

The Deflection of Charged Particles in a Ferromagnetic Medium

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The deflection is recomputed which a beam of charged particles will undergo when penetrating a ferromagnetic medium whose magnetism is due to electronic spins. The extreme importance of the "interior of the electron" for the effective magnetic field \mathbf{b} is brought out by a quantum calculation which is analogous to well-known classical considerations. Only if the interpenetration of beam particle and electron has random probability will one find $\mathbf{b}=\mathbf{B}$. Otherwise deviations are to be expected which can be estimated in special cases. A section is added which develops the conditions under which magnetic deflection can be observed in competition with multiple scattering. Very high energy protons and deuterons offer the best possibilities (see Fig. 2).

1. GENERAL NATURE OF THE PROBLEM

IN recent years, experiments have come to the fore¹ in which the magnetic deflection of a beam inside a magnet plays a role. Such experiments involve the question of the magnitude of the "effective field" inside a polarized medium. Such a field has to be specified in terms of a process. In the case which is to be studied in this paper a fast charged particle traverses a piece of magnetized matter and experiences a very large number of small magnetic deflections. The resultant effect can be described in terms of an "effective magnetic field" \mathbf{b} by the formula

$$\mathbf{F} = (ze/c)\mathbf{v} \times \mathbf{b}, \quad (1)$$

where \mathbf{F} is the force on a beam particle, ze its charge, and \mathbf{v} its velocity.

There is one simple statement we can make about \mathbf{b} . If the velocity of the beam particles is increased sufficiently so that their paths can be considered to be straight lines, then \mathbf{b} will approach the magnetic induction \mathbf{B}

$$\lim_{v \rightarrow \infty} \mathbf{b} = \mathbf{B}. \quad (2)$$

This is a consequence of the generally accepted theory that all magnetic dipoles are in nature equivalent to circulating currents. That this view is correct for the Dirac electron has been shown by Weizsäcker.²

Many people believe that (2) is not just a limiting relation, but a generally valid equation.

The case against this supposition has been very well stated by Swann.³ His considerations, although expressed in classical language, are valid in quantum mechanics also. They can be expressed as follows: If the magnetization of a ferromagnet is orbital in nature then there is little doubt that (2) will hold under very wide conditions. However, the more common case is that of ferromagnetism arising from the electron spin. In other words, the magnetization \mathbf{M} originates in dipoles which are very small even on an atomic scale. The field which they produce varies over a very wide range of magnitudes and even the sign of its component along \mathbf{M} is variable, with regions of very large negative and positive contributions adjoining near the electron. This situation is schematically represented in Fig. 1. The quantity \mathbf{b} defined through (1) is an average value of this true field for a beam particle. It is well known that the extreme fields in the neighborhood of a small dipole enter decisively in the determination of such an average. The one simple result mentioned earlier, giving $\mathbf{b}=\mathbf{B}$, results from the averaging process only if there is no force capable of favoring or hindering the interpenetration of the two particles. In addition, it is necessary for a statistical average that each beam particle enter several times the "interior" of an electron. Such interpenetration does not pose any problems of principle in present day physical theory, but it does remain a practical question in each individual case. Among the forces which might enter into play there is of course the Coulomb force

* The greater part of this work was carried out at the University of Iowa, Iowa City, Iowa.

¹ F. Rasetti, *Phys. Rev.* **66**, 1 (1944).

² C. F. v. Weizsäcker, *Ann. d. Physik* **17**, 869 (1933).

³ W. F. G. Swann, *Phys. Rev.* **49**, 574 (1936).

between the two charges. In addition, there is the possibility of short range forces which might not otherwise be observable.

2. CALCULATION OF THE AVERAGE FIELD \mathbf{b}

Since ferromagnetism is associated with the electron spin we must use Dirac wave functions for the electron. For the beam particles, no structure will be assumed except the possession of a charge. This is not so much an assumption as a definition of what we wish to call the magnetic deflection of a charged particle.

The calculation which follows differs from the standard in that it computes the expectation value of a force rather than an energy. The force is the force \mathbf{F} in Eq. (1). Reading the latter as an operator equation, we can see that finding the expectation value of the force \mathbf{F} is equivalent to finding the expectation value of the field \mathbf{b} , provided the velocity is an approximate constant of the motion; this must necessarily be so if we want to observe a small deflection at all. The correct operator for \mathbf{b} can be obtained from two independent approaches. Either we take the law of Biot-Savart and make the substitution for the electron velocity \mathbf{w}

$$\mathbf{w} \rightarrow -\alpha c,$$

where $\alpha_x, \alpha_y, \alpha_z$ are the three Dirac matrices; or else we take the Breit Hamiltonian, translate it into classical language, compute the magnetic part of the force, and translate back into quantum language. Either approach gives the same result. The operator giving the field $\mathbf{b}(\mathbf{r})$ at the position \mathbf{r} equals

$$\mathbf{b}(\mathbf{r}) = \sum_{\nu=1}^n \frac{e\alpha_{\nu} \times (\mathbf{r} - \mathbf{r}_{\nu})}{|\mathbf{r} - \mathbf{r}_{\nu}|^3}. \quad (3)$$

Here the summation extends over all ferromagnetic electrons in the magnet. In order to obtain the expectation value of (3) we require the knowledge of the spinor wave function

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \dots, \mathbf{r}_n; \mathbf{r}),$$

where \mathbf{r} stands for the Cartesian coordinates of a beam particle, and $\mathbf{r}_1, \mathbf{r}_2, \dots$ are the Cartesian and spin coordinates of the ferromagnetic electrons.

