A Class of Exact Solutions of Einstein's Field Equations

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The work of Weyl on the gravitational field occasioned by an axially symmetric distribution of matter and charge is generalized to the case in which g_{44} and ϕ for an electrostatic field are functionally related, with or without spatial symmetry. It is shown that the most general electrostatic field in which g_{44} and ϕ are related by an equation of the form $g_{44}=\frac{1}{2}(\phi+c)^2$ can be represented by a line element of the form $(ds)^2=-e^{-w}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+e^w(dt)^2$. Certain of the field equations are then identically satisfied while the remaining ones reduce to a single equation for w. The substitution $w=-2\log(1+v)$ transforms this into Laplace's equation for v, so that the solution can be expressed in terms of harmonic function.

1. INTRODUCTION

In a well-known paper H. Weyl[1] showed that the gravitational field in empty space caused by *any* static axially symmetric distribution of matter and charge can be represented by the line element

$$(ds)^2=-e^u[(dz)^2+(dr)^2]-r^2e^{-w}(d\theta)^2+e^w(dt)^2 \quad (1)$$

with a suitable choice of coordinates. Using these coordinates, which he calls "canonical coordinates," Weyl completely solved the problem of the pure gravitational field with axial symmetry and also obtained a particular class of solutions for an axially symmetric electrostatic field; namely, those involving a functional relation between the electrostatic potential, ϕ, and the component g_{44} of the metric tensor.

In this paper the more general problem of any electrostatic field (with or without spatial symmetry) in which g_{44} and ϕ are functionally related, has been considered. It is found that the *only* type of functional relation that can exist between g_{44} and ϕ is of the form

$$g_{44}=A+B\phi+\tfrac{1}{2}\phi^2, \quad (2)$$

where A and B are arbitrary constants and $\phi/(8\pi)^{\frac{1}{2}}$ is the electrostatic potential in relativistic Lorentz units. If now the constants, A, B, be so chosen that the right-hand side of (2) becomes a perfect square, then the relation (2) reduces to

$$g_{44}=\tfrac{1}{2}(\phi+C)^2. \quad (3)$$

It is then shown that the *most general* electrostatic field in which g_{44} and ϕ are connected by the relation (3) can be represented by the line element

$$(ds)^2=-e^{-w}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+e^w(dt)^2 \quad (4)$$

with a suitable choice of coordinates. Some of the field equations are then identically satisfied, and the remaining ones reduce to a single equation for determining w. The substitution $w=-2\log(1+v)$ transforms this differential equation into Laplace's equation for the function v. The solution can, therefore, be expressed in terms of harmonic functions.

The solution thus obtained is, in a sense, more general than that of Weyl, since in this case there need not be spatial symmetry of the field. In another sense, however, it is less general. For, the functional relation (3) adopted here is a special case of the most general relation (2) adopted by Weyl in the case of axial symmetry.

The solution outlined above is worked out in details in Sections 4 to 6. In Section 5 it has been pointed out that the nature of the solution is such that if the field be due to several charged bodies separated by empty space then to the Newtonian approximation they remain in equilibrium because of their mutual gravitational attraction and electrostatic repulsion.

In Section 7 Weyl's axially symmetric solutions have been examined, and it has been shown that in certain cases no solution of the field equations for empty space exists. A necessary and sufficient condition for the existence of *static* axially symmetric solutions has also been ob-

[1] H. Weyl, Ann. d. Physik [4] **54**, 117 (1917).

tained. It has been proved that static field is possible only when the masses are in equilibrium under the influence of one another. This brings to relief a distinct superiority of the relativistic theory over the older Newtonian theory.

In the last section an attempt has been made to bring out the strong analogy between the old and the new theories by establishing a few simple theorems on the uniqueness of solutions and dependence of solutions on boundary values. The general boundary value problem for any static field is, however, far from being solved.

2. THE FUNDAMENTAL EQUATIONS

As usual we represent the static field by the line element

$$(ds)^2=g_{ab}dx^a dx^b+g_{44}(dt)^2, \quad (a, b=1, 2, 3) \quad (5)$$

the coordinate system so chosen that $g_{14}=g_{24}=g_{34}=0$, and the metric tensor g_{ij} is independent of t. Since no magnetic field is present, the four-potential has the components $(0, 0, 0, \phi/(8\pi)^{\frac{1}{2}})$, where $\phi/(8\pi)^{\frac{1}{2}}$ is the electrostatic potential in relativistic Lorentz units. The only surviving components of the antisymmetrical tensor, F_{ij}, and its associate F^{ij}, representing the electromagnetic field-strengths, are, therefore,

