be superconducting at a lower temperature is not excluded. The data on samarium give an atomic volume of 19.4 and indicate a Debye temperature in excess of 130°K. Although the atomic volume, 19.4, would lie within the region on the atomic volume *versus* atomic number graph, because of the relatively large value of Debye temperature, greater than 130°K, it is not likely that samarium is a superconductor.

Protoactinium occupies a position in the periodic table indicating an atomic volume in the neighborhood of 16 and a Debye temperature of the order of 150°K. Protoactinium belongs to the electro-negative group for which the data in Fig. 1 is very sketchy. The best that can be inferred from the data available is that there is some likelihood that protoactinium is superconducting and will have a transition temperature within reach of helium cryostats.

It would be interesting to see if something could be done along these lines with the superconducting compounds and alloys.

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On the Kinematics of Uniformly Accelerated Motions and Classical Electromagnetic Theory

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In continuation of earlier work a study is made of the 4-dimensional conformal group of transformations in space-time as the extension of the Lorentz group permitting the introduction of uniformly accelerated reference frames into relativity theory. The problem of the motion of a particle is discussed, as well as the implications for the classical-type electron theory developed by Dirac.

1. INTRODUCTION

THE problem of extending the special theory of relativity to permit the introduction of euclidean systems of uniformly accelerated reference axes has been shown[1-4] to depend on the generalization from the group of inhomogeneous Lorentz transformations L_4 to the group of conformal transformations C_4 in space-time. This group is characterized by a line-element of the form[5]

$$ds^2=\lambda^2(d\tau^2-dx^2-dy^2-dz^2)=\lambda^2\eta_{ij}dx^idx^j, \quad (1)$$

in which the function λ is determinable from the group properties of C_4. The mathematical characterization of this group was first given by Lie[6] who showed that it consists of a 15-parameter family, within which L_4 forms a 10-parameter subgroup. In the earlier discussion by the writer,[4] the detailed proof of the association of C_4 with uniformly accelerated motions was established only for the one-dimensional case, that for motion in three dimensions being obtained by generalization only. In the present paper the complete solution of this problem will be given.

The interest for physical theory in this extension of the special theory of relativity rests on three main foundations. In the first place, it supplies a direct procedure for the study of uniformly accelerated motions, in a relativistic sense, by their reduction to analytical coordinate transformations. Secondly, it has been known for a number of years from the work of Cunningham[7] and of Bateman[8] that C_4 is the general symmetry group of point transformations of the Maxwell-Lorentz field equations. The association with the kinematical interpretation of uniformly accelerated motions provides a direct approach to the study of the radiations from accelerated charged

[1] L. Page, Phys. Rev. **49**, 254 (1936).
[2] H. P. Robertson, Phys. Rev. **49**, 755 (1936).
[3] H. T. Engstrom and M. Zorn, Phys. Rev. **49**, 701 (1936).
[4] E. L. Hill, Phys. Rev. **67**, 358 (1945).
[5] The notation here is $x^0=\tau=ct$, $x^1=x$, $x^2=y$, $x^3=z$, with $\eta_{ij}=0$ if $i\neq j$, $=+1$ if $i=j=0$, $=-1$ if $i=j=1, 2, 3$.
[6] S. Lie, *Theorie der Transformationsgruppen* (Teubner, Leipzig, 1930).
[7] E. Cunningham, Proc. Lond. Math. Soc. [2]**8**, 77 (1910).
[8] H. Bateman, Proc. Lond. Math. Soc. [2]**8**, 223 (1910).

particles.[9] Lastly, the fact that the group C_4 is larger than the corresponding group of classical kinematical theory, which depends on only 13 parameters,[4] leads one to certain model universes of interest in cosmological theory. This aspect of C_4, which was first discussed briefly by Robertson,[2] more recently has been noted independently by Infeld[10] and by the writer[11] and has been analyzed comprehensively by Infeld and Schild.[12]

It may be noted that although our discussion necessarily has much in common with those of Robertson and of Infeld and Schild, insofar as the properties of the group C_4 are in question, the motivation here is quite different. In the cosmological application attention is directed primarily to those subgroups of C_4 which satisfy the so-called principle of homogeneity. Our analysis of the general kinematical problem, as well as the applications to electrodynamics, require a treatment of the full group.