In looking for a reasonable approximation to ψ we can first convince ourselves that a product

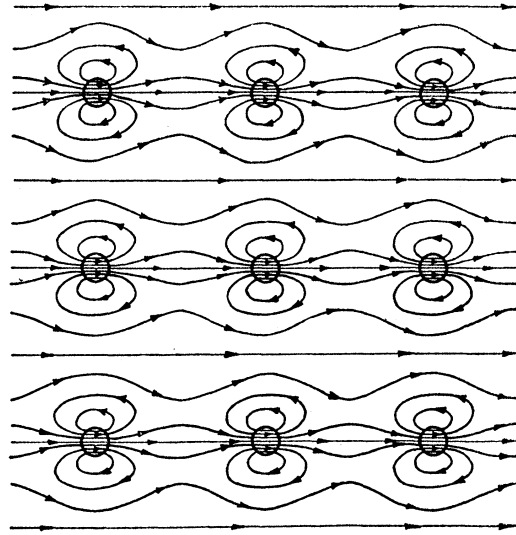


FIG. 1. A schematic picture of the magnetic field inside a ferromagnet.

wave function

$$\psi_1(\mathbf{r}_1, \mathbf{r})\psi_2(\mathbf{r}_2, \mathbf{r})\psi_3(\mathbf{r}_3, \mathbf{r}) \cdots \psi_n(\mathbf{r}_n, \mathbf{r})\chi(\mathbf{r})$$

is adequate. The reason is that the ferromagnetic electrons are inner shell electrons and can be assumed to be localized and independent of each other in their Cartesian coordinates. For their spins, on the other hand, we may assume complete coupling with a pre-assigned field direction. This justifies the product assumption and shows that the individual factors ψ_{ν} differ from each other only in the location of their origin. The result should be anti-symmetrized, of course, but as this will have no effect on an interaction of the type (3) we can dispense with it. The factor $\chi(\mathbf{r})$ at the end is added for convenience. It is essentially a modified plane wave which is normalized over the whole ferromagnet. The functions ψ_{ν} can then be considered normalized in their first coordinate.

With this type of wave function, the calculation is reduced to a two-body problem. We get for the expectation value \mathbf{b} of (3)

$$\mathbf{b} = \int_{\Omega} \chi^*(\mathbf{r}')\chi(\mathbf{r}')d\tau' \\ \sum_{\nu=1}^n \int_{\Omega} \left\{ \psi_{\nu}(\mathbf{r}, \mathbf{r}') \frac{e\alpha_{\nu} \times (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \psi_{\nu}(\mathbf{r}, \mathbf{r}') \right\} d\tau. \quad (4)$$

The sum extends over all atoms of the ferromagnet.

It is useful at this point to use the Darwin-Pauli approximation to the correct Dirac wave functions.⁴ In this approximation we introduce a Schrödinger wave function of the type

$$\varphi_\nu(\mathbf{r}, \mathbf{r}'), \quad (5)$$

and then get the spinor components in the following way

$$u_{\nu 1}(\mathbf{r}, \mathbf{r}') = \frac{i\hbar}{2mc} \left(\frac{\partial \varphi_\nu}{\partial x} - i \frac{\partial \varphi_\nu}{\partial y} \right),$$

$$u_{\nu 2}(\mathbf{r}, \mathbf{r}') = -\frac{i\hbar}{2mc} \frac{\partial \varphi_\nu}{\partial z},$$

$$u_{\nu 3}(\mathbf{r}, \mathbf{r}') = 0,$$

$$u_{\nu 4}(\mathbf{r}, \mathbf{r}') = \varphi_\nu.$$

Substitution of these values into (4) yields

$$\mathbf{b} = \int_{\Omega} \rho(\mathbf{r}') d\tau' \times \int_{\Omega} \frac{[\mathbf{I}_o(\mathbf{r}, \mathbf{r}') + \mathbf{I}_s(\mathbf{r}, \mathbf{r}')] \times (\mathbf{r}' - \mathbf{r})}{c|\mathbf{r}' - \mathbf{r}|^3} d\tau, \quad (6)$$

