The Relativistic Clock Problem

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The theory of uniformly accelerated motions based on the conformal group of transformations in space-time is applied to the clock problem of relativity theory. Two solutions are found, both of which are at variance with the usual theory. The bearing of the problem on the relation between mechanics and electromagnetic theory is discussed briefly.

1. INTRODUCTION

ONE of the most interesting problems furnishing a link between the special and the general theory of relativity is the so-called *clock paradox*. This was first introduced into relativity theory by Einstein[1] and has since been discussed many times in the literature; for present purposes we may take the easily accessible treatment of Tolman[2] as standard. The interest in the problem centers around the fact that it leads to Einstein's important relation between gravitational potential and the rate of an ideal clock.

In the problem as ordinarily stated,[2] two identical clocks are initially in coincidence and at rest in the laboratory system, and it is assumed that their readings have been adjusted and their rates synchronized. One of the clocks, B, is then given a strong acceleration for a short time, which brings it quickly up to a velocity u with respect to clock A, while the latter remains at rest in the laboratory system. Clock B continues to move with this velocity for a long time, after which it is given a further strong acceleration which reverses its motion and returns it towards A with the velocity $-u$. Just before reaching the latter, B is decelerated in such a manner as to bring it into coincidence and relative rest with respect to A. The readings of the two clocks are then compared. According to the usual theory, the elapsed time intervals, as measured by the two clocks, are related by the equation

$$\Delta t_A = \Delta t_B(1+u^2/2c^2), \tag{1}$$

to terms in $(u/c)^2$. The elapsed time, as read by the moving clock is thus *less* than that recorded by the stationary clock. The apparent paradox arising from taking the local-system of clock B as a standard of rest, in which A is then supposed to perform the contrary motion to that sketched above, is explained as an effect equivalent to that of the gravitational field which can be used to provide the apparent acceleration of clock A in the local-system of B.

A variant of the problem, in which the clocks are not required to be at relative rest when in coincidence, has been discussed by Møller.[3] Apart from the clock problem itself, for which he appears to concur in the result of the usual theory, Møller takes the occasion to discuss the problem of determining transformations between reference systems in uniform acceleration and, in fact, for even more general motions. We shall return to this aspect of Møller's discussion later in this paper.

Despite the fact that in its kinematical aspects the problem appears to be entirely symmetrical between the two clocks, Einstein's theory introduces a dissimilarity between them by the assumption that one of them is in an inertial system; the local-system of the other clock then cannot be an inertial system. However, the persistent difficulty is that there is no *a priori* way of making a decision as to which, if either, of the two clocks is in an inertial system. Its resolution in the relativistic argument appears to be quite as arbitrary as is the corresponding assumption in Newtonian mechanics.

It has been shown in previous work[4] that one can introduce uniformly accelerated reference systems into relativity theory by employing the conformal group of transformations C_4 in space-

[1] A. Einstein, Ann. d. Physik [4] 35, **898** (1911).
[2] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, New York, 1934), p. 192.
[3] C. Møller, Det. Kgl. Danske Videnskabernes Selskab, Matematisk-Fysiske Meddelelser, Bind XX, Nr. 19 (1943).
[4] E. L. Hill, Phys. Rev. **67**, 358 (1945); **72**, 143 (1947).

time, as a generalization of the Lorentz group L_4, which establishes transformations between systems in uniform relative motion. It is the purpose in the present paper to discuss the application of this theory to the clock problem; in this way we shall be able to lift one of the deficiencies of the current theory by establishing explicitly the transformations between the local-systems of the two clocks and so can make the kinematical treatment symmetrical between them. However, as the theory is kinematical in character, and makes no direct appeal to mechanical principles, we can draw no particular conclusions concerning the *causes* of the apparent motions of the clocks, nor can we infer any relationship to any particular type of force field such as gravitation.[5]

Our treatment will be limited to the case in which either of the clocks appears to perform a linear uniformly accelerated motion in the local-system of the other. If the clocks have *two* coincidences, they are then not at relative rest at either coincidence. An important point in the analysis must then be the method of establishing a comparison of the natural rates of the two clocks. We shall require that the rate at which either of the clocks appears to run in the local-system of the other, at the instant at which it comes to rest in that system, agrees with the rates of other neighboring clocks which are permanently at rest and synchronized in that system. The possibility of establishing extended covering times in the local-systems of *both* clocks, despite the inherent uncertainty as to whether either system is inertial in a mechanical sense, is founded on an appeal to the known form invariance of the electromagnetic field equations under C_4; if such a covering time can be established in *either* system by purely optical means, it can be established in *both*.