The theory developed here is based on the consideration that the *form-invariance* of physical equations under finite continuous groups of transformations such as the translation, rotation, Lorentz, and conformal groups may be of more fundamental significance than general *tensor covariance.* This is suggested by the fact that under the transformations of parameter groups there is no fundamental way in which one can distinguish one set of reference axes from another. So far as the present paper is concerned, this is recognized by the condition that although the line-element of Eq. (1) may be considered as an invariant of the group C_4, and although there exists a set of coordinates in which the function λ reduces identically to unity, we refrain from identifying this system with any particular physical set of coordinates which may arise in the discussion.

2. THE CHARACTERISTIC DIFFERENTIAL EQUATION OF UNIFORMLY ACCELERATED MOTION

The first studies of the uniformly accelerated motion of a particle in one dimension, in a relativistic sense, were made by Einstein, Minkowski, Born, and Sommerfeld; references to this work will be found in the books by Pauli[13] and von Laue.[14] In the applications of the conformal group to the electromagnetic field equations by Cunningham, Bateman, and Hassé, it was realized that there existed an association with accelerated motions on the basis of the form of the infinitesimal transformations of the group, but the full analysis of this association was lacking.

We consider as given *a priori* a "laboratory" system of euclidean axes S, in which a particle is observed to perform a type of motion which we wish to characterize as of uniformly accelerated type. By a homogeneous Lorentz transformation we can, at any instant, introduce a rest-system of coordinates such that in it the particle is instantaneously at rest, although in general it will still have an acceleration, as well as higher ordered derivatives of its motion. We now give the following:

Definition. *The motion of a particle will be considered to be uniformly accelerated if the time rate of change of its acceleration, as measured in an instantaneous rest-system, vanishes identically.*

To put this definition into effect, we consider a coordinate system S' which is obtained from the laboratory system by the general homogeneous Lorentz transformation[15]

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} - \boldsymbol{\beta}_0\{(\mathbf{r}\cdot\boldsymbol{\beta}_0/\beta_0^2)[1-(1-\beta_0^2)^{-\frac{1}{2}}] \\ &\qquad + \tau(1-\beta_0^2)^{-\frac{1}{2}}\}, \\ \tau' &= (\tau - \mathbf{r}\cdot\boldsymbol{\beta}_0)(1-\beta_0^2)^{-\frac{1}{2}}. \end{aligned} \quad (2)$$

We now consider a particle, for which the motion is observable in both S and S', and compute the transformation equations for its velocity, acceleration, and time rate of change of acceleration, employing the definitions

$$\boldsymbol{\beta} = d\mathbf{r}/d\tau, \quad \boldsymbol{\alpha} = d\boldsymbol{\beta}/d\tau, \quad \boldsymbol{\gamma} = d\boldsymbol{\alpha}/d\tau \quad (3)$$

with similar relations for S'. We shall need to write down explicitly only the transformation equation for the vector $\boldsymbol{\gamma}$.

For convenience in the later argument we express the vector $\boldsymbol{\gamma}'$ of the S'-system in terms of the variables of the laboratory system S,

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[9] Calculations in this direction, on the basis of specific models of the electron, were made by H. Hassé, Proc. Lond. Math. Soc. [2]**12**, 181 (1912).

[10] L. Infeld, Nature **156**, 114 (1945).

[11] E. L. Hill, Phys. Rev. **68**, 232 (1945).

[12] L. Infeld and A. Schild, Phys. Rev. **68**, 250 (1945); *ibid.* **70**, 410 (1946).

[13] W. Pauli, *Relativitätstheorie* (Teubner, Leipzig, 1921), sec. 26.

[14] M. von Laue, *La Théorie de la Relativité* (Gauthier-Villars, Paris, 1922), Tome 1, p. 166 *et seq.*

[15] E. Madelung, *Die Mathematischen Hilfsmittel des Physikers* (Springer, Berlin, 1936), p. 272.

obtaining

$$\begin{aligned}\boldsymbol{\gamma}' = {} & (1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0)^{-3}\{\boldsymbol{\gamma}(1-\beta_0^2)^{\frac{3}{2}} \\ & +\boldsymbol{\beta}_0[(\boldsymbol{\gamma}\cdot\boldsymbol{\beta}_0/\beta_0^2)(1-\beta_0^2)(1-(1-\beta_0^2)^{\frac{1}{2}})]\} \\ & +3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}_0)(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0)^{-4}\{\boldsymbol{\alpha}(1-\beta_0^2)^{\frac{3}{2}} \\ & +\boldsymbol{\beta}_0[(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}_0/\beta_0^2)(1-\beta_0^2)(1-(1-\beta_0^2)^{\frac{1}{2}})]\} \\ & +(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0)^{-5}\{(\boldsymbol{\gamma}\cdot\boldsymbol{\beta}_0)(1-\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0)+3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}_0)^2\} \\ & \times\{\boldsymbol{\beta}(1-\beta_0^2)^{\frac{3}{2}}-\boldsymbol{\beta}_0[1-(\boldsymbol{\beta}\cdot\boldsymbol{\beta}_0/\beta_0^2) \\ & \qquad\times(1-(1-\beta_0^2)^{\frac{1}{2}})](1-\beta_0^2)\}. \end{aligned} \tag{4}$$