with

$$\rho(\mathbf{r}') = \chi^*(\mathbf{r}') \chi(\mathbf{r}'), \quad (7)$$

$$\mathbf{I}_o(\mathbf{r}, \mathbf{r}') = \frac{ie\hbar}{2m} \sum_{\nu} (\varphi_{\nu}^* \nabla \varphi_{\nu} - \varphi_{\nu} \nabla \varphi_{\nu}^*), \quad (8)$$

$$I_{sx} = \frac{e\hbar}{2m} \sum_{\nu} \frac{\partial}{\partial y} (\varphi_{\nu}^* \varphi_{\nu}),$$

$$I_{sy} = -\frac{e\hbar}{2m} \sum_{\nu} \frac{\partial}{\partial x} (\varphi_{\nu}^* \varphi_{\nu}), \quad (9)$$

$$I_{sz} = 0.$$

Equation (6) states the law of Biot-Savart for a beam particle which has a probability density $\rho(\mathbf{r}')$ to be at a given place. $\mathbf{I}_o + \mathbf{I}_s$ is the current density producing the magnetic field. This current density breaks up into two parts. The part (8) is the current produced by the orbital motion of the electrons, part (9) is the contribution of the spin. We shall neglect the contribution (8) in the future.

⁴H. A. Bethe, *Handbuch der Physik*, Vol. XXIV, 1, p. 304.

To facilitate further discussion we shall define as the true magnetization $\mathbf{M}(\mathbf{r}, \mathbf{r}')$ the following quantity

$$\mathbf{M}(\mathbf{r}, \mathbf{r}') = \mathbf{k}(e\hbar/2mc) \sum_{\nu} \varphi_{\nu}^* \varphi_{\nu}, \quad (10)$$

where \mathbf{k} is a unit vector in the z -direction. $\mathbf{M}(\mathbf{r}, \mathbf{r}')$ gives the magnetization at the point \mathbf{r} . Because of the assumption of perfect ferromagnetism, it has the same direction everywhere and can be treated as a scalar if convenient. Its magnitude varies periodically in the crystal lattice and depends on the location \mathbf{r}' of the beam particle as a parameter. With the help of definition (10), Eq. (9) takes the form

$$\mathbf{I}_s(\mathbf{r}, \mathbf{r}') = c \nabla \times \mathbf{M}(\mathbf{r}, \mathbf{r}') \quad (11)$$

which is a well-known relationship of classical electrodynamics.⁵ Using (11) and neglecting (8), we get (6) in the form

$$\mathbf{b} = \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \frac{(\nabla \times \mathbf{M}) \times (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} d\tau. \quad (12)$$

In addition to (10) we shall introduce several averages of this true magnetization \mathbf{M} . The first will be called the atomic magnetization $M_a(\mathbf{r})$

$$M_a(\mathbf{r}) = \Omega^{-1} \int_{\Omega} M(\mathbf{r}, \mathbf{r}') d\tau'. \quad (13)$$

This average removes the dependence on the beam particle, but it still varies rapidly within atomic distances. Another average that can be formed is the dynamic magnetization $M_d(\mathbf{r}, \mathbf{r}')$.

$$M_d(\mathbf{r}, \mathbf{r}') = (\Delta\Omega)^{-1} \int_{\Delta\Omega} M(\mathbf{r} + \boldsymbol{\rho}, \mathbf{r}' + \boldsymbol{\rho}) d\tau_{\rho}. \quad (14)$$

Here $\Delta\Omega$ is a volume large compared to atomic dimensions, but small compared to the size Ω of the magnet. The final average is the gross magnetization \bar{M} which is obtained by carrying out the missing averaging process on (13) or (14)

$$\begin{aligned} \bar{M}(\mathbf{r}) &= (\Delta\Omega)^{-1} \int_{\Delta\Omega} M_a(\mathbf{r} + \boldsymbol{\rho}) d\tau_{\rho} \\ &= \Omega^{-1} \int_{\Omega} M_d(\mathbf{r}, \mathbf{r}') d\tau'. \end{aligned} \quad (15)$$

⁵R. Becker, *Theorie der Elektrizität* (B. G. Teubner, Leipzig, 1933), Vol. 2, Chapter C.

Both M_d and \bar{M} have the crystalline structure averaged out, but (14) still shows the dynamic effect of the beam particle. The final average (15) is variable only in the walls of the magnet; the current (11) reduces for this case to the "equivalent solenoid" current which, in turn, produces a field equal to the induction \mathbf{B} . Thus Eq. (12) reduces for this case to

$$\mathbf{B} = \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \frac{(\nabla \times \bar{\mathbf{M}}) \times (\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} d\tau. \quad (16)$$

The two last averages can be related to each other through an equation of the form

$$M_d(\mathbf{r}, \mathbf{r}') = \bar{M}(\mathbf{r}) f(\mathbf{r} - \mathbf{r}'), \quad (17a)$$

where

$$\int_{\Omega} f(\mathbf{r}) d\tau = \Omega, \quad (17b)$$

and

$$f(\mathbf{r}) = 1, \text{ (if } \mathbf{r} \text{ is larger than atomic dimensions).} \quad (17c)$$

The reason for this is that inside the crystal the only variation of M_d not yet averaged out is that arising from the proximity of the beam particle. This effect involves coordinate differences only as is indicated in (17a) and is necessarily of short range as is shown by (17c); in the walls of the magnet, the two averages vary the same way.

