PHYSICAL REVIEW VOLUME 72, NUMBER 3 AUGUST 1, 1947

The Physical Significance of Birkhoff's Gravitational Equations

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(Received December 11, 1946)

Birkhoff's gravitational equations are put in terms of dt in place of the local time ds used by him. The transformed equations show that Lorentzian mass has been used, and to the Newtonian attractive force is added a force normal to the direction of motion, v^2/c^2 times the component of the gravitational force normal to the motion.

1. INTRODUCTION

THE theory of gravitation proposed by the late G. D. Birkhoff was developed by him from a very general mathematical standpoint, and the physical significance of the theory is difficult to grasp in the form in which it has been presented. As indicated by the title of his most detailed presentation, "El Concepto Matematico de Tiempo y la Gravitacion,"[1] Birkhoff laid stress on the use of ds (local time) in place of dt (Newtonian time); the "forces" figuring in his development are Minkowski forces instead of physical (Lorentzian) forces. Yet his solution for a planetary orbit (which gives the correct advance of perihelion of Mercury) contains no terms in ds, raising the question whether this result does in fact depend on his adherence to this variable. Actuated by the desire to see what form his "force" equations would take if expressed in terms of dt and ordinary physical force (time rate of change of momentum), I have carried through the necessary transformations, with the results given below which exhibit some features of interest.

2. THEORY

Birkhoff's gravitational force, which is the third term of an expansion which he states[2] "it is natural to set" as typifying "all force vectors," gives, in his hands, for the motion of a particle moving in a plane in the gravitational field of a *mass M*,

$$\frac{d^2x}{ds^2}=-\frac{GMx}{r^3}-2\frac{GMx}{r^3c^2}\left[\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right]+\frac{GM}{r^2c^2}\frac{dx}{ds}\cdot\frac{dr}{ds}, \quad (1)$$

$$\frac{d^2y}{ds^2}=-\frac{GMy}{r^3}-2\frac{GMy}{r^3c^2}\left[\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right]+\frac{GM}{r^2c^2}\frac{dy}{ds}\cdot\frac{dr}{ds}, \quad (2)$$

It is these equations that we wish to put in terms of dt. To transform the right-hand side we use the relations

$$ds=dt(1-v^2/c^2)^{\frac{1}{2}},\quad v^2=\dot{x}^2+\dot{y}^2,$$

where the dots indicate differentiation with respect to t. We note that the left-hand side is the Minkowski force,

$$F_L(1-v^2/c^2)^{-\frac{1}{2}}$$

in which F_L is the Lorentzian force, that is, the time variation of momentum where mass varies with velocity according to the relation

$$m=m_0(1-v^2/c^2)^{-\frac{1}{2}}.$$

We then have

$$(F_L)_x=[-(GMm_0x/r^3)(1+v^2/c^2)+(GMm_0\dot{x}\dot{r}/r^2c^2)](1-v^2/c^2)^{-\frac{1}{2}}, \quad (3)$$

$$(F_L)_y=[-(GMm_0y/r^3)(1+v^2/c^2)+(GMm_0\dot{y}\dot{r}/r^2c^2)](1-v^2/c^2)^{-\frac{1}{2}}. \quad (4)$$

Putting $r\dot{r}=x\dot{x}+y\dot{y}$ we get

$$(F_L)_x=-\frac{GMm_0}{r^2}\left[\frac{x}{r}+\dot{y}\frac{(\dot{y}x-\dot{x}y)}{rc^2}\right](1-v^2/c^2)^{-\frac{1}{2}}, \quad (5)$$

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[1] G. D. Birkhoff, Boletin de la Sociedad Matematica Mexicana 1, 1 (1944).

[2] A. Barajas, G. D. Birkhoff, C. Graef and M. S. Vallarta, Phys. Rev. 66, 142 (1944).

$$(F_L)_y = -\frac{GMm_0}{r^2}\left[\frac{y}{r} - \dot{x}\frac{(\dot{y}x-\dot{x}y)}{rc^2}\right](1-v^2/c^2)^{-\frac{1}{2}}. \quad (6)$$

Comparing these with the Newtonian forces

$$F_x = -GMm_0x/r^3, \quad (7)$$

$$F_y = -GMm_0y/r^3, \quad (8)$$

it appears that the Birkhoff equations differ in the use of the Lorentzian mass, and the addition of terms in $(\dot{y}x-\dot{x}y)$.

It is instructive to put these equations in polar coordinates through the relations

$$\dot{y}x-\dot{x}y = r^2\dot{\theta}$$

and

$$F_R = (F_xx+F_yy)/r, \quad F_\theta = (-F_xy+F_yx)/r$$

used on the right-hand side, using on the left the statement of the Lorentzian force as given by Eddington,[3] yielding

$$m_0\left\{(\ddot{r}-r\dot{\theta}^2)[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}} + \dot{r}\frac{d}{dt}[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}\right\}$$
$$= -(GMm_0/r^2)(1+r^2\dot{\theta}^2/c^2) \times[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}, \quad (9)$$

$$m_0\left\{(2\dot{r}\dot{\theta}+r\ddot{\theta})[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}} + r\dot{\theta}\frac{d}{dt}[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}\right\}$$
$$= (GMm_0/r^2)(\dot{r}r\dot{\theta}/c^2) \times[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}. \quad (10)$$

These equations are solved by multiplying (9) by $\dot{r}$, and (10) by $r\dot{\theta}$, and adding, which gives

$$\frac{\frac{d}{dt}[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]}{[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]} = \frac{2GM\dot{r}}{r^2c^2} \quad (11)$$

from which

$$[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2] = \exp[-(2GM/rc^2)+k]. \quad (12)$$

[3] A. S. Eddington, Phil. Mag. [6] **34**, 321 (1917).