2. THE SUBGROUP OF THE CLOCK PROBLEM

Our first task is to abstract from the full group, C_4, that subgroup which describes the type of motion appropriate to our version of the problem. This can be built up from a translation, a uniform velocity, and a uniform acceleration along the x-axis. A study of the commutator table[4] of C_4 shows that we must adjoin three further transformations, representing (a) a translation of the time origin, (b) a dilatation in space-time, and (c) a "red-shift" transformation.[6] This gives us, in total, a 6-parameter subgroup of C_4. However, the two translations can be eliminated at once by taking as our fundamental particle representing a clock, that one which corresponds to the origin of coordinates in space-time; the moving clock will then cross the origin of coordinates at the zero of local-time, moving along the x-axis, and the clocks will be at the origins of space-coordinates in their respective local-systems. By including translations we should merely shift to other points representing the clocks.

The general transformation of this type has the subsidiary differential equations[7]

$$\begin{aligned}
dx/d\mu &= -\tau\rho_1 - \tfrac{1}{2}(\tau^2+x^2-y^2-z^2)\rho_2 - x\tau\rho_3 + x,\\
dy/d\mu &= y(-x\rho_2-\tau\rho_3+1),\\
dz/d\mu &= z(-x\rho_2-\tau\rho_3+1),\\
d\tau/d\mu &= -x\rho_1 - x\tau\rho_2 - \tfrac{1}{2}(\tau^2+x^2+y^2+z^2)\rho_3 + \tau.
\end{aligned} \tag{2}$$

For notation, we let (x_0, y_0, z_0, and τ_0) be the space-time coordinates in the local-system S_B of clock B, while (x, y, z, and τ) are those for the local-system S_A of A. We have now to establish the relationship between these sets of coordinates by integration of Eqs. (2), with adjustment of the constants to suit the conditions of the prescribed motion.

We start by considering only those points which move along the common x-axis of the two systems. On setting $y=z=0$ in Eqs. (2) we obtain

$$\begin{aligned}
d(\tau+x)/d\mu &= (\tau+x)(1-\rho_1) - \tfrac{1}{2}(\tau+x)^2(\rho_2+\rho_3),\\
d(\tau-x)/d\mu &= (\tau-x)(1+\rho_1) + \tfrac{1}{2}(\tau-x)^2(\rho_2-\rho_3),
\end{aligned}$$

[5] Even should one bring in considerations concerning an external force field, it does not appear mandatory that it be interpreted as a gravitational field; the invariance of the Maxwell-Lorentz field equations suggests a formal interpretation in terms of electromagnetic fields.

[6] E. L. Hill, Phys. Rev. **68**, 232L (1945).

[7] For comparison with the notation of reference 4, we have the correspondence

$$\rho_1\mu\rightarrow\alpha_8/c, \quad \rho_2\mu\rightarrow\alpha_{11}/c^2, \quad \rho_3\mu\rightarrow\alpha_{14}/c, \quad \mu\rightarrow\alpha_{15}.$$

The theory of the differential equations involved in the integration of a continuous group can be found in the books by J. E. Campbell, *Theory of Continuous Groups* (Oxford University Press, New York, 1903), p. 47; L. P. Eisenhart, *Continuous Groups of Transformations* (Princeton University Press, Princeton, New Jersey, 1933), Chapter 1.