Equations (12) and (16) show that \mathbf{b} will equal \mathbf{B} if it is permissible to replace M by its double average \bar{M} . The two intermediate averages will have a certain usefulness in the following discussion.

3. A GENERAL THEOREM CONCERNING \mathbf{b}

A classical calculation of the average field \mathbf{b} leads³ to certain general statements emphasizing the importance of the interior of the electron. One introduces as a device the true di-pole type of an elementary magnet. Let us call \mathbf{h} the average field which would be produced if all the electronic di-poles were of this type. This quantity \mathbf{h} is of course fictitious, but it is useful in that we can infer that it would be very accidental to find $\mathbf{b} = \mathbf{B}$ unless we find $\mathbf{b} - \mathbf{h} = 4\pi\bar{\mathbf{M}}$. Of this difference $\mathbf{b} - \mathbf{h}$, one can show that it arises entirely from contributions inside the magnetic di-poles. The quantum mechanical analog of this calculation will now be carried through.

It has been pointed out that Eq. (12) can be arrived at without the use of Dirac theory. Simple electrodynamics predicts Eq. (11), and Eq. (12) results from it by the law of Biot-Savart. The only assumption needed is that \mathbf{M} is due to circulating electric charges. Now, if we assume instead that \mathbf{M} is due to di-poles, then a variable magnetization would produce a pole density P which equals⁵

$$P = -\nabla \cdot \mathbf{M} = -\partial M / \partial z. \quad (18)$$

The average field resulting from the same magnetization (10) would then produce a different field \mathbf{h} . It equals

$$\mathbf{h} = - \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \frac{\partial M}{\partial z} \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} d\tau. \quad (19)$$

Following the reasoning which led from (12) to (16) we get an expression for the conventional magnetic field \mathbf{H} as due to the pole faces:

$$\mathbf{H} = - \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \frac{\partial \bar{M}}{\partial z} \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} d\tau. \quad (20)$$

Now we take (12) and (19) and form their difference. We find

$$b_z - h_z = \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \frac{(\mathbf{r}' - \mathbf{r}) \cdot \nabla M}{|\mathbf{r}' - \mathbf{r}|^3} d\tau,$$

or, expressing the second integrand as a divergence,

$$b_z - h_z = \int_{\Omega} \rho(\mathbf{r}') d\tau' \int_{\Omega} \nabla \cdot \left[M \nabla \frac{1}{|\mathbf{r}' - \mathbf{r}|} \right] d\tau.$$

We can apply Gauss' theorem provided we exclude the point $\mathbf{r}' = \mathbf{r}$. The expression then reduces to

$$b_z - h_z = \int_{\Omega} \rho(\mathbf{r}') d\tau' 4\pi M(\mathbf{r}', \mathbf{r}). \quad (21)$$

In a similar manner we find $b_x - h_x = 0$, $b_y - h_y = 0$. If the same calculation is applied to (16) and (20), instead of (12) and (19), we get, of course

$$B_z - H_z = 4\pi\bar{M}, \quad B_x - H_x = 0, \quad B_y - H_y = 0, \quad (22)$$

as expected.

On Eq. (21) we can make the following observation. Consider

$$\sum_{\nu} \varphi_{\nu}^*(\mathbf{r}, \mathbf{r}') \varphi_{\nu}(\mathbf{r}, \mathbf{r}')$$

and its average

$$\Omega^{-1} \int_{\Omega} \sum_{\nu} \varphi_{\nu}^*(\mathbf{r}, \mathbf{r}') \varphi_{\nu}(\mathbf{r}, \mathbf{r}') d\tau'$$

For physical reasons the two expressions must be equal when $\mathbf{r}' - \mathbf{r}$ is made large in the first one. The second expression gives, therefore, the electron density at \mathbf{r} if the beam particle is very far away. On the other hand

$$\sum_{\nu} \varphi_{\nu}^*(\mathbf{r}, \mathbf{r}) \varphi_{\nu}(\mathbf{r}, \mathbf{r})$$

gives the electron density at \mathbf{r} if the beam particle is also at \mathbf{r} . We can write, therefore,

$$\begin{aligned} \sum_{\nu} \varphi_{\nu}^*(\mathbf{r}, \mathbf{r}) \varphi_{\nu}(\mathbf{r}, \mathbf{r}) \\ = p(\mathbf{r}) \Omega^{-1} \int_{\Omega} \sum_{\nu} \varphi_{\nu}^*(\mathbf{r}, \mathbf{r}') \varphi_{\nu}(\mathbf{r}, \mathbf{r}') d\tau', \end{aligned}$$

where

$$p(\mathbf{r}) = \frac{\text{chance of finding the electron at } \mathbf{r} \text{ if beam particle is also at } \mathbf{r}}{\text{chance of finding the electron at } \mathbf{r} \text{ if beam particle is far away}}. \quad (23)$$

