pi/G)^{\frac{1}{2}}(f_0/K_0)\epsilon^{\frac{1}{2}}[u_n{}^*\gamma_5 u_m + (c\mathbf{p}/\epsilon)(u_n{}^*\boldsymbol{\sigma}u_m)], \quad (3a)$$

M_1 (transverse):

$$(2\pi/G)^{\frac{1}{2}}f_1\hbar c\epsilon^{-\frac{1}{2}}[(\epsilon/\mu_1c^2)(u_n{}^*\boldsymbol{\gamma}\mathbf{e}_m u_m) + [(\mathbf{p}/\mu_1 c)\times\mathbf{e}_m](u_n\beta\boldsymbol{\sigma}u_m)], \quad (3b)$$

where $(\mathbf{p}, \epsilon)$ are the momentum and energy of the meson, while its polarization is $\mathbf{e}_m$. When the meson is charged, there are additional terms depending on $\mathbf{A}$; these new terms are not needed before Eq. (28). Matrix elements for the emission and absorption of a photon by a proton are also required:

$$(2\pi/G)^{\frac{1}{2}}e\hbar c\epsilon^{-\frac{1}{2}}(u_n{}^*\boldsymbol{\alpha}\mathbf{e}u_m). \quad (3c)$$

The functions u represent plane wave solutions of the Dirac equation. These matrix elements correspond to processes in which a nucleon in one state is created while a nucleon in another state is annihilated, and at the same time a meson or photon is created or annihilated. If both nucleon states have positive (or negative) energy, the corresponding process can be described as the emission or absorption of a photon or meson by a nucleon (or antinucleon). If one energy state is positive while the other is negative, the process described is the creation or annihilation of a nucleon pair with the simultaneous emission or absorption of a photon or meson.

Terms of exactly the same form as (2) couple the meson field to the electron-neutrino. The β-decay of a free meson is an illustration of pair creation associated with the matrix elements (3a) or (3b), if u is interpreted as belonging to an electron-neutrino instead of to a proton-neutron. In this process a charged meson disappears, while simultaneously a neutrino in a negative energy state is annihilated, and an electron in a positive energy state is created. Hence the initial charged meson is replaced by an electron and an antineutrino, which is equivalent to a neutrino. One should expect an analogous process with nucleons; instead of $M^+\rightarrow e^+ + n$, one has $M^+\rightarrow P^+ + N$, or if the meson is neutral, $M^0\rightarrow P^+ + P^-$. Since the mass of the meson is less than that of the nucleon, these transitions can take place only virtually; but because the coupling constant is 10^8 times greater for the nucleon, the decomposition into heavy particles is correspondingly more frequent than the decay into light particles. Although such a decomposition cannot lead directly to a final state in which the nucleons separate, it can play an intermediate role in a higher order process which is still very fast, since the virtual decomposition is so frequent.

3. DECOMPOSITION OF A NEUTRAL PSEUDO-SCALAR MESON INTO PHOTONS[3]

The lifetime, τ, is given by the usual expression

$$\tau^{-1}=(2\pi/\hbar)\rho|V|^2, \tag{4}$$

where ρ is the density of final states and V is the matrix element for the process. Both photons get the same energies and opposite momenta so that

$$\rho=G4\pi\mu_0^2c/16h^3, \tag{5}$$

where G is the fundamental volume and μ_0 is the mass of the meson.

The decomposition of a neutral meson into two photons requires three steps, e.g., $M_0\rightarrow P^+ + P^-$, $P^+\rightarrow P_1^+ + \gamma_1$, and $P_1^+ + P^- \rightarrow \gamma_2$, where P_1^+ means that the momentum of P^+ has been altered by the emission of γ_1. These three steps can take place in any one of six orders; furthermore, since the theory is symmetrical with respect to nucleons and antinucleons, each of these six possibilities gives rise to another if the roles of the nucleon and antinucleon are interchanged. Hence there are 12 terms in V, the probability amplitude. Further, since the conservation laws do not fix the momentum of the proton pair, there is an infinity of intermediate states, and the probability for the transition from the initial to the final state must be integrated over the momentum space of an intermediate pair particle.

$$V=(G/h^3)\int_0^\infty\int_\omega(\sum N_iD_i^{-1})P^2dPd\omega, \tag{6}$$

where the N_i are products of matrix elements, the D_i are energy denominators, i runs over the 12 possibilities mentioned, and ω is solid angle. In order that (6) be convergent it is necessary that N vanish faster than $1/E$.