$$(8\pi)^{\frac{1}{2}}F_{4i}=\frac{\partial\phi}{\partial x^i}\equiv\phi_i,$$

$$(8\pi)^{\frac{1}{2}}F^{4i}=e^{-w}g^{ia}\phi_a, \quad (i=1, 2, 3)$$

where e^w has been written for g_{44}. From these the components of the electromagnetic energy tensor E_i^j are calculated by means of the formula

$$E_i^j=-F^{ja}F_{ia}+\tfrac{1}{4}g_i^j F^{ab}F_{ab}.$$

The gravitational tensor G_{ij} is calculated by the formula

$$G_{ij}=-\frac{\partial}{\partial x^a}\{ij, a\}+\{ia, b\}\{jb, a\}+\frac{\partial^2}{\partial x^i\partial x^j}\log(-g)^{\frac{1}{2}}-\{ij, a\}\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}.$$

The condition for the absence of electric charge and current then comes out to be

$$\frac{\partial}{\partial x^a}(e^{-w}g^{ab}\phi_b(-g)^{\frac{1}{2}})=0, \quad (6)$$

which corresponds to the classical equation $\nabla^2\phi=0$.

In addition we have the seven field equations

$$G_{ij}=-8\pi E_{ij} \quad (i, j=1, 2, 3) \quad (7)$$

$$G_{44}=-8\pi E_{44}, \quad (8)$$

the remaining three equations $G_{i4}=-8\pi E_{i4}$ for $i=1, 2, 3$ having been identically satisfied. The eight equations, (6)–(8) form the set of fundamental equations of the electrostatic field.

3. SOLUTIONS INVOLVING A FUNCTIONAL RELATION BETWEEN g_{44} AND ϕ

In this paper we shall be mainly concerned with those solutions of the field equations in which g_{44} and ϕ are functionally related. First we shall show that the *only* type of functional relation that can exist between them is of the form

$$g_{44}=A+B\phi+\tfrac{1}{2}\phi^2,$$

where A and B are arbitrary constants. In order to prove this we shall require only the two Eqs. (6) and (8).

$$G_{44}=-\frac{\partial}{\partial x^a}\{44, a\}+2\{4a, 4\}\{44, a\}-\{44, a\}\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}$$

$$=-\frac{1}{2}\frac{\partial}{\partial x^a}(g^{ab}f_b)-\tfrac{1}{2}\cdot f^{-1}f_a g^{ab}f_b+\frac{1}{2}g^{ab}f_b\cdot\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}, \quad (9)$$

where f is written for g_{44}. Also

$$-8\pi E_{44}=\tfrac{1}{2}g^{ab}\phi_a\phi_b. \quad (10)$$

Therefore, by Eq. (8)

$$\frac{\partial}{\partial x^a}(g^{ab}f_b)-f^{-1}\cdot g^{ab}f_a f_b+g^{ab}f_b\cdot\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}=g^{ab}\phi_a\phi_b.$$

Since f is a function of ϕ, this equation can be written as

$$\frac{\partial}{\partial x^a}(g^{ab}(-g)^{\frac{1}{2}}\phi_b)$$
$$=(f')^{-1}(1+f^{-1}f'^2-f'')\cdot g^{ab}(-g)^{\frac{1}{2}}\phi_a\phi_b, \quad (11)$$

where $f'=df/d\phi$ and $f''=d^2f/d\phi^2$. Equation (6) can be written as

$$\frac{\partial}{\partial x^a}(g^{ab}(-g)^{\frac{1}{2}}\cdot\phi_b)=f^{-1}f'\cdot g^{ab}(-g)^{\frac{1}{2}}\phi_a\phi_b. \quad (12)$$

Now Eqs. (11) and (12) can hold simultaneously only if

$$(1+f^{-1}f'^2-f'')/f'=f'/f,$$

that is, if $1-f''=0$. From this it follows that

$$f\equiv g_{44}=A+B\phi+\tfrac{1}{2}\phi^2$$

is the *only* functional relation possible between f and ϕ.

Since any metric approaches that of the special relativity theory at great distances from matter and charge, g_{44} must tend to unity at infinity with the proper choice of the unit of time. And if we choose the arbitrary constant in ϕ so as to make ϕ vanish at infinity, then the above relation may be written in the form

$$g_{44}=1+B\phi+\tfrac{1}{2}\phi^2. \quad (13)$$

From general considerations, solutions of this type are expected to be a bit easier to obtain, since the relation (13) makes the Eqs. (6) and (8) identical, so that there is one equation less to be satisfied. The utmost simplification is expected to result from taking the right-hand side of (13) to be a perfect square. The relation (13) then reduces to

$$g_{44}=\tfrac{1}{2}(\phi\pm\sqrt{2})^2. \quad (14)$$

4. A CLASS OF EXACT SOLUTIONS

It has been possible to find the *most general* solution of (6)–(8) involving the relation (14). The relation (14) itself, however, restricts the generality of the solutions to a great extent.