We shall call it the coincidence probability at \mathbf{r} . Now, using (10) and (13), the previous considerations give us

$$M(\mathbf{r}, \mathbf{r}) = p(\mathbf{r}) M_a(\mathbf{r}).$$

This transforms (20) into

$$\mathbf{b} - \mathbf{h} = 4\pi \int_{\Omega} \rho(\mathbf{r}) p(\mathbf{r}) \mathbf{M}_a(\mathbf{r}) d\tau, \quad (24)$$

or in words: $\mathbf{b} - \mathbf{h}$ is equal to 4π times the average value of the atomic magnetization along the path of a beam particle, *provided the atomic magnetization at each point is multiplied with the coincidence probability at that point.*

The same relation was stated in a simpler form in an earlier paper.⁶ If the effect of the crystalline field upon the beam is negligible $\rho(\mathbf{r})$ and $p(\mathbf{r})$ become constants in the magnet. The latter is obviously connected with $f(\mathbf{r})$, as defined in (17).

⁶ G. H. Wannier, Phys. Rev. **67**, 364 (1945).

It equals

$$p = f(\mathbf{0}). \quad (25)$$

In view of (15) Eq. (24) gives then

$$\mathbf{b} - \mathbf{h} = 4\pi p \bar{\mathbf{M}}, \quad (26)$$

where $\bar{\mathbf{M}}$ is now the gross magnetization. Equation (26), in turn, reduces to (22) if the coincidence probability is unity, that is for undisturbed wave functions.

The conditions under which (24) or (26) holds were enumerated formally in an earlier paper.⁶ A short review of them may be in order:

(a) "That the magnetic interaction is sufficiently small to be treated as a first order perturbation." This condition is implicit in picking a wave function (5) not involving the magnetic interaction and computing from it the expectation value of (3). It appears reasonable enough for charged particles, but seems inadmissible for neutrons. It is not possible, of course, to find wave functions containing magnetic effects unless "cut-off" methods are used.

(b) "That the magnetic field is due entirely to the electronic spin." It arises from the neglect of (8) and appears generally accepted for ferromagnetic metals.

(c) "That the ferromagnetic electrons move in orbits independent of each other." This is a reasonable assumption which is necessary to get Eq. (4).

(d) "That it is sufficiently accurate to solve the Dirac equation in the Schrödinger-type approximation of Darwin-Pauli." This assumption transforms (4) into (6) and needs some discussion. The true Dirac wave functions are very weakly infinite at the origin; they will produce divergent results in the present case unless combined with a cut-off procedure. As soon, however, as a cut-off radius of the order of the electron radius is introduced the wave functions differ only insignificantly from those used here. The assumption, therefore, appears to be justified.

(e) "That the test charge is much heavier than the electron." Having a p different from unity in (26) implies scattering. This scattering will affect the beam particle unless it is heavier or much faster than the electron with which it collides. Even if this condition is broken the equations from (3) on may still be applicable;

but Eq. (1) will break down and make our results meaningless unless a new empirical definition is found justifying (3). This remark applies for instance to the proposal of Webster⁷ to apply considerations of this type to conduction electrons in iron.

A further assumption stressed by Swann³ happens to be unnecessary. The difficulty raised there is that coincidence of beam particle and electron would be so uncommon statistically that a strict average would have little meaning. In quantum theory the "coincidence" part of a wave arises entirely from the *S*-part of the wave function. The chance for such an *S*-collision can be obtained as follows: we estimate the maximum angular momentum possible for a beam particle passing a monatomic layer by taking it to be (reduced mass) × (impact parameter) × (relative velocity), and limiting the impact parameter to one half of the interatomic distance. The maximum quantum number *J* thus obtained lies between 10 and 100, depending on the velocity. This gives a probability for *S*-collisions lying between 10⁻² and 10⁻⁴. Thus even in a 0.01-mm foil each beam particle experiences at least 10 *S*-collisions.

Under the restrictions just listed, Eqs. (24) and (26) state in quantum language the same thing as the classical considerations of Swann.³ The difference **b** - **h** arises entirely from the interior of the electrons and thus depends on the result of head-on collisions. If **b** is to equal **B** this difference must equal 4π \bar{M} in spite of the facts just stated.