It is convenient to expand the ratios N/D in powers of $x=\mu_0/M$:

$$N_iD_i^{-1}=({}^0N_i+{}^1N_ix)({}^0D_i-{}^1D_ix).$$

Writing ${}^0D_i=1/4E^2$, where E is the energy of a pair particle, and anticipating the result that ${}^0N_i=0$, one has to the first order in x

$$\sum N_iD_i^{-1}=x(\sum {}^1N_i)/4E^2. \tag{7}$$

[3] This instability has also been considered in unpublished calculations by Oppenheimer, Christy, and Lewis.

By Eqs. (4)–(7)

$$\tau=512\pi^4(\hbar/\mu_0c^2)(\hbar^3c^3/e^4f_0^2)J^{-2}, \tag{8}$$

where

$$J=\int_1^\infty(\sum\omega_i)(y^2-1)^{\frac{1}{2}}(dy/y), \tag{8a}$$

$${}^1\omega_i=\int{}^1n_id\omega, \tag{8b}$$

$${}^1n_i={}^1N_i/[2e^2f_0h^3(2\pi/G\mu_0)^{\frac{1}{2}}], \tag{8c}$$

$$y=E/Mc^2, \tag{8d}$$

n_i, y, and J are dimensionless.

We now calculate the dimensionless numerators n_i. These may be distinguished by three subscripts having the order in which the three intermediate steps occur. For example, by (3),

$$n_{0ij}=\sum(u^-|\gamma_5|u)(u|\boldsymbol{\alpha}\mathbf{e}_i|u_i)(u_i|\boldsymbol{\alpha}\mathbf{e}_j|u^-),$$

where the sum is over the spins of the intermediate states. The quantities u and u^- are plane wave functions having momentum P and energies $+E$ and $-E$, respectively. Similarly u_i corresponds to momentum P_i and energy E_i. To satisfy the conservation of momentum, one must have $\mathbf{P}_i=\mathbf{P}-\mathbf{p}_i$, where $\mathbf{p}_i$ is the momentum of the ith photon. Here $\mathbf{e}_i$ and $\mathbf{e}_j$ are unit polarization vectors of the corresponding photons. Then

$$n_{0ij}=Sp\lambda^-\gamma_5\lambda(\boldsymbol{\alpha}\mathbf{e}_i)\lambda_i(\boldsymbol{\alpha}\mathbf{e}_j), \tag{9}$$

where the λ's are the projection operators defined by

$$\lambda^\pm=\pm\tfrac{1}{2}(\pm1+\boldsymbol{\alpha}(\mathbf{V}/c)+\beta/y), \tag{9a}$$

$$\lambda_i=\lambda(\mathbf{V}_i,E_i). \tag{9b}$$

Evaluation of n_{0ij} gives

$$n_{0ij}=i[\mathbf{e}_i\times\mathbf{e}_j]\{[y^{-1}y_i^{-1}-y^{-2}](\mathbf{V}/c)-y^{-2}(\mathbf{V}_i-\mathbf{V})/c\}.$$

Expansion of E_i and $\mathbf{V}_i$ to the first order in x gives

$$n_{0ij}=(ix/2y^3)[\mathbf{e}_i\times\mathbf{e}_j]\mathbf{k}_i, \tag{10}$$

where $\mathbf{k}_i$ is the unit propagation vector of the ith photon. Since $\mathbf{k}_1=-\mathbf{k}_2$, it follows from (10) that

$$n_{012}=n_{021}.$$

In addition $n_{ij0}=Sp[\lambda^-(\boldsymbol{\alpha}\mathbf{e}_i)\lambda_i(\boldsymbol{\alpha}\mathbf{e}_j)\lambda(\gamma_5)]$. Take the complex conjugate:

$$n_{ij0}^* = Sp[\lambda^- \gamma_5 \lambda(\boldsymbol{\alpha}\mathbf{e}_j)\lambda_i(\boldsymbol{\alpha}\mathbf{e}_i)] = -n_{0ij},$$
$$\therefore \quad n_{ij0} = n_{0ij}.$$

Hence $n_{012} = n_{021} = n_{120} = n_{210}$. All of these numerators lead to convergent integrals. The situation is quite different if the meson is annihilated in the second step; the corresponding integral is then indeterminate. One rather natural way of evaluating this integral leads to a cancellation and a finite result again equal to n_{012}. We shall tentatively assume this result, but will afterwards return to it. The six n_i values obtained by permutation of 0, 1, and 2 are then equal, and by Furry's theorem are also equal to six others, obtained by interchanging the roles of nucleon and antinucleon. Hence

$$\sum \omega_i = (12)(4\pi)(ix/2y^3)[\mathbf{e}_1 \times \mathbf{e}_2]\mathbf{k}_1. \tag{11}$$