and by integration we obtain the transformation equations for this class of points in the form

$$y=z=0,$$

$$\tau+x=\frac{(1-\rho_1)(\tau_0+x_0)}{(1-\rho_1)e^{-(1-\rho_1)\mu}+\frac{1}{2}(\rho_2+\rho_3)(\tau_0+x_0)[1-e^{-(1-\rho_1)\mu}]},$$

$$\tau-x=\frac{(1+\rho_1)(\tau_0-x_0)}{(1+\rho_1)e^{-(1+\rho_1)\mu}-\frac{1}{2}(\rho_2-\rho_3)(\tau_0-x_0)[1-e^{-(1+\rho_1)\mu}]}.$$

To find the trajectory of the point representing the moving clock, B, in the local-system of clock A, we set $x_0=0$ and eliminate τ_0. This yields the equation

$$(x+\omega^{-1}e^{\mu}\cosh\rho_1\mu)^2-(\tau-\omega^{-1}e^{\mu}\sinh\rho_1\mu)^2=(e^{\mu}/\omega)^2,$$

where

$$\omega=\frac{\rho_2-\rho_3}{2(1+\rho_1)}[1-e^{(1+\rho_1)\mu}]+\frac{\rho_2+\rho_3}{2(1-\rho_1)}[1-e^{(1-\rho_1)\mu}].$$

In order to synchronize the clocks we now require that for $x_0=0$

$$(d\tau_0/d\tau)=1 \quad \text{at} \quad (x,\tau)=(x_*,\tau_*),$$

where x_* and τ_* are, respectively, the point and the instant at which the moving clock, B, comes to rest in the local-system of clock A. They are given by the relations

$$x_*=\omega^{-1}e^{\mu}(1-\cosh\rho_1\mu), \quad \tau_*=\omega^{-1}e^{\mu}\sinh\rho_1\mu.$$

On writing out this condition in full, we find the relation

$$(\rho_2-\rho_3)(1-\rho_1)[-\sigma+\sigma e^{(1+\rho_1)\mu}+e^{\frac{1}{2}(1-\rho_1)\mu}-e^{\frac{1}{2}(3+\rho_1)\mu}]+(\rho_2+\rho_3)(1+\rho_1)[-\sigma+\sigma e^{(1-\rho_1)\mu}+e^{\frac{1}{2}(1+\rho_1)\mu}-e^{\frac{1}{2}(3-\rho_1)\mu}]=0$$

with $\sigma=\pm1$.

Now when ρ_1 is considered to have an indeterminate numerical value, the exponentials in this expression are linearly independent functions of μ, so that this condition can be satisfied only by assuming

$$(\rho_2-\rho_3)(1-\rho_1)=0$$

and

$$(\rho_2+\rho_3)(1+\rho_1)=0. \tag{3}$$

These relations yield just two solutions, which we designate as

Case a: $\rho_1=+1, \quad \rho_2=-\rho_3=\rho_a,$

Case b: $\rho_1=-1, \quad \rho_2=+\rho_3=\rho_b.$

In the next two sections we shall examine the solutions for these cases separately.

3. SOLUTION FOR CASE a.

The differential Eqs. (2) reduce to the form

$$\begin{aligned} d(\tau+x)&=\rho_a(y^2+z^2)d\mu,\\ d(\tau-x)&=(\tau-x)[2+\rho_a(\tau-x)]d\mu,\\ dy&=y[1+\rho_a(\tau-x)]d\mu,\\ dz&=z[1+\rho_a(\tau-x)]d\mu. \end{aligned} \tag{4}$$