4. SPECIAL RESULTS FOR THE COULOMB FIELD

The analysis of the previous section shows that there are two effects which may produce a **b** which is not equal to **B**:

(a) The crystalline field will modify the wave function of the beam particle so that encounters between beam particle and ferromagnetic electron no longer have random probability.

(b) When there is an encounter, the chance of interpenetration is not average because of the intervention of repulsive or attractive forces.

The effect (b) opens a possibility of detecting short range forces which may not otherwise be observable. For such hypothetical forces, Eq.

(26) goes about as far as we can hope to go. There is, however, such an effect produced by the Coulomb field of the two charges. We shall calculate this effect in the following, assuming effect (a) to be negligible, as it probably is for beams of reasonable energy.

The first thing we can do for this case is to examine the relationship of the previous section. The simpler form (26) applies here. The value of *p* is given by (25); it is well known for the Coulomb field⁸ and equals

$$p = x/(1 - e^{-x}), \quad (27a)$$

where

$$x = (4\pi^2ze^2/hv) = 1.39 \cdot 10^9(z/v). \quad (27b)$$

If the absolute value of *x* is large, *p* will be very large for positive charges and zero for negative charges. For high velocities, on the other hand, *x* will be small and *p* will equal unity. A list of values of *p* in the transition region is given in Table I of reference 6.

The calculation can be completed without any difficulty to yield **b** itself. Our assumptions permit substitution of the smoothed out M_a for M in Eqs. (12) or (19). The quantity M_a , in turn, can be expressed through Eq. (17). Taking the difference between Eqs. (12) and (16) or between Eqs. (19) and (20) we get formulas of the form

$$\begin{aligned} \mathbf{h} - \mathbf{H} = & - \int_{\Omega} \rho(\mathbf{r}') d\tau' \\ & \times \int_{\Omega} \frac{\partial}{\partial z} (\bar{M} \{f(\mathbf{r}' - \mathbf{r}) - 1\}) \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} d\tau. \end{aligned}$$

\bar{M} can be treated as a constant, because in the region in which it is variable the second factor is zero by virtue of Eq. (17c). The integral then breaks up in two independent factors and gives

$$\mathbf{h} - \mathbf{H} = \bar{M} \int_{\Omega} \frac{\partial f(\mathbf{r})}{\partial z} \frac{\mathbf{r}}{r^3} d\tau.$$

For Coulomb forces, *f*(**r**) is expressible in parabolic coordinates, and the calculation can be made in closed form. The result involves the same quantity *p* = *f*(0) that was written down in

⁷ D. L. Webster, Am. J. Phys. **14**, 360 (1946).

⁸ Mott and Massey, *Theory of Atomic Collisions* (Oxford University Press, New York, 1933), p. 36.

(27). It reads

$$\mathbf{b} = \mathbf{B} + 2\pi\bar{\mathbf{M}}(p-1), \quad (28a)$$

or

$$\mathbf{b} = \mathbf{H} + 2\pi\bar{\mathbf{M}}(p+1). \quad (28b)$$

Strictly speaking, this equation is only found for the field transverse to the beam, while for the longitudinal field an equation

$$b_l = B_l + 4\pi\bar{M}_l(p-1)$$

results. There seems to be no physical application, however, in which this component could be observed.

It is interesting to notice that (28) gives for positive particles an average field which is larger than \mathbf{B} , while for negative particles it is smaller. This asymmetry gives some hope of observing the effect, which is, unfortunately, not in the range of easy observation (see next section).

Webster⁷ has called attention to a consequence of (28) for conduction electrons in a ferromagnet. For them p equals zero and we get

$$\mathbf{b} = \frac{1}{2}(\mathbf{B} + \mathbf{H}). \quad (29)$$

This formula has been inferred earlier by Swann.³

The first objection to this formula is that the crystalline field will certainly modify the chance of encounters at these low speeds. The second difficulty is the lack of an experimental definition of the type (1) which would make \mathbf{b} at least in principle a measurable quantity. However, Webster⁷ suggests some indirect approaches in his paper which may be of value in this situation.

5. MAGNETIC DEFLECTION AND MULTIPLE SCATTERING

Observations on magnetic deflection have to compete in practice with multiple scattering. It is therefore useful to study the two effects concurrently. The study is based on the following three equations:

(a) The range-energy formula⁹

$$-\frac{dE}{dx} = \frac{4\pi\epsilon^2 z^2 ZN}{mv^2} \ln\left(\frac{2mv^2}{I(1-\beta^2)e\beta^2}\right); \quad (30)$$