After squaring and summing over both polarizations of both photons, one has by (11) and (8)

$$\tau = 4\pi^2(\hbar/\mu_0 c^2)(\hbar^3 c^3/e^4 f_0)^2. \tag{12}$$

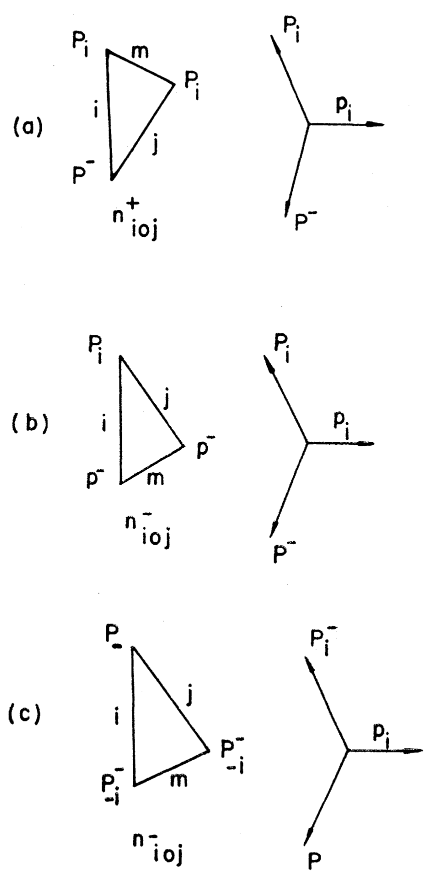

FIG. 1. Transition schemes (on left) and momenta corresponding to n_{i0j}^+ and n_{i0j}^-. On the left the minus superscript means a state of negative energy and on the right it refers to a negatron. P_i means momentun $P - p_i$, while P_{-i} and P_- means momenta $-(P-p_i)$ and $-P$, respectively.

The Elements n_{i0j}

Let n^+ denote the numerator when P^+ absorbs a meson, and let n^- designate the numerator when P^- absorbs a meson. The two transition schemes are shown in Fig. 1a and 1b. Then

$$n_{i0j}^+ = Sp[\lambda^-(\boldsymbol{\alpha}\mathbf{e}_i)\lambda_i\gamma_5\lambda_i(\boldsymbol{\alpha}\mathbf{e}_j)], \tag{13}$$

$$n_{i0j}^- = -Sp[\lambda^-(\boldsymbol{\alpha}\mathbf{e}_i)\lambda_i(\boldsymbol{\alpha}\mathbf{e}_j)\lambda^-\gamma_5]. \tag{13a}$$

The negative sign is introduced in (13a) because two of the vacuum protons are interchanged in the transition to which this equation refers. Evaluation of the spurs gives

$$n_{i0j}^+ = i[\mathbf{e}_i \times \mathbf{e}_j]\{-[2 - y_i^{-2} + y_i^{-1}y^{-1}](\mathbf{V}/c) - [1 + y_i^{-1}y^{-1}](\mathbf{V}_i - \mathbf{V})/c\},$$

$$n_{i0j}^- = i[\mathbf{e}_i \times \mathbf{e}_j]\{[2 - y^{-2} + y^{-1}y_i^{-1}](\mathbf{V}/c) + [1 - y^{-2}](\mathbf{V}_i - \mathbf{V})/c\},$$

and

$$n_{i0j}^+ + n_{i0j}^- = i[\mathbf{e}_i \times \mathbf{e}_j]\{(y_i^{-2} - y^{-2})(\mathbf{V}/c) + (-y^{-2} - y_i^{-1}y^{-1})(\mathbf{V}_i - \mathbf{V})/c\}.$$

Both n^+ and n^- contain the logarithmically divergent term $(\mathbf{V}_i - \mathbf{V})/c$ which cancels in the addition. To the first order in x this sum is $2n_{012}$. But according to Furry's theorem n^+ and n^- are of the same absolute value so that they either double or cancel and never give the partial cancellation just described. The reason for this difference is that Furry calculates the n^- by reversing all momenta, energies, and directions of transition. This procedure is equivalent to choosing the momenta as shown in Fig. 1c. Since there is an ultimate integration over solid angle, the relative directions of the protonic momentum in n^+ and n^- should not affect the integral $(n^+ + n^-)$, if this integral is well defined. But since it is not well defined, its value can be changed by reordering the terms in the integrand. Hence there is an uncertainty in the corresponding lifetimes. The terms in which the meson is annihilated in the second step are similar to those appearing in the self energy, since one photon is used to create the pair and the other to collapse it. As usual, there is no satisfactory procedure for dealing with this situation.