By integration we find the transformation equations between the local-systems S_A and S_B of the two clocks to be

$$\begin{aligned} \tau+x&=(\tau_0+x_0)+\lambda_a\rho_a(y_0^2+z_0^2)\sinh\mu,\\ \tau-x&=\lambda_a e^{\mu}(\tau_0-x_0),\\ y&=\lambda_a y_0, \quad z=\lambda_a z_0 \end{aligned} \tag{5}$$

with

$$\lambda_a=1/[e^{-\mu}-\rho_a(\tau_0-x_0)\sinh\mu].$$

The trajectory of the moving clock, B, in S_A is found by setting $x_0=y_0=z_0=0$ and eliminating τ_0. This gives us

$$(x-\rho_a^{-1}\coth\mu)^2-(\tau+\rho_a^{-1})^2=(1/\rho_a\sinh\mu)^2. \tag{6}$$

In the local-system of clock A, the moving clock, B, crosses the origin at time $\tau=0$, with a velocity $\beta_1=-\tanh\mu$, proceeds to the point $x_*=\rho_a^{-1}\tanh(\mu/2)$ on the x-axis, which it reaches at time $\tau_*=-1/\rho_a$, coming to rest at this point. On reversing its motion it again reaches the origin, at which clock A is fixed, at time $2\tau_*$, after which it continues its motion indefinitely along the x-axis, its speed approaching the value c asymptotically. By adjustment of the parameters one can cause the ultimate motion to take place in either direction. The whole motion is symmetrical between the two clocks, since the transformation of Eqs. (5) is inverted by interchanging the coordinates and changing the sign of the group parameter μ.

The relation between the local-times of the two clocks is obtained directly from Eqs. (5) on setting $x_0=y_0=z_0=0$, which yields

$$\tau_0=\tau+x. \qquad (7)$$

We see from this that $\tau_0=\tau$ for $x=0$, so that the readings of the two clock agree at *both* coincidences, in contrast with the result expressed in Eq. (1).

It follows also from Eq. (7) that $d\tau_0/d\tau$ is greater than or less than unity, according as $dx/d\tau$ is greater than, or less than, zero. This relation is peculiar in that it does not depend on the algebraic signs of ρ_a and of μ. There appears to be an absolute distinction between left and right directions along the x-axis; this may also be interpreted as an absolute distinction between past and future which is conditioned by the imposed temporal ordering of the coincidences as *first* and *second.*

It is of interest to note that we can arrive at a transformation corresponding to *uniform* relative motion of the two systems if we take the limiting case $\rho_a\rightarrow 0$, with $\beta_1=-\tanh\mu=\text{const.}$, with which we find

$$\begin{aligned} x&=(x_0+\beta_1\tau_0)/(1+\beta_1),\\ \tau&=(\tau_0+\beta_1 x_0)/(1+\beta_1),\\ \begin{Bmatrix} y \\ z \end{Bmatrix}&=\left[\frac{1-\beta_1}{1+\beta_1}\right]^{\frac{1}{2}}\begin{Bmatrix} y_0 \\ z_0 \end{Bmatrix}. \end{aligned} \qquad (8)$$

The transformation of Eqs. (8) is formed by a homogeneous Lorentz transformation with the velocity parameter, β_1, followed by a dilatational transformation of all coordinates with a scale factor $[(1-\beta_1)/(1+\beta_1)]^{\frac{1}{2}}$. The complete group of transformations of this type is obtained by adjoining dilatations to the inhomogeneous Lorentz group, giving rise to an 11-parameter subgroup of C_4. From the point of view of physical interpretation this result throws new light on the meaning of the usual simple Lorentz transformation, which provides no mechanism for the synchronization of clocks in uniform relative motion.

We observe that our transformation does not satisfy the *a priori* requirement imposed by Møller[3] that it be of the form

$$x=f(x_0,\tau_0),\quad y=y_0,\quad z=z_0,\quad \tau=h(x_0,\tau_0).$$

There appears to be no reason to believe that transformations of this functional form have any wider validity than the Lorentz group.