If we now require that S' be a rest-system for the particle at the instant considered and apply our definition of uniformly accelerated motion, we have

$$\boldsymbol{\beta}'=0,\quad \boldsymbol{\beta}_0=\boldsymbol{\beta},\quad \boldsymbol{\gamma}'=0. \tag{5}$$

On reduction of Eq. (4) with the values in Eq. (5) we find the *characteristic differential equation of uniformly accelerated motion*[16]

$$\boldsymbol{\gamma}(1-\beta^2)+\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})=0. \tag{6}$$

The foundation of our argument now is that C_4 gives just the group of point transformations under which Eq. (6) is invariant. Since this is a purely mathematical question which the reader may be prepared to take for granted, the detailed proof is relegated to Appendix A. An alternative proof is provided by the discussion of section 3. The general integral of Eq. (6), which describes the class of uniformly accelerated motions, is given in Appendix B.

Our kinematical interpretation of C_4 arises from the fact that we can find within it a transformation which will introduce a co-moving or *local-system*[17] for a particle performing any motion governed by Eq. (6), which will be such that the particle will be permanently at rest in it. Conversely, starting with a particle at rest we can find a system in which it will appear to describe any motion of this type.

In order to accomplish this, we consider a coordinate system S_0, having space and time variables (x_0, y_0, z_0, τ_0), and think of each point of it as occupied by a particle, the whole family composing the points of S_0. Suppose now that S_0 is related to the laboratory system S by a transformation of C_4 such that

$$x=F_1(x_0, y_0, z_0, \tau_0),\quad \tau=G(x_0, y_0, z_0, \tau_0) \tag{7}$$

with similar relations for y and z.

If we now fix our attention on a particular particle P^* of S_0, having space coordinates (x_0^*, y_0^*, z_0^*), then the expression $\tau=G(x_0^*, y_0^*, z_0^*, \tau_0)$, gives the relation between the laboratory time τ and the *local-time* τ_0 of the particle. On solving this relation for τ_0 in the form $\tau_0=g(x_0^*, y_0^*, z_0^*, \tau)$ and substituting in the first three of Eqs. (7), we obtain the equation for the trajectory of the particle P^* in the laboratory system in the form

$$x(\tau)=F_1[x_0^*, y_0^*, z_0^*, g(x_0^*, y_0^*, z_0^*, \tau)], \tag{8}$$

with similar equations for $y(\tau)$ and $z(\tau)$.

In S_0 the "trajectory" of each particle is a degenerate form of uniformly accelerated motion, corresponding to constant values of its space variables; therefore in S each particle will appear to move on a trajectory of the family obeying Eq. (6). There is no loss of generality in picking out a particular one of the particles of S_0, say that one P_0^* at the origin of coordinates in S_0 for convenience, as a *fundamental particle*. By choosing various transformations from C_4 we can cause this particle to perform any of the possible types of uniformly accelerated motion in the laboratory system S.

It is clear that, just as for the Lorentz group,

[16] The statement in reference 4 that the characteristic differential equation for 3-dimensional motion is complex arose originally from a definition which was later shown to be inadequate, since it led to an equation which was not even invariant under the Lorentz group. The essential point in the present definition is that the time-derivative of the acceleration must be computed before S' is identified with a rest-system; in the 1-dimensional motion discussed in reference 4 this complication did not arise. Equation (6) of the present paper is a simple generalization of Eq. (6) of reference 4.