4. DECAY OF A NEUTRAL VECTOR MESON INTO THREE PHOTONS

According to Furry's theorem a vector meson cannot decompose into two γ-quanta because the n^+ and the n^- cancel.[4] Hence we consider the decomposition into three photons, suggested by Sakata and Tanikawa.[2]

The momenta of the three photons close a triangle. For given $\mathbf{p}_1$ the ergodic surfaces are ellipsoids with foci at the two ends of $\mathbf{p}_1$. The density of final states is determined by the number of states common to the ellipsoidal shell $(E, E+dE)$ and the spherical shell (p_2, p_2+dp_2). The volume of this ring is

$$dR = 2\pi y J\left(\frac{x,\ y}{p_2,\ \eta}\right) dp_2 d\eta,$$

where (x, y) are Cartesian coordinates in the plane of the moments, and $\eta = p_2 + p_3$. Hence

$$\rho = \frac{G^2}{6ch^6}\frac{dR}{d\eta} p_1^2 dp_1 d\omega_1 = 2\pi G^2 \frac{p_1 p_2 p_3}{6ch^6} dp_1 dp_2 d\omega_1. \quad (14)$$

The factor 6 appears in the denominator of ρ because the photons are indistinguishable. The product $p_1p_2p_3$ favors a symmetrical decomposition of the meson, but this initial bias is lost when ρ is combined with V^2. The lifetime now is given by

$$\tau^{-1} = (1/96\pi^5)(\mu_1 c^2/\hbar)(e^6 f^2/c^4\hbar^4) \times \int_0^{\frac{1}{2}} \int_{\frac{1}{2}-f_1}^{\frac{1}{2}} J^2 df_1 df_2, \quad (15)$$

where

$$8J = \int_1^\infty \left(\sum \omega_i d_i^{-1}\right)(y^2-1)^{\frac{1}{2}} y^{-2} dy, \quad (15a)$$

$$\omega_i = \int n_i d\omega, \quad (15b)$$

$$D_i = 8E^2 d_i, \quad (15c)$$

$$n_i = N_i / [(e\hbar c)^3 (f_1 \hbar c)(\epsilon_1\epsilon_2\epsilon_3\mu_1 c^2)^{-\frac{1}{2}} (2\pi/G)^2]. \quad (15d)$$

Here $\epsilon_i (= \mu_1 c^2 f_i)$ is the energy of the ith photon. The n_i and d_i are the dimensionless numerators and energy denominators. It is again convenient to expand in powers of $\mu_1 c^2/E(=x/y)$.

$$d_i = 1 + {}^1d_i(x/y), \quad n_i = {}^0n_i + {}^1n_i(x/y).$$

The n_i will now be considered in detail.

The decomposition takes place in four steps of which one is needed to annihilate the meson and three to create the three photons. These four steps may be permuted in 24 ways. Again new possibilities may be realized because of symmetry between nucleons and antinucleons; it is possible to choose three intermediate states of positive (negative) energy and one with negative (positive) energy or two with positive and two with negative energy. Hence there are $24\times 3 = 72$ chains in all. A typical numerator is

$$n_{mijk} = Sp[\lambda^-(\beta\boldsymbol{\alpha}\mathbf{e}_m)\lambda(\boldsymbol{\alpha}\mathbf{e}_i)\lambda_i(\boldsymbol{\alpha}\mathbf{e}_j)\lambda_{ij}(\boldsymbol{\alpha}\mathbf{e}_k)], \quad (16)$$

where

$$\lambda_i = \lambda(v_i), \quad \mathbf{P}_i = \mathbf{P} - \mathbf{p}_i, \quad \mathbf{P}_{ij} = \mathbf{P} - \mathbf{p}_i - \mathbf{p}_j.$$

Equation (16) may be written

$$n_{mijk} = \sum A_{st} = \sum a_{st} Sp(\beta^s \alpha^t).$$

$A_{st} = 0$ unless $t = 4$ or 6. A_{s4} involves x only through the energy of the virtual proton. But $E/E_i = 1 + A_i x/y$, where $A_i = f_i \mathbf{V}\mathbf{k}_i/c$ and where $\mathbf{k}_i$ is the unit propagation vector. Hence terms linear in x, being also linear in $\mathbf{V}$, vanish in the angular integration. A similar argument is applicable to the linear portion of A_{s6}, insofar as x appears in the energy. But A_{s6} also has terms linear in x in virtue of

$$\mathbf{V}_i/c = \mathbf{V}/c + f_i[-\mathbf{k}_i + (\mathbf{V}\mathbf{k}_i/c)(\mathbf{V}/c)](x/y).$$