4. SOLUTION FOR CASE b

It will not be necessary to give the calculational details for this case, since the work is similar to that of Section 3. The transformation equations are

$$\begin{aligned} \tau+x&=\lambda_b e^{\mu}(\tau_0+x_0),\\ \tau-x&=(\tau_0-x_0)-\lambda_0\rho_b(y_0^2+z_0^2)\sinh\mu, \qquad (9)\\ y&=\lambda_b y_0,\quad z=\lambda_b z_0, \end{aligned}$$

with

$$\lambda_b=1/[e^{-\mu}+\rho_b(\tau_0+x_0)\sinh\mu].$$

This transformation can be obtained from that of Eqs. (5) by either of the formal substitutions

(a) $(x, y, z, \tau)\rightarrow(-x, y, z, \tau),\quad \rho_a\rightarrow-\rho_b,\quad \mu\rightarrow\mu,$

(b) $(x, y, z, \tau)\rightarrow(x, y, z, -\tau),\quad \rho_a\rightarrow+\rho_b,\quad \mu\rightarrow\mu.$

The remainder of the anaylsis proceeds exactly as in Section 3. The two clocks again appear to register the same time interval between coincidences. The moving clock appears to gain when it moves to the left and to lose when it moves to the right, which is just the reverse of the behavior found for case a.

5. DISCUSSION

The existence of two solutions of the clock problem shows that the conceptions of classical kinematical theory are not capable of characterizing the problem uniquely. Mathematically this arises from the circumstance that C_4 is a 15-parameter group, while the corresponding group of classical kinematical theory has but 13 parameters.[4] In order to see the nature of the influence of the enlarged character of C_4, let us examine the behavior of the particles which move along the x-axis. We find the formula for the trajectory of such a particle from Eqs. (5) and (9) on setting $y_0=z_0=0$ and eliminating τ_0. We obtain equations of the form

$$(x-x_0-\rho^{-1}\coth\mu)^2-(\tau\mp x_0\pm\rho^{-1})^2=(1/\rho\sinh\mu)^2.$$

For the apparent velocities of these particles we have

$$dx/d\tau = \pm(\tau \mp x_0 \pm \rho^{-1})/[(\tau \mp x_0 \pm \rho^{-1})^2 + (\rho \sinh\mu)^{-2}]^{\frac{1}{2}}.$$

If we now consider those particles at great distances in either direction along the axis, for any fixed time τ, we find that $|dx/d\tau| \rightarrow 1$ as $|x_0| \rightarrow \pm\infty$. In this way we are brought quite directly to a kinematical connection between the clock problem and the "expanding universe." This is, in fact, simply another apsect of the relationship between C_4 and cosmological theory, which has been discussed by Robertson[8] and by Infeld and Schild.[9]

To return to the comparison of the present analysis with the current relativistic theory, we observe that the divergence between the two mathematical procedures rests on the interpretation assigned to the "line-element" associated with the group C_4. The group is, in fact, characterized by a differential form of the type[4]

$$\lambda^2(x, y, z, \tau)(d\tau^2 - dx^2 - dy^2 - dz^2). \qquad (10)$$

In the relativistic theory an arbitrary assignment is made of the "inertial" system in which $\lambda = 1$. In the present theory we refrain from making a unique assignment of line-element to the local-systems of the clocks; our discussion leaves the line-element of any particular coordinate system indefinite to the extent indicated by the form (10). We are concerned only with relations *between* coordinate systems, but not with the absolute specification of either system. From the mathematical point of view the present procedure is more closely related to Weyl's theory[10] than to that of Einstein.

From the point of view of physical interpretation, this bifurcation is just that existing between mechanical and electrical theories, and it seems to the writer that by the apposition of the two procedures in the clock problem we are enabled to see the basic divergence between the two types of theory in a particularly elementary, but fundamental, light. The equations of mechanics are form invariant under transformations to systems moving with constant relative velocity, but the electromagnetic equations are insensitive to transformations involving uniform accelerations. The failure of the electromagnetic theory of mass seems to be a real measure of the incompatibility between the two theories. We seem here to be close to the root mathematical difficulties confronting the attempt to correlate quantum mechanics and the theory of relativity on any wider basis than that provided by the Lorentz group.

[8] H. P. Robertson, Phys. Rev. **49**, 755 (1936).

[9] L. Infeld and A. Schild, Phys. Rev. **68**, 250 (1945); *ibid.*, **70**, 410 (1946).

[10] H. Weyl, *Space-Time-Matter* (Methuen Press, London, 1922) Section 35.