We proceed to obtain the expressions for the various tensors in terms of g_{ij}, w, and ϕ with the help of the formulas given in Section 2.

$$-8\pi E_{ij}=e^{-w}\phi_i\phi_j-\tfrac{1}{2}e^{-w}g_{ij}g^{ab}\phi_a\phi_b, \quad (i,j=1,2,3), \quad (15)$$

$$-8\pi E_{44}=\tfrac{1}{2}g^{ab}\phi_a\phi_b, \quad (16)$$

$$G_{44}=\frac{1}{2}\frac{\partial}{\partial x^a}(g^{ab}e^w w_b)-\tfrac{1}{2}e^w g^{ab}w_a w_b+\tfrac{1}{2}e^w g^{ab}w_b\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}. \quad (17)$$

On account of the relation (14), Eqs. (15), (16), and (6) can be written as

$$-8\pi E_{ij}=\tfrac{1}{2}w_i w_j-\tfrac{1}{4}g_{ij}g^{ab}w_a w_b, \quad (i,j=1,2,3) \quad (18)$$

$$-8\pi E_{44}=\tfrac{1}{4}e^w g^{ab}w_a w_b, \quad (19)$$

$$\frac{\partial}{\partial x^a}(e^{-\frac{1}{2}w}g^{ab}w_b(-g)^{\frac{1}{2}})=0. \quad (20)$$

From (8), (17), (19), and (20) we find that the field equations, (6) and (8), become identical on account of the relation (14).

Next we shall show that the relation (14) makes it possible to reduce the general static line element (5) to the isotropic form

$$ds^2=-e^{-w}[(dx^1)^2+(dx^2)^2+(dx^3)^2]+e^w(dt)^2 \quad (21)$$

by suitable transformation of coordinates. In order to do this, we introduce two sets of functions $^*g_{ij}$ and $\bar{g}_{ij}$ $(i,j=1,2,3)$ defined by the equations

$$g_{ij}=-{}^*g_{ij}=-e^{-w}\bar{g}_{ij}, \quad (i,j=1,2,3)$$

whence

$$g^{ij}=-{}^*g^{ij}=-e^w\bar{g}^{ij}.$$

$^*g_{ij}$ is the metric tensor of the three-space in (5) and $\bar{g}_{ij}$ that of an associated three-space *conformal* to it. The contracted Riemann tensors $^*G_{ij}$ and $\bar{G}_{ij}$ of these two spaces in conformal correspondence are connected by the equations

$$^*G_{ij}=\bar{G}_{ij}-\tfrac{1}{2}\bar{w}_{,ij}-\tfrac{1}{4}w_i w_j+\bar{g}_{ij}[-\tfrac{1}{2}\Delta_2 w+\tfrac{1}{4}\Delta_1 w], \quad (22)$$

where

$$\Delta_1 w=\bar{g}^{ab}w_a w_b, \quad \Delta_2 w=\bar{g}^{ab}\bar{w}_{,ab}$$

and $\bar{w}_{,ij}$ is the second co-variant derivative of the scalar function w in the associated space.

The relation between the contracted Riemann tensors G_{ij} $(i, j=1, 2, 3)$ and $^*G_{ij}$ of the four-space and three-space in (5) is

$$G_{ij}={}^*G_{ij}+\tfrac{1}{2}w_{ij}+\tfrac{1}{4}w_iw_j-\tfrac{1}{2}{}^*\{ij, a\}w_a$$
$$={}^*G_{ij}+\tfrac{1}{2}{}^*w_{,ij}+\tfrac{1}{4}w_iw_j. \quad (i, j=1, 2, 3) \quad (23)$$

Further we have the relations[2]

$$^*\{ij, k\}={}^{(-)}\{ij, k\}+\tfrac{1}{2}[-w_j\delta_i{}^k-w_i\delta_j{}^k+w_a\bar{g}_{ij}\bar{g}^{ak}]$$
$$^*\{ij, a\}w_a={}^{(-)}\{ij, a\}w_a-w_iw_j+\tfrac{1}{2}\bar{g}_{ij}\Delta_1 w. \quad (24)$$

From Eqs. (22)–(24) we have the relations

$$G_{ij}=\bar{G}_{ij}+\tfrac{1}{2}w_iw_j-\tfrac{1}{2}\bar{g}_{ij}\Delta_2 w. \quad (i, j=1, 2, 3). \quad (25)$$