Here the factor of x is even in $\mathbf{V}$, and so are all other factors in the spur except $\mathbf{V}$ itself. Hence the term is odd and may be ignored. In addition

[4] W. H. Furry, Phys. Rev. **51**, 125 (1937). The original statement of this theorem refers primarily to transitory electron-positron pairs interacting with the electromagnetic field. The more general statement of Furry's theorem needed here is the following. Let O be any Dirac matrix such that $(k\tau|O|k'\tau') = \pm(-k'-\tau'|O|-k-\tau)$, where k is the momentum and τ is the sign of the energy. The theorem now states that a process is forbidden if it involves an odd number of even matrix elements, where "even" means that the positive sign is correct in the equation just written for O. The even matrix elements are contained in the vector and antisymmetric tensor interactions; those contained in the scalar, pseudoscalar, and pseudovector combinations are odd.

one has $\sum{}^0\omega_i=0$ (see Eq. (18)). It follows that

$$\sum\omega_i d_i^{-1}=(x/y)\sum{}^0\omega_i{}^1c_i, \tag{17}$$

where ${}^1c_i$ is the part of ${}^1d_i$ independent of $\mathbf{V}$. It is therefore only necessary to determine the zero-order part of n_i and one can put $\lambda_i=\lambda_{ij}=\lambda$ in all the spurs. The n_i contain terms independent of $\mathbf{V}/c$ and in addition terms of the type $Sp[(\boldsymbol{\alpha}\mathbf{e}_m)(\boldsymbol{\alpha}\mathbf{V}/c)(\boldsymbol{\alpha}\mathbf{e}_i)(\boldsymbol{\alpha}\mathbf{e}_k)(\boldsymbol{\alpha}\mathbf{V}/c)(\boldsymbol{\alpha}\mathbf{e}_j)]$, which are quadratic in $\mathbf{V}/c$ and must be integrated over solid angle. Let

$$\tilde{\omega}_i=2{}^0\omega_i'+{}^0\omega_i'',$$

where ω' corresponds to those processes in which one intermediate state is positive (negative) and the other three states are negative (positive); and where ω'' refers to those processes in which there are two positive and two negative states.

The results are

$$\tilde{\omega}_{mijk}=\{[mijk]+[mjik]\}F(y),$$
$$\tilde{\omega}_{imjk}=\{-[mjki]+[mjik]\}F(y),$$

where

$$F(y)=\pi(y^{-3}-y^{-1})$$

and

$$[mijk]=Sp[(\boldsymbol{\alpha}\mathbf{e}_m)(\boldsymbol{\alpha}\mathbf{e}_i)(\boldsymbol{\alpha}\mathbf{e}_j)(\boldsymbol{\alpha}\mathbf{e}_k)],$$
$$[mijk]=[mkji]=4\{(\mathbf{e}_m\mathbf{e}_i)(\mathbf{e}_j\mathbf{e}_k)+(\mathbf{e}_m\mathbf{e}_k)(\mathbf{e}_i\mathbf{e}_j)-(\mathbf{e}_m\mathbf{e}_j)(\mathbf{e}_i\mathbf{e}_k)\}.$$

The remaining elements can be found from the following relations, which can be established by taking the complex conjugate of (16) and the corresponding equation for ω_{imjk}.

$$\tilde{\omega}_{mijk}=-\tilde{\omega}_{kjim}, \tag{18a}$$

$$\tilde{\omega}_{imjk}=-\tilde{\omega}_{kjmi}. \tag{18b}$$

The ${}^1c_i$ are also antisymmetric in the same indices, and

$${}^1c_{mijk}=1+(f_k-f_i)/2,$$

$${}^1c_{imjk}=\tfrac{1}{2}+f_k/2-f_i.$$

Hence

$$\sum_1^{72}{}^0\omega_i{}^1c_i=16FE, \tag{19a}$$

where

$$E=\sum_{ijk}(\mathbf{e}_m\mathbf{e}_i)(\mathbf{e}_j\mathbf{e}_k)[1+2(f_k-f_i)]. \tag{19b}$$

By (15a), (17) and (19)

$$J=-2\pi xE\int_1^\infty(y^2-1)^{\frac{1}{2}}y^{-6}dy=-2\pi xE/5$$

and by (15)

$$\tau^{-1}=(1/600\pi^3)(\mu_1/M)^2(e^2f_1^2/c^4\hbar^4)(\mu_1c^2/\hbar)\times\int_0^{\frac{1}{2}}\int_{\frac{1}{2}-f_1}^{\frac{1}{2}}E^2(f_1,f_2)df_1df_2. \tag{20}$$