Using this result in (10) gives the areal constant

$$r^2\dot{\theta} = h\exp[-(2GM/rc^2)+k], \quad (13)$$

leading to the solution, replacing r by $1/u$

$$\frac{d^2u}{d\theta^2}+u = (c^2/h^2)[\exp(k+2GMu/c^2)-1] \times\exp(k+2GMu/c^2) \quad (14)$$

which is identically Birkhoff's expression, from which the advance of perihelion of Mercury is correctly indicated.

Equations (9) and (10) may be put in simpler, although less instructive form, by inserting the value of $[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{\frac{1}{2}}$ from (12), giving

$$\ddot{r}-r\dot{\theta}^2 = -\frac{GM}{r^2}\left[1-\frac{\dot{r}^2-r^2\dot{\theta}^2}{c^2}\right], \quad (15)$$

$$d(r^2\dot{\theta})/dt = 2GM\dot{r}\dot{\theta}/c^2. \quad (16)$$

These are identically the expressions given by Fernández[4] who also puts his equations in terms of dt instead of ds, in discussing the problem of two bodies in Birkhoff's theory.

3. DISCUSSION

Examination of (9) and (10) shows that Birkhoff's equations for a planetary orbit are the equations which one would obtain by using Lorentzian masses throughout in place of the invariant masses of the simple Newtonian theory, *with the addition* of terms in $\dot{r}$, $r\dot{\theta}^2$, and c^2 on the gravitational side of the equations.

First consider the appearance of the Lorentzian masses. Birkhoff in his presentation lays great stress on the use of *local time*. The idea that a planet is controlled in its course by the time indicated on a clock carried on it lacks substantiality. Physically, forces act on masses. Actually the factor by which local time is distinguished from absolute time is the "contraction factor," $(1-v^2/c^2)^{\frac{1}{2}}$. Now this is also the factor, appropriately placed, by which *local mass* is distinguished from stationary mass. Our transformed equations are force equations in terms of Lorentzian local mass. Hence although Birkhoff

[4] C. G. Fernández, Boletin de la Sociedad Matematica Mexicana **1**, 25 (1944), especially p. 36.

talks of local time as the significant factor, he is really stressing the importance of the contraction factor,[5] and his development could just as well, but with real physical significance, be founded on local mass.

Next consider the added terms on the right in Eqs. (5) and (6). Their characteristic is that when multiplied respectively by $\dot{x}$ and $\dot{y}$ and added they cancel exactly, leaving us with the simple Lorentzian Eq. (11) which may be written

$$\frac{1}{2}m_0\frac{\frac{d}{dt}(1-v^2/c^2)}{(1-v^2/c^2)^{\frac{3}{2}}}+\frac{GMm_0}{r^2c^2(1-v^2/c^2)^{\frac{1}{2}}}=0.$$

This equation states that the sum of the derivatives of kinetic and potential energies is zero which may be accepted as a fundamental requirement. An infinite number of such additional terms could be set down, which cancelling out, would leave the energy equation unchanged. Any term of this sort has, however, the result of changing the value of $r^2\dot{\theta}$, which is obtained by integration of the second of the equations in polar coordinates (10), alone. The *whole difference* between the Birkhoff solution (14) and the simple Lorentzian solution comes about from the value of $r^2\dot{\theta}$, the "areal constant," which occurs (squared) in the denominator of (14), the numerator having always the same fixed value determined by (12). The terms added by Birkhoff in his force equations change this from

$$r^2\dot{\theta}=h\exp[\tfrac{1}{2}k-GM/rc^2],$$

the Lorentzian value, to

$$r^2\dot{\theta}=h\exp[k-2GM/rc^2],$$

thereby changing the predicted advance of perihelion from $\frac{1}{3}$, the value obtained from the Newton-Lorentz solution, to the full observed value. The prediction of these particular added terms is, in the light of this analysis, the key contribution of Birkhoff's theory.

Let us study these added terms more in detail. They are completely described by the following

[5] The contraction factors are the coefficients of mass, length, and clock rate, which are demanded to insure the conservation of energy and momentum in radiation-matter interactions. Cf. H. E. Ives, Phil. Mag. [7] **36**, 392 (1945).