Equation (18) can also be written in the form

$$-8\pi E_{ij}=\tfrac{1}{2}w_iw_j-\tfrac{1}{4}\bar{g}_{ij}\Delta_1 w. \quad (i, j=1, 2, 3) \quad (26)$$

By the use of the expressions (25) and (26) for G_{ij} and E_{ij}, the field equation (7) can now be written as

$$\bar{G}_{ij}-\tfrac{1}{2}\bar{g}_{ij}[\Delta_2 w-\tfrac{1}{2}\Delta_1 w]=0. \quad (i, j=1, 2, 3) \quad (27)$$

The expression (17) for G_{44} can also be written in a very compact form

$$G_{44}=-\frac{1}{2}\frac{\partial}{\partial x^a}(e^{2w}\bar{g}^{ab}w_b)+\tfrac{1}{2}e^{2w}\Delta_1 w-\tfrac{1}{2}e^{2w}\bar{g}^{ab}w_b\frac{\partial}{\partial x^a}\log(-g)^{\frac{1}{2}}$$
$$=-\frac{1}{2}e^{2w}\left[\bar{g}^{ab}w_{ab}+\frac{\partial\bar{g}^{ab}}{\partial x^a}w_b+\bar{g}^{ab}w_b\frac{\partial}{\partial x^a}\log(\bar{g})^{\frac{1}{2}}\right]. \quad (28)$$

By use of the relations

$$\frac{\partial\bar{g}^{ij}}{\partial x^k}=-{}^{(-)}\{ak, i\}\bar{g}^{aj}-{}^{(-)}\{ak, j\}\bar{g}^{ia},$$

$$\frac{\partial}{\partial x^i}\log(\bar{g})^{\frac{1}{2}}={}^{(-)}\{ia, a\},$$

Eq. (28) can be written as

$$G_{44}=-\tfrac{1}{2}e^{2w}\Delta_2{}^w. \quad (29)$$

Because of (19) and (29) the field equation (8) now takes the form

$$-\tfrac{1}{2}e^{2w}\Delta_2 w=-\tfrac{1}{4}e^{2w}\Delta_1 w, \quad \text{or} \quad \Delta_2 w=\tfrac{1}{2}\Delta_1 w. \quad (30)$$

From (27) and (30) follows the interesting result

$$\bar{G}_{ij}=0. \quad (i, j=1, 2, 3). \quad (31)$$

Since in a *three*-dimensional space the vanishing of the contracted Riemann tensor implies the vanishing of the Riemann tensor itself, we arrive at the conclusion that *the associated three-space is Euclidean.* We are, therefore, justified in adopting (21) as the line element of the *most general* static field in which the relation (14) holds. According to (30) the field equations will then reduce to the single equation

$$\nabla^2 w=\tfrac{1}{2}(w_1{}^2+w_2{}^2+w_3{}^2). \quad (32)$$

The substitution,

$$w=-2\log(1+v), \quad (33)$$

transforms it into Laplace's equation

$$\nabla^2 v=0 \quad (34)$$

for the function v.

Also according to (14) and (33) the relation between v and ϕ is

$$v=-\phi/(\phi\pm\sqrt{2}). \quad (35)$$

5. NATURE OF THE SOLUTION OBTAINED

The Newtonian approximation of the above solutions is obtained by neglecting all powers of v higher than the first. We then have

$$g_{44}\equiv e^w=1-2v=1-2\Omega,$$

where Ω is the gravitational potential in gravitational units. Also

$$\phi/(8\pi)^{\frac{1}{2}}=\pm v/2(\pi)^{\frac{1}{2}}.$$

The potential ϕ' in electrostatic units is, therefore,

$$\phi'=\pm v,$$

whence

$$\Omega=\pm\phi'. \quad (36)$$

[2] Editor's note: for typographical reasons the Christoffel symbols associated with the metric coefficients $^*g_{ij}$ and $\bar{g}_{ij}$ are indicated with the superposed signs * and $^{(-)}$.

From this we see that if the field be due to several charged bodies separated by empty space, then to the Newtonian approximation they must remain in equilibrium under their mutual interaction. In fact it is because of this interesting property of the solutions that a reduction of the field equations to Laplace's equation has been possible.

6. VANISHING OF THE MATERIAL STRESSES FOR A PROPER CHOICE OF THE INTERNAL FIELD

In the Newtonian theory if the densities of charge and matter bear the same constant proportion everywhere, and if in the above-mentioned units the constant of proportionality is unity, then owing to the exact balancing of the gravitational and electric forces on every piece of matter, the stresses in matter will vanish. It is remarkable that such a result exactly holds in the relativistic theory also. Of course, given an external field in empty space the distribution of matter and charge producing that field is far from being uniquely determined. We shall show below that, corresponding to the external solution obtained above, a particular internal solution can always be constructed so that the stresses in matter may vanish at every point. We choose the internal solution in the following manner.