An approximate integration of this equation followed by a sum over both directions of polarization of the three photons and by an average over the polarization of the vector meson yields the final result

$$\tau^{-1}=(2.4/600\pi^3)(\mu_1/M)^2(e^6f_1^2/c^4\hbar^4)(\mu_1c^2/\hbar). \tag{21}$$

5. DECAY OF A NEUTRAL VECTOR MESON INTO A NEUTRAL PSEUDOSCALAR MESON AND A PHOTON

The density of states is now

$$\rho=G(4\pi/h^3)(\mu_1^2c)F_\gamma^3, \tag{22}$$

where the photon carries off the fraction F_γ of the rest energy of the meson. The reciprocal lifetime is now

$$\tau^{-1}=(1/128\pi^4)(\mu_1c^2/\hbar)(e^2f_0^2f_1^2/c^3\hbar^3)\times(M/\mu_0)^2(F_\gamma^4/1-F_\gamma)J^2, \tag{23}$$

where

$$J=\int_1^\infty(\sum\omega_id_i^{-1})[y^2-1]^{\frac{1}{2}}y^{-1}dy. \tag{23a}$$

The total process requires three steps; for example, $M_1\rightarrow P^++P^-$, $P^+\rightarrow P_i^++\gamma_i$, $P_i^++P^-\rightarrow M_0$. Again there are six permutations, each of which appears twice in the way previously mentioned. This case is slightly different from those already considered, because the pseudoscalar particle is created in a state of motion, and both non-relativistic and relativistic terms in the Hamiltonian must be considered. The relativistic numerator, except for a factor of β, is the same as the numerator appearing in case (A). For example,

$$n_{1\gamma0}=Sp[\lambda^-(\beta\boldsymbol{\alpha}\mathbf{e}_1)\lambda(\boldsymbol{\alpha}\mathbf{e}_\gamma)\lambda_\gamma(\gamma_5)].$$

One finds

$$n_{1\gamma0}=i[\mathbf{e}_1\times\mathbf{e}_\gamma]y^{-1}(\mathbf{V}-\mathbf{V}_\gamma)/c,$$
$$n_{10\gamma}=i[\mathbf{e}_1\times\mathbf{e}_\gamma]y^{-1}(\mathbf{V}+\mathbf{V}_0)/c,$$
$$n_{01\gamma}=-i[\mathbf{e}_1\times\mathbf{e}_\gamma](y^{-1}+y_0^{-1})(\mathbf{V}_0/c),$$
$$n_{\gamma01}=i[\mathbf{e}_1\times\mathbf{e}_\gamma]y^{-1}(\mathbf{V}+\mathbf{V}_\gamma)/c,$$
$$n_{0\gamma1}=i[\mathbf{e}_1\times\mathbf{e}_\gamma]y^{-1}(\mathbf{V}-\mathbf{V}_0)/c,$$
$$n_{\gamma10}=-i[\mathbf{e}_1\times\mathbf{e}_\gamma](y^{-1}+y_\gamma^{-1})(\mathbf{V}_\gamma/c).$$

The leading term of $\sum n_i d_i^{-1}$ caused by the relativistic part of n_i is of order $(\mu/M)^2$. Hence we consider only the non-relativistic term which is of the order $(\mu/M)^0$. A typical non-relativistic numerator is

$$n_{1\gamma 0}=Sp[\lambda^-(\beta\boldsymbol{\alpha}\mathbf{e}_1)\lambda(\boldsymbol{\alpha}\mathbf{e}_\gamma)\lambda(\gamma_5\boldsymbol{\alpha}\mathbf{k}_0)].$$

The subscripts on the projection operators have been omitted because only zero-order terms need be calculated. The non-vanishing terms of the spur contain α as a factor three or five times. Those containing α five times are quadratic functions of $\mathbf{V}$ and must be averaged over all directions of $\mathbf{V}$. The results are

$$\omega_{1\gamma 0}=\omega_{01\gamma}=(8\pi i/3)(y^{-3}-y^{-1})[\mathbf{e}_1\times\mathbf{e}_\gamma]\mathbf{k}_0,$$
$$\omega_{10\gamma}=(8\pi i/3)(y^{-3}+2y^{-1})[\mathbf{e}_1\times\mathbf{e}_\gamma]\mathbf{k}_0.$$