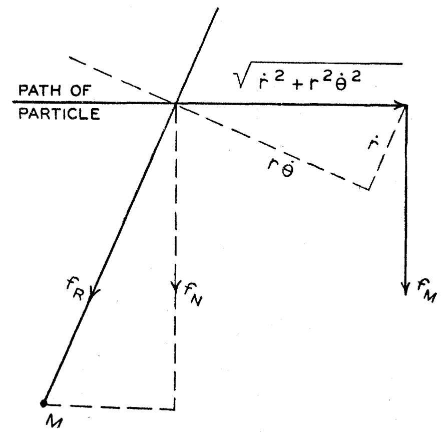

FIG. 1. Resolution of the force on a particle.

statement: *A particle in motion in a gravitational field experiences an additional force, normal to the direction of motion, of value v^2/c^2 times the component of the gravitational force normal to the direction of motion.*

The proof of this theorem is exhibited in the accompanying Fig. 1, in which

f_N = gravitational force normal to direction of motion—

$$f_R r\dot{\theta}(\dot{r}^2+r^2\dot{\theta}^2)^{-\frac{1}{2}};$$

f_M = force due to motion, normal to direction of motion—

$$f_N v^2/c^2=f_R(r\dot{\theta}/c^2)(\dot{r}^2+r^2\dot{\theta}^2)^{\frac{1}{2}}.$$

The component of added force in the radial direction is

$$f_M r\dot{\theta}(\dot{r}^2+r^2\dot{\theta}^2)^{-\frac{1}{2}}=f_R r^2\dot{\theta}^2/c^2,$$

and the component of added force in the direction normal to the radius is

$$-f_M\dot{r}(\dot{r}^2+r^2\dot{\theta}^2)^{-\frac{1}{2}}=-f_R\dot{r}r\dot{\theta}/c^2.$$

Putting $-(GMm_0/r^2)[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}$ for f_R we get the added terms in (9) and (10).

Denoting the Lorentzian forces (time variation of momentum, mass varying with velocity) by F_L, and the Newton-Lorentz gravitational force $-(GMm_0/r^2)[1-(\dot{r}^2+r^2\dot{\theta}^2)/c^2]^{-\frac{1}{2}}$ by G_L, the Birkhoff gravitational equations for a planetary orbit may be concisely expressed as follows

$$(F_L)_R=[G_L+(G_L)_N v^2/c^2]_R, \qquad (17)$$

$$(F_L)_\theta=[G_L+(G_L)_N v^2/c^2]_\theta, \qquad (18)$$

where the subscript N denotes components normal to the path of the particle, and the subscripts R and θ denote components in the radial direction and perpendicular to the radius, respectively.

As to the origin or nature of this transverse force produced by motion, it is of interest to observe that it is similar to the "fundamental law" proposed in a posthumous note by Gauss[6] for the mutual action of two elements of electricity in relative motion. The occurrence of c^2 stems from the idea of the attraction being transmitted with the speed of light.

The Birkhoff force equations for a planetary orbit can be summarized, according to this analysis, as follows: They are the equations one would obtain from a Newtonian attractive force acting on the Lorentzian local mass of the planet, with the addition of forces caused by motion, transverse to the path of the planet, which do not affect the conservation of energy but alter the areal constant. The precise value of these added forces and the method of obtaining them is thus of crucial importance. Apparently these forces are not introduced by Birkhoff with conscious resort to physical concepts, but they are present because of his choice for gravity of the third term[2] of a formal expansion in rational and integral components of a typical force function, in which successive terms are of increasing complexity and hence provide for additional force components. An independent physical derivation of these transverse forces would be welcome.

[6] C. F. Gauss, *Werke* (Göttingen, 1863–74), Vol. 5, p. 616.

PHYSICAL REVIEW VOLUME 72, NUMBER 3 AUGUST 1, 1947

Refraction of Plane Non-Uniform Electromagnetic Waves between Absorbing Media

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(Received August 6, 1946)

When a plane non-uniform electromagnetic wave is refracted between two conducting media, there are two possible positions for the propagation vector in the second medium. Consideration of the energy flow shows that each solution holds within a certain range of values of the (complex) angle of incidence, the transition from one to the other occurring in a discontinuous way. The two cases of the electric vector perpendicular and parallel to the plane of incidence are discussed.

(1) INTRODUCTION

THE problem of the refraction of a plane non-uniform electromagnetic wave at the plane boundary between two conducting media is not generally fully discussed in textbooks where it is pointed out that, with the use of complex angles of incidence and refraction and of complex propagation vectors, the problem is formally identical to the usual one in which perfect dielectrics are involved.[1]

It is the purpose of this paper to complete this treatment discussing the new physical features which appear when both media are conducting, in particular, a discontinuity occurring in the (complex) propagation vector in the second medium at the (complex) angle of incidence for which there is no average flow of energy across the boundary.

[1] See, for instance, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 500–524.

(2) THE PROPAGATION VECTOR IN THE SECOND MEDIUM

Let the boundary be the plane $y-z$, and the plane of incidence the plane $x-z$, the x-axis being directed from the first medium into the second. Let any field component be represented by

$$Ee^{-\mathbf{k}_i\cdot\mathbf{r}+i\omega t} \quad (1)$$

with

$$\mathbf{k}_1=\mathbf{a}+i\mathbf{b};\quad \mathbf{k}_2=\mathbf{A}+i\mathbf{B}, \quad (2)$$

E being a complex amplitude.