In the interior of matter: (i) there exists a line element of the form (21), (ii) v is a solution of Poisson's equation, (iii) g_{44} and ϕ are connected by the relation (14), (iv) the first, second (and whatever derivative may be required), of w are continuous at the boundary surface, separating matter from empty space.

The internal field should satisfy the equation

$$G_i^j - \tfrac{1}{2}g_i^j G = -8\pi(M_i^j + E_i^j), \qquad (37)$$

where M_i^j is the material and E_i^j the electromagnetic energy tensor. Since Eqs. (25) and (29) hold for any static field, and Eqs. (19) and (26) follow from the assumption (iii) alone, they continue to hold in this case also. On account of the assumption (i) we can put

$$\bar{G}_{ij} = 0, \quad \bar{g}_{ij} = \delta_{ij}, \quad (i, j = 1, 2, 3)$$

in (25), (26), (29), and (19) and obtain the expressions

$$\left.\begin{aligned} G_{ij} &= \tfrac{1}{2}w_i w_j - \tfrac{1}{2}\delta_{ij}\nabla^2 w \\ -8\pi E_{ij} &= \tfrac{1}{2}w_i w_j - \tfrac{1}{4}\delta_{ij}\Sigma w_a^2 \end{aligned}\right\}, \quad (i, j = 1, 2, 3)$$

$$G_{44} = -\tfrac{1}{2}e^{2w}\nabla^2 w,$$

$$G = -\tfrac{1}{2}e^w \Sigma w_a^2 + e^w \nabla^2 w,$$

$$G_i^j - \tfrac{1}{2}g_i^j G = -\tfrac{1}{2}e^w w_i w_j + \tfrac{1}{4}\delta_{ij} e^w \Sigma w_a^2, \quad (i, j = 1, 2, 3) \qquad (38)$$

$$-8\pi E_i^j = -\tfrac{1}{2}e^w w_i w_j + \tfrac{1}{4}e^w \delta_{ij}\Sigma w_a^2. \quad (i, j = 1, 2, 3) \qquad (39)$$

Combining Eqs. (37)–(39) we arrive at the interesting result

$$M_i^j = 0, \quad (i, j = 1, 2, 3),$$

that is, the material stresses all vanish at every point. But the situation is quite different in the case of the component M_4^4 which represents the density of matter.

By (19), (29)

$$G_4^4 - \tfrac{1}{2}g_4^4 G = -e^w \nabla^2 w + \tfrac{1}{4}e^w \Sigma w_a^2,$$

$$-8\pi E_4^4 = -\tfrac{1}{4}e^w \Sigma w_a^2.$$

Because of Eq. (37) we, therefore, have

$$-8\pi M_4^4 = -e^w[\nabla^2 w - \tfrac{1}{2}\Sigma w_a^2],$$

whence

$$\nabla^2 v = -4\pi(1+v)^3 M_4^4.$$

For small v this reduces to Poisson's equation. In empty space $M_4^4 = 0$, and we get back Laplace's Eq. (34).

7. EXAMINATION OF WEYL'S AXIALLY SYMMETRIC SOLUTIONS

(A) The Case of a Pure Gravitational Field

Weyl has shown that the most general static axially symmetric gravitational field in empty space can be represented by the line element

$$(ds)^2 = -e^u[(dz)^2 + (dr)^2] - r^2 e^{-w}(d\theta)^2 + e^w(dt)^2. \qquad (40)$$

Weyl chooses the internal line element also in this form. We then have the following expressions for the components of the material energy tensor:

$$8\pi T_1^1 = -8\pi T_2^2$$
$$= \tfrac{1}{4}e^{-u}(2u_2/r + 2w_2/r + w_1^2 - w_2^2), \quad (41)$$
$$-8\pi T_3^3 = \tfrac{1}{2}e^{-u}(u_{11} + u_{22} + w_{11} + w_{22} + \tfrac{1}{2}w_1^2 + \tfrac{1}{2}w_2^2),$$
$$-8\pi T_4^4 = -\tfrac{1}{2}e^{-u}(w_{11} + w_{22} - u_{11} - u_{22} + 2w_2/r - \tfrac{1}{2}w_1^2 - \tfrac{1}{2}w_2^2),$$
$$-8\pi T_1^2 = -8\pi T_2^1 = -\tfrac{1}{2}e^{-u}(w_1 w_2 - w_1/r - u_1/r). \quad (42)$$