Again

$$\omega_{1\gamma 0}=\omega_{0\gamma 1},\quad \omega_{01\gamma}=\omega_{\gamma 10},\quad \omega_{10\gamma}=\omega_{\gamma 01}.$$

Hence

$$\sum_1^{12}\omega_i=(32\pi i/y^3)[\mathbf{e}_1\times\mathbf{e}_\gamma]\mathbf{k}_0. \tag{24}$$

In this case also, logarithmically divergent terms cancel. After squaring J, summing over the two possible polarizations of $\mathbf{e}_\gamma$, and averaging over $\mathbf{e}_1$, one finds by (23) and (24) that the reciprocal lifetime is

$$\tau^{-1}=(16/27\pi^2)(\mu_1c^2/\hbar)(e^2f_0^2f_1^2/c^3\hbar^3)\times(M/\mu_0)^2F_\gamma^4(1-F_\gamma)^{-1}, \tag{25}$$

and by the conservation laws

$$F_\gamma=(1/2)[1-(\mu_0/\mu_1)^2]. \tag{25a}$$

6. DECAY OF A CHARGED VECTOR MESON

The decomposition (D) is very similar to (C). Although the electric charge present in (D) introduces some new complexity, it alters the γ-lifetime only very slightly because the processes permitted by the charge are strongly forbidden by selection rules. Before describing the effects of the charge, however, we consider the decompositions common to both neutral and charged mesons. For example, one such possibility is: $M_1^+\to P^++(N)$, $P^+\to P_i^++\gamma_i$, $P_i^++(N)\to M_0^+$, where (N) means antineutron. There are six permutations and the usual doubling. In the neutral case all of these twelve combinatorial possibilities are realizable, but in the charged case only six of them are permitted because (a) a neutron cannot emit a photon if its magnetic moment is neglected, and (b) reactions of the type $P^++M^+\to P^{++}$ are forbidden. The result is that the matrix element due to these processes for the charged case is only half its value for the neutral case, and the lifetime for either charged meson would be

$$\tau_D=4\tau_C, \tag{26}$$

if the charged and neutral cases were otherwise the same.

However, the charged meson, by interacting directly with the electromagnetic field, can decompose in other ways. It is convenient to divide these new processes which lead to decay into two classes according to whether they require two or three steps. The three-step processes are

$$\begin{array}{llll}
\text{(a)} & M_1^+\to M_1^++\gamma & M_1^+\to P^++(N) & P^++(N)\to M_0^+\\
\text{(b)} & M_1^+\to M_1^++\gamma & 0\to P^-+N+M_0^+ & M_1^++P^-+N\to 0\\
\text{(c)} & 0\to P^-+N+M_0^+ & M_0^+\to M_0^++\gamma & P^-+M_1^++N\to 0\\
\text{(d)} & 0\to P^-+N+M_0^+ & M_1^+\to M_1^++\gamma & P^-+M_1^++N\to 0\\
\text{(e)} & 0\to P^-+N+M_0^+ & P^-+M_1^++N\to 0 & M_0^+\to M_0^++\gamma\\
\text{(f)} & M_1^+\to P+(N) & P+(N)\to M_0^+ & M_0^+\to M_0^++\gamma.
\end{array} \tag{27}$$

It will now be shown that all processes (27) are forbidden. The nucleons are involved in only two of the three steps. In (c), (e), and (f) M_1 disappears and M_0 is created in a state of rest. The corresponding spur is $Sp[\lambda^-(\beta\boldsymbol{\alpha}\mathbf{e}_m)\lambda\gamma_5]$. This must be multiplied by the matrix element describing the emission of a photon by M_0^+; but since this spur vanishes, the corresponding process may be ignored. In (a), (b), and (d) both M_0 and M_1 move and the relevant spur is

$$\begin{aligned}Sp\{\lambda^-[A_0\gamma_5+\mathbf{B}_0\gamma_5\boldsymbol{\alpha}]\lambda_i[\mathbf{A}_1\beta\boldsymbol{\alpha}+\mathbf{B}_1\beta\gamma_5\boldsymbol{\alpha}]\}\\
=Sp\{\lambda^-[A_0\gamma_5\lambda_i(\mathbf{A}_1\beta\boldsymbol{\alpha})+(\mathbf{B}_0\gamma_5\boldsymbol{\alpha})\lambda_i(\mathbf{A}_1\beta\boldsymbol{\alpha})\\
+A_0\gamma_5\lambda_i(\mathbf{B}_1\beta\gamma_5\boldsymbol{\alpha})+(\mathbf{B}_0\gamma_5\boldsymbol{\alpha})\lambda_i(\mathbf{B}_1\beta\gamma_5\boldsymbol{\alpha})]\},\end{aligned}$$

where the A and B coefficients are given in Eq. (3). The first term vanishes identically; the fourth term vanishes since $\mathbf{B}_0 \perp \mathbf{B}_1$. The second term reduces to a linear function of $V_\perp = [\mathbf{A}_1 \times \mathbf{B}_0] \cdot \mathbf{V}$; and since the energy denominator is an even function of $V_\perp$, this term vanishes after the angular integration. The third term vanishes for a similar reason, and hence all processes (27) are forbidden.