(b) The magnetic deflection formula

$$\frac{d\varphi}{dx} = \frac{zeb}{pc}; \quad (31)$$

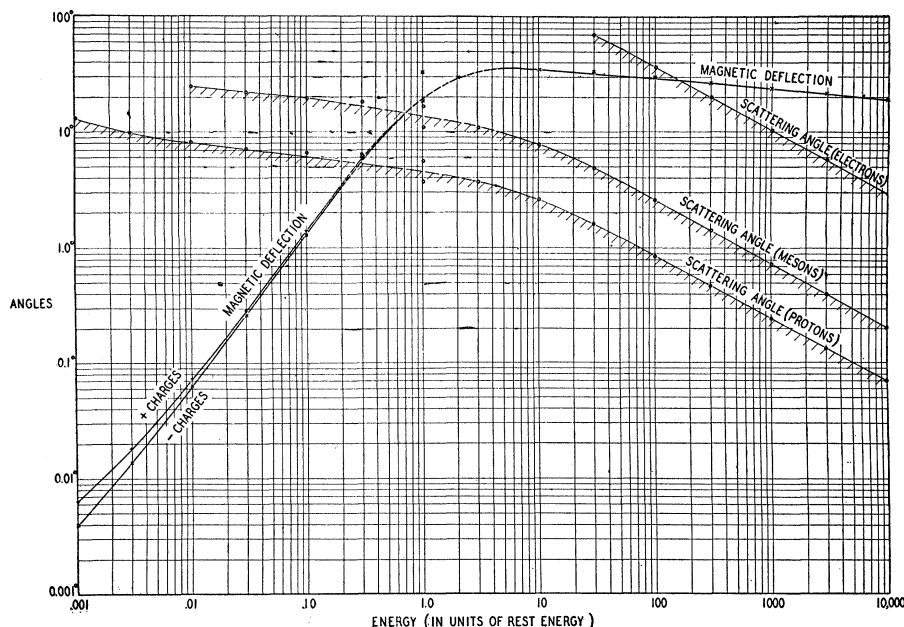


FIG. 2. Angles of magnetic deflection and scattering for various particles as a function of energy. Thickness is picked to maximize the ratio of the two quantities. Results near $E = Mc^2$ are rough interpolations only.

⁹ M. Livingston and H. A. Bethe, Rev. Mod. Phys. 9, 263 (1937), Eqs. (749) and (750).

(c) The multiple scattering formula¹⁰

$$\frac{d\langle\theta^2\rangle}{dx} = \frac{8\pi z^2 Z^2 \epsilon^4 N}{p^2 v^2} \ln(181Z^{-\frac{1}{2}}). \quad (32)$$

Here the symbols have the following meaning:

- e = basis of natural logarithms,
- ϵ = elementary charge,
- m = electronic mass,
- M = mass of beam particle,
- ze = charge of beam particle,
- E = energy of beam particle,
- φ = angle of magnetic deflection of beam particle,
- θ = root mean square angle of scattering of beam particles
- p = momentum of beam particle,
- v = velocity of beam particle,
- $\beta = v/c$,
- x = path length in foil material,
- Z = atomic number of foil material,
- b = effective magnetic field inside foil material,
- N = number of atoms per unit volume in foil material,
- I = mean ionization potential of foil material.

As a preliminary calculation, we can try to determine the thickness of the ferromagnet in such a way as to make the quantity φ/θ a maximum. The result is that the thickness must be chosen in such a way that in the non-relativistic region ($E \ll Mc^2$)

$$E/E_0 = 0.432,$$

and in the relativistic region ($E \gg Mc^2$)

$$E/E_0 = 0.203,$$

where E = energy of beam particle when leaving foil, E_0 = energy of beam particle when entering foil. A somewhat more complicated result is obtained in the neighborhood of $E = Mc^2$. The maximum of φ/θ is sufficiently flat to make a choice

$$E = \frac{1}{4}E_0 \quad (33)$$

reasonable. This choice was made for the numerical results of this section. It determines the thickness of the magnet to be used when the energy E_0 of the beam is given. The actual value of the thickness is obtained by integrating (30) and is easily available in the literature.

For the present purpose, it is preferable to eliminate x from the Eqs. (30), (31), and (32) and to get φ and θ as functions of the beam energy.

¹⁰ B. Rossi and K. Greisen, Rev. Mod. Phys. **13**, 263 (1941), Eq. (1.53a).

The integration can be carried out for the ranges $E \ll Mc^2$ and $E \gg Mc^2$. The results are for $E \ll Mc^2$

$$\varphi = \frac{Ib}{16\pi z Z \epsilon^2 N} \left(\frac{I}{2mc^2} \right)^{\frac{1}{2}} \times \left[li \left\{ \frac{2mv_0^2}{I} \right\}^{\frac{1}{2}} - li \left\{ \frac{2mv^2}{I} \right\}^{\frac{1}{2}} \right], \quad (34)$$

$$\theta^2 = Z \frac{m}{M} \ln(181Z^{-\frac{1}{2}}) \left[\ln \ln \frac{2mv_0^2}{I} - \ln \ln \frac{2mv^2}{I} \right]; \quad (35)$$

for $E \gg Mc^2$

$$\varphi = \frac{mc^2 b}{8\pi z Z N \epsilon^3} \left[\ln \ln \frac{2mE_0^2}{IM^2 c^2 e} - \ln \ln \frac{2mE^2}{IM^2 c^2 e} \right], \quad (36)$$

$$\theta^2 = Z \frac{m}{M} \left(\frac{2mc^2}{Ie} \right)^{\frac{1}{2}} \ln(181Z^{-\frac{1}{2}}) \left[li \left\{ \frac{IM^2 c^2 e}{2mE_0^2} \right\}^{\frac{1}{2}} - li \left\{ \frac{IM^2 c^2 e}{2mE^2} \right\}^{\frac{1}{2}} \right]. \quad (37)$$