Weyl makes the assumption that the stress component T_3^3 vanishes everywhere. This leads to the two differential equations,

$$\Delta w \equiv w_{11} + w_{22} + w_2/r = 8\pi T_4^4 e^u, \quad (43)$$

$$\nabla^2 v \equiv v_{11} + v_{22} = -\tfrac{1}{2}(w_1^2 + w_2^2), \quad (v = u + w) \quad (44)$$

for determining u and w. In empty space $T_4^4 = 0$, and Eq. (43) reduces to Laplace's equation for axial symmetry. This equation is first solved for w and the solution substituted in the right-hand side of (44). The latter equation can then be treated as a two-dimensional Poisson's equation for the function v. The solution for v is made unique by the condition that it vanishes on the axis and at infinity. Thus we see that the two assumptions, $T_3^3 = 0$ and $T_4^4 = 0$, made by Weyl enable us to find any desired solutions for u and w which, when substituted in (41) and (42), will determine the stress components T_1^1, T_2^2, and T_1^2 uniquely. In order that the solutions thus obtained may be the correct solutions for empty space, it is essential that T_1^1, T_2^2, and T_1^2 should vanish in addition to T_3^3 and T_4^4. Weyl's method of solution, in many cases, does not lead to the vanishing of these components. In order, therefore, to arrive at the correct solutions for empty space, we set the left-hand sides of Eqs. (41) to (42) equal to zero. The functions w and v then satisfy the equations

$$\Delta w = 0, \quad (45)$$

$$v_1 = r w_1 w_2, \quad (46)$$

$$v_2 = \tfrac{1}{2}r(w_2^2 - w_1^2). \quad (47)$$

As before, we first integrate Eq. (45) by introducing an arbitrary axially symmetric density distribution σ. This solution for w is then substituted in Eqs. (46) and (47). If a solution for v can now be found satisfying Eqs. (46) and (47), then Weyl's Eq. (44) will be automatically satisfied in the empty regions of space. But the converse is not true. A necessary and sufficient condition that a function v may exist satisfying (46) and (47) is that the line integral

$$I = \int (v_1 dz + v_2 dr). \quad (48)$$

(i) along *any* closed curve, A, which lies entirely in empty space should vanish, and (ii) along *any* curve, B, which starts from the axis and terminates on the axis and lies entirely in empty space, should vanish.

First we consider curves of the type A which may surround matter distributed in the shape of a ring. By Stokes' theorem,

$$I = \int\int (v_{12} - v_{21}) dS,$$

where dS is an element of plane area in the $(r-z)$ half-plane enclosed by the curve, and

$$v_{12} = \partial v_1/\partial r, \quad v_{21} = \partial v_2/\partial z.$$

v_1, v_2 are looked upon as two different functions and not as the derivatives of a single function v.

By (46) and (47)

$$v_{12} - v_{21} = r w_1 \cdot \Delta w.$$

If the enclosed area is empty then $\Delta w = 0$ and, therefore, I vanishes. If, however, the area contains matter, then

$$\Delta w = -4\pi\sigma,$$

and

$$I = -4\pi \int\int r w_1 \sigma dS$$
$$= -2\int\int\int r w_1 \sigma dz dr d\theta$$
$$= -2\int\int\int w_1 \sigma d\tau,$$

the last integral having been taken over the entire ring. Now w_1 is the component of force in the z direction acting on the mass element $\sigma d\tau$, and w_1 consists of two parts,

$$w_1 = w_{i,1} + w_{e,1},$$

where $w_{i,1}$ is the internal force attributable to the remaining part of the ring and $w_{e,1}$ is the resultant of external forces acting on the element $\sigma d\tau$. The contribution to the integral I due to $w_{i,1}$ therefore vanishes, and we have

$$I = -2\int\int\int w_{e,1}\sigma d\tau. \qquad (49)$$

Thus we obtain as the necessary and sufficient condition for the existence of v that the "resultant force acting on the ring according to classical calculations must vanish."

Next we consider a curve of the type B which starts from a point P on the axis, terminates at a point Q on the axis, and lies entirely in empty space. After going from P to Q along the curve if we go back to P along the axis we complete a circuit, and the above proof can be easily extended to this case also.

From this we conclude that solutions corresponding to two spheres separated by empty space do not exist, whereas solutions corresponding to a sphere at the center of a circular ring do exist.