Finally the meson may interact simultaneously with the electromagnetic and mesonic fields in the following ways.

$$\begin{array}{lll} \text{(a)} & 0 \to P^- + N + \gamma + M_0^+ & P^- + N + M_1^+ \to 0 \\ \text{(b)} & M_1^+ \to P^+ + (N) & P^+ + (N) \to M_0^+ + \gamma \\ \text{(c)} & M_1^+ \to P^+ + (N) + \gamma & P^+ + (N) \to M_0^+ \\ \text{(d)} & 0 \to M_0^+ + P^- + N & P^- + N + M_1^+ \to \gamma. \end{array} \tag{27}$$

In each case one step involves a pair, a photon, and a meson. The corresponding matrix elements follow from (2), where it is no longer permissible to replace $\mathbf{D}$ by ∇; they are (except for constants which do not concern us)

$$(u_n^* \boldsymbol{\sigma} \mathbf{e}_\gamma u_m), \tag{28a}$$

$$[\mathbf{e}_\gamma \times \mathbf{e}_m](u_n^* \beta \gamma_5 \boldsymbol{\alpha} u_m). \tag{28b}$$

The probabilities for the processes (a) and (b) vanish, as one can verify by direct computation or by the following observation. The interchange of the two members of the pair is equivalent to a reflection with respect to the origin of momentum space, since they have equal and opposite momenta. But this interchange reverses the sign of the probability amplitude;[4] hence the probability is an odd function in momentum space and vanishes when integrated over all directions.

In (c) and (d) M_1 is at rest so that it is only necessary to find

$$n_{10} = Sp[\lambda^-(\mathbf{C}_1 \beta \gamma_5 \boldsymbol{\alpha}) \lambda_\gamma (A_0 \gamma_5 + \mathbf{B}_0 \gamma_5 \boldsymbol{\alpha})],$$

where $\mathbf{C}_1$ is given in (28b). The symbol n_{10} means that M_1 is absorbed before M_0 is emitted. It is also necessary to find n_{01}. The results are

$$n_{10} = A_0 \mathbf{C}_1 \{ [y^{-1}(\mathbf{V}_\gamma/c) + y_\gamma^{-1}(\mathbf{V}/c)] + (c\mathbf{p}_0/\epsilon_0)(y^{-1} - y_\gamma^{-1}) \},$$

$$n_{01} = A_0 \mathbf{C}_1 \{ y^{-1}(\mathbf{V}_0/c) + y_0^{-1}(\mathbf{V}/c)] + (c\mathbf{p}_0/\epsilon_0)(y^{-1} - y_0^{-1}) \}.$$

On expanding $n_{10}D_{10}^{-1} + n_{01}D_{01}^{-1}$ in powers of y^{-1} one finds that the leading term contains y^{-4}; hence the integral of $\sum n_1 D_1^{-1}$ over momentum space is convergent even though there is only one energy denominator for these processes. Furthermore, the leading term, which is of order $(\mu/M)^2$, can be neglected in agreement with our previous neglect of the relativistic terms, also of the second order in (μ/M), which appear in case (C). Hence the lifetime in case (D) is given by Eq. (26).

7. NUMERICAL RESULTS

The significance of our numerical results is quite uncertain because of doubts concerning the correctness of meson theory and the applicability of quantum theory. For consistency, however, the following parameters were used in all cases: $\mu_1/\mu_0 = 1.6$, $f_0^2/\hbar c = 0.05$, $f_0/\mu_0 = f_1/\mu_1$. By Eqs. (12), (21), (25), and (26), the lifetimes given in Eq. (1) were computed. These parameters are given by Jauch and Hu[5] for the Schwinger mixture. The two constants μ_1/μ_0 and f_0 are chosen so as to give the correct ^3S binding energy and the correct ^1S virtual level of the deuteron, while the mass μ_0 is assumed to have the (low) value of 177 electron-masses determined by cosmic ray measurements.

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[5] J. M. Jauch and N. Hu, Phys. Rev. **65**, 289 (1944).