Here lix is the logarithmic integral

$$lix = \int_0^x dx / \ln x$$

or, more properly, its real part.

The numerical evaluation of (34), (35), (36), and (37) is shown in Fig. 2 for the case of iron. On the abscissa axis is plotted the energy in units of Mc^2 ; this gives the same φ -curve for all particles of unit charge. A somewhat arbitrary connection is made between the regions $E \ll Mc^2$ and $E \gg Mc^2$; it is marked by dotted lines. The value of b was taken as large as possible, namely, 21,900 oersted. A reasonable averaging process¹¹ was used for I and gave

$$I/mc^2 = 0.000385.$$

Scattering curves are given for the electron, meson, and proton. The meson mass was taken as 200 electron masses. The proton gives best results; actually the deuteron is still more favorable, as it has the scattering diminished by a factor 0.7 compared to the proton. The Coulomb splitting discussed in Section 4 is shown for the φ -curve. It appears difficult to observe.

¹¹ The average is a logarithmic average of the type discussed by Livingston and Bethe (reference 9, p. 265).

In conclusion we can say that the magnetic deflection of a beam of charged particles in magnetized iron should be quantitatively observable if high energy protons or deuterons are used instead of mesons.¹ A modification of the generally accepted result

$$\mathbf{b} = \mathbf{B}$$

may be expected if there are short range forces modifying interpenetration of proton and electron. The Coulomb force alone will give such an effect, but it appears barely at the threshold of observation.

I wish to express my thanks at this point to Dr. John Eldridge, Dr. J. R. Dunning, and Dr. H. A. Bethe for valuable discussions.

Interpretation of the Triton Moment*

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In order to account for the measured magnetic moment of the triton it is necessary to assume that the wave function in the ground state is a linear combination of 2S , 2P , 4P , and 4D functions. An attempt is made to determine the amplitudes of these functions from the magnetic moment on the assumption that the intrinsic nucleon moments are additive and relativistic effects are negligible. With certain reasonable assumptions concerning the nature of the wave functions, it is found that the relative probabilities for finding the system in the 2P , 4P , and 4D states satisfy the relation shown by the curves in Fig. 1. Wherever the results would otherwise be arbitrary, the wave functions have been chosen in such a way as to minimize the amount of P state, with the exception that only the lowest one-particle configurations have been considered. If the amplitude of the 2S state is taken to be as large as possible, the wave function contains no 4D state, 8 percent 4P state, and 17 percent 2P state. A wave function of this form would seem to indicate that there is a spin-orbit coupling other than the tensor interaction acting among nuclear particles. In the other extreme case that the wave function contains a maximum of the 4D function, the 2S state probability is zero, the 4D probability is 22 percent, the 4P is 30 percent, and the 2P is 48 percent. If the wave function of He^3 has the same form as that of H^3 , the He^3 moment would be expected to lie on one of the curves shown in Fig. 2.

1. INTRODUCTION

THE recent measurements^{1,2} of the magnetic moment of the triton give a value about 6.7 percent greater than that of the proton. If the ground state of the triton were a pure 2S_3 state, it would be expected that the moment would be equal to the proton moment. It is believed, of course, that the ground state is not a pure 2S state but contains an admixture of 2P , 4P , and 4D states.³ A theory based on simplifying assumptions leads⁴ to the conclusion that

the presence of these states should result in a reduction of the moment instead of the observed increase. However, it has been pointed out⁵ that cross terms between the various states in the expression for the magnetic moment have been neglected in the simple theory. These may be positive and could, therefore, account for the large moment.

It is the purpose of this paper to obtain a general expression for the magnetic moment in terms of the amplitudes of the various wave functions and thereby to gain some information concerning the nature of the ground state wave

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¹H. L. Anderson and A. Novick, *Phys. Rev.* **71**, 372 (1947).

²F. Bloch, A. C. Graves, M. Packard, and R. W. Spence, *Phys. Rev.* **71**, 373 and 551 (1947).

³E. Gerjuoy and J. Schwinger, *Phys. Rev.* **61**, 138 (1942).

⁴R. G. Sachs and J. Schwinger, *Phys. Rev.* **70**, 41 (1946).

⁵R. G. Sachs, *Phys. Rev.* **71**, 457 (1947).