(B) The Case of an Electrostatic Field

Weyl has shown that the most general axially symmetric electrostatic field in empty space can be represented by the line element (40). He takes the internal line element also in this form and obtains a class of solutions of the type (13). As his solutions may not lead to the vanishing of all the components of stress in empty space, we proceed, as before, to investigate the conditions for existence of solutions for strictly empty space. The field equations in this case are

$$8\pi E_1^1 = -8\pi E_2^2 = \tfrac{1}{2}e^{-u-w}(\phi_1^2-\phi_2^2) = \tfrac{1}{4}e^{-u}(2v_2/r+w_1^2-w_2^2),$$

$$-8\pi E_3^3 = \tfrac{1}{2}e^{-u-w}(\phi_1^2+\phi_2^2) = \tfrac{1}{2}e^{-u}(\nabla^2 v+\tfrac{1}{2}w_1^2+\tfrac{1}{2}w_2^2),$$

$$-8\pi E_4^4 = -\tfrac{1}{2}e^{-u-w}(\phi_1^2+\phi_2^2) = -\tfrac{1}{2}e^{-u}(2\Delta w-\nabla^2 v-\tfrac{1}{2}w_1^2-\tfrac{1}{2}w_2^2),$$

$$-8\pi E_1^2 = -8\pi E_2^1 = -e^{-u-w}\phi_1\phi_2 = -\tfrac{1}{2}e^{-u}(w_1w_2-v_1/r).$$

The equations to be satisfied by w and ϕ are, therefore,

$$\Delta w = e^{-w}(\phi_1^2+\phi_2^2), \qquad (50)$$

$$\Delta\phi = w_1\phi_1+w_2\phi_2. \qquad (51)$$

We shall consider Weyl's solutions only, which are of the type (13). The two equations, (50) and (51), then become identical. Weyl introduces an auxiliary function χ defined by the equation

$$\chi = \int(1+B\phi+\tfrac{1}{2}\phi^2)^{-1}d\phi. \qquad (52)$$

We then have

$$\Delta\chi = e^{-w}[\Delta\phi-w_1\phi_1-w_2\phi_2] = (B+\phi)^{-1}[\Delta w-e^{-w}\phi_1^2-e^{-w}\phi_2^2]. \qquad (53)$$

Hence (50) and (51) reduce to Laplace's equation

$$\Delta\chi = 0.$$

Inside matter we take the auxiliary function χ to be a solution of Poisson's equation. The equations to be satisfied by v are

$$v_1 = rw_1w_2-2e^{-w}\phi_1\phi_2, \qquad (54)$$

$$v_2 = \tfrac{1}{2}r(w_2^2-w_1^2)+re^{-w}(\phi_1^2-\phi_2^2). \qquad (55)$$

In this case,

$$\begin{aligned} v_{12}-v_{21} &= rw_1\Delta w+re^{-w}[w_1\phi_1^2-w_1\phi_2^2 -2\phi_1\Delta\phi+2w_2\phi_1\phi_2] && \text{by (54) and (55)}\\ &= r[w_1(B+\phi)-2\phi_1]\Delta\chi && \text{by (53)}\\ &= r(B^2-2)\chi_1\cdot\Delta\chi && \text{by (52).}\end{aligned}$$

Therefore, the line integral

$$I = -2(B^2-2)\int\int\int\chi_{e,1}\sigma d\tau. \qquad (56)$$

We therefore arrive at the same condition for existence of solution as before. From (56) we draw the interesting conclusion that *if $B=\pm\sqrt{2}$, then a solution always exists irrespective of the distribution of the masses*. This is in agreement with what has been said in Section (5).

8. BOUNDARY VALUE PROBLEMS

If g_{44} and ϕ are functionally related, then every level-surface of g_{44} (thereby meaning a surface on which g_{44} is constant) is also a level-surface of ϕ. We shall now prove the proposition that if there exists one surface S (either closed or extending to infinity) on which g_{44} and ϕ are both constant, and if one of the two domains into which this surface divides the entire space be completely free from matter and charge, then in this domain, D, every level-surface of g_{44} will also be a level-surface of ϕ, and therefore g_{44} and ϕ will be connected by an equation of the type (13).

To prove this we shall require only Eqs. (6) and (8). Equation (8) can be written as

$$\frac{\partial}{\partial x^a}[f^{-1}(-g)^{\frac{1}{2}}g^{ab}f_b]=f^{-1}(-g)^{\frac{1}{2}}g^{ab}\phi_a\phi_b. \quad (57)$$

Multiplying Eq. (6) by ϕ and subtracting from Eq. (57), we have

$$\frac{\partial}{\partial x^a}[f^{-1}(-g)^{\frac{1}{2}}g^{ab}f_b]-\phi\frac{\partial}{\partial x^a}[f^{-1}(-g)^{\frac{1}{2}}g^{ab}\phi_b]-f^{-1}(-g)^{\frac{1}{2}}g^{ab}\phi_a\phi_b=0$$

or

$$\frac{\partial}{\partial x^a}[f^{-1}(-g)^{\frac{1}{2}}g^{ab}(f-\tfrac{1}{2}\phi^2)_b]=0. \quad (58)$$

Equations (6) and (58) are of the form

$$a^{ik}u_{[ik]}+b^iu_{[i]}=0, \quad (59)$$

where

$$a^{ik}=-f^{-1}(-g)^{\frac{1}{2}}g^{ik}, \quad (60)$$

$$b^i=-\frac{\partial}{\partial x^k}[f^{-1}(-g)^{\frac{1}{2}}g^{ik}], \quad (61)$$

$$u=\phi-A, \quad \text{or} \quad f-A, \quad \text{or} \quad f-\tfrac{1}{2}\phi^2-B\phi-1,$$

$$u_{[i]}=\partial u/\partial x^i, \quad u_{[ik]}=\partial^2u/\partial x^i\partial x^k.$$

We now require the help of a theorem given by Courant and Hilbert[3] on the uniqueness of the solutions of linear elliptic differential equations. The proof, as given by these authors, is for a finite domain only, but the following modification holds for both finite and infinite domains.

We make the substitution

$$u=zv, \quad z=1-e^{-\mu r}, \quad \mu>0$$

with

$$r=[(x^1)^2+(x^2)^2+(x^3)^2]^{\frac{1}{2}}=[\delta_{ij}x^ix^j]^{\frac{1}{2}}.$$

The origin of space-coordinates is chosen to be outside the domain D, so that r is everywhere positive in D. Equation (59) is then transformed into

$$a^{ik}v_{[ik]}+\beta^iv_{[i]}+cv=0, \quad (62)$$

where

$$\beta^i=2z^{-1}a^{ik}z_{[k]}+b^i=2z^{-1}\mu e^{-\mu r}(a^{ik}\delta_{kj}x^j/r)+b^i, \quad (63)$$

$$c=z^{-1}(a^{ik}z_{[ik]}+b^iz_{[i]})=-z^{-1}e^{-\mu r}\left\{\mu^2\frac{a^{ik}\delta_{ij}\delta_{km}x^jx^m}{r^2}-\mu\left[\left(\frac{\delta_{ik}}{r}-\frac{\delta_{ij}\delta_{km}x^jx^m}{r^3}\right)a^{ik}+\frac{b^i\delta_{ij}x^j}{r}\right]\right\}. \quad (64)$$

Now we choose the space coordinates in such a way that the g_{ik}'s approach Galilean values at infinity. We then notice that: (1) the a^{ik}'s tend to δ^{ik} at infinity, and (2) b^i, β^i, c all tend to zero at infinity. Concerning the sign of c, which is very important, we see first of all that the a^{ik}'s form the matrix of a positive definite quadratic form. Therefore, the quantity $a^{ik}\delta_{ij}\delta_{km}x^jx^m/r^2$ in the first term of (64) is everywhere positive in D and approaches the value unity at infinity, and hence possesses a positive lower bound. Similarly the coefficient of μ in the bracketed expression in (64) tends to zero at infinity, and, therefore, its absolute value possesses a positive upper bound. Hence the quantity μ can be chosen so large as to make this expression positive throughout D. Since z is positive throughout D, it follows that c is negative throughout D.

We have thus established the two main requirements for the proof, namely, that a^{ik} should form the matrix of a positive definite quadratic form and that c should be negative. The proof now follows the same course as that given by Courant and Hilbert,[3] and we arrive at the conclusion that if v (and therefore u) vanishes on the boundary of D (and at infinity if the domain be infinite) then v (and therefore u) vanishes throughout D.

[3] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Julius Springer, Berlin, 1937), pp. 274–276.

DEDUCTIONS FROM THE ABOVE THEOREM

(1) First we apply the theorem to the function

$$u = f - \tfrac{1}{2}\phi^2 - B\phi - 1.$$

If there exists a surface S on which f and ϕ are both constant, then we can always choose the constant, B, so as to make u vanish on S. Hence it follows that if one of the two domains into which the entire space is divided by this surface be empty, then in this domain u vanishes and, therefore, f and ϕ are functionally related. That is, f and ϕ have the same family of level-surfaces in D. This is the theorem stated at the beginning of this section.

(2) Applying the theorem to the function $u = \phi - A$ we draw similar conclusions.

(3) In the *absence* of an electric field we apply the theorem to the function $u = f - A$ and draw similar conclusions. In this case it can be easily shown that space-time is flat in D.

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