[17] In reference 4 the term *proper-system* was used to denote a system of space and time variables, which might be considered to belong to an "observer" thought of as permanently riding with a particle. Since this terminology comes into conflict with the established usage of the term *proper-time*, we have used here the designation *local-system*, which stems from the older term *local-time* used by Lorentz. The concept of such a local-system is frequently employed in qualitative discussions for general motions of a particle, but seldom with an attempt at defining it accurately in the sense of establishing the transformation equations between the laboratory system and the local-system. The fact that its accomplishment is possible for uniformly accelerated motions provides a keystone in our physical analysis. It is to be noted, however, that the essential point really is the establishment of local-systems for "observers"; the application of Lorentz transformations to the Dirac quantum mechanical equations for an electron shows that the interpretation of the observer as the actual physical particle is not essential to the development of a quantum mechanical theory. In the present paper we consider only the phases of the problem related to classical theory, in which we can speak of the "observer" as a physical particle.

the transformation of C_4 which introduces a local-system for a particle is not uniquely defined. This is of little consequence in the special theory of relativity in which all particles moving with a given velocity can be reduced to rest simultaneously by a single Lorentz transformation, but a similar situation does not hold for accelerated motions. The trajectories of a set of particles must be related as is indicated in Eq. (8) in order that they may *all* be reduced to permanent rest by a single transformation of C_4.

3. THE ABRAHAM 4-VECTOR

The kinematical considerations of section 2 can be put in an invariant form by the introduction of a 4-vector which was originally discovered by Abraham in connection with his theory of the radiation reaction of an accelerated charged particle; we shall refer to it as the *Abraham 4-vector*.

We write Eq. (1) in the form

$$ds^2=\lambda^2(d\varphi)^2=\lambda^2(1-\beta^2)d\tau^2, \tag{9}$$

and define the 4-vectors for the velocity, acceleration, and time rate of change of acceleration as usual by differentiation along the world-line of the particle, the formulas being[18]

$$\beta^i=dx^i/ds,\quad \alpha^i=d\beta^i/ds+\left\{\begin{matrix} i \\ jk \end{matrix}\right\}\beta^j\beta^k,$$
$$\gamma^i=d\alpha^i/ds+\left\{\begin{matrix} i \\ jk \end{matrix}\right\}\alpha^j\beta^k \tag{10}$$

The Abraham 4-vector is now defined as

$$\Gamma^i=\gamma^i+\beta^i(g_{jk}\alpha^j\alpha^k). \tag{11}$$

We now introduce the quantities

$$\bar{\beta}^i=dx^i/d\varphi,\quad \bar{\alpha}^i=d\bar{\beta}^i/d\varphi, \tag{12}$$

which do not form 4-vectors in the relativistic sense, but which may be considered as the *apparent* velocity and acceleration which would be defined for the particle if we neglected the appearance of the scaling factor λ in Eq. (9).

On introduction of Eqs. (9), (10), and (12) into Eq. (11) we obtain the result

$$\Gamma^i=\frac{1}{\lambda^3}\left[\frac{d^2\bar{\beta}^i}{d\varphi^2}+\bar{\beta}^i\left(\eta_{jk}\frac{d\bar{\beta}^j}{d\varphi}\frac{d\bar{\beta}^k}{d\varphi}\right)\right]+\frac{\bar{\beta}^i\bar{\beta}^m}{\lambda^3}\left[\frac{d}{d\varphi}\left(\frac{\partial\ln\lambda}{\partial x^m}\right)-\frac{d\ln\lambda}{d\varphi}\frac{\partial\ln\lambda}{\partial x^m}\right]-\frac{\eta^{in}}{\lambda^3}\left[\frac{d}{d\varphi}\left(\frac{\partial\ln\lambda}{\partial x^n}\right)+\frac{d\ln\lambda}{d\varphi}\frac{\partial\ln\lambda}{\partial x^n}\right]. \tag{13}$$

The general form of the function λ can be calculated, and is found to be[19]

$$\lambda=k/[1+2(\eta_{ij}a^ix^j)+(\eta_{mn}a^ma^n)(\eta_{rs}x^rx^s)], \tag{14}$$

in which the a's and k are arbitrary constants. When this is substituted into Eq. (13) it is found that the second and third members on the right hand side cancel identically, so that we get finally

$$\Gamma^i=\frac{1}{\lambda^3}\left[\frac{d^2\bar{\beta}^i}{d\varphi^2}+\bar{\beta}^i\left(\eta_{jk}\frac{d\bar{\beta}^j}{d\varphi}\frac{d\bar{\beta}^k}{d\varphi}\right)\right]. \tag{15}$$

On lowering the index and separating into space and time parts we find the expressions[20]

$$\Gamma_0=(\boldsymbol{\beta}\cdot\boldsymbol{\sigma})/\lambda(1-\beta^2),$$
$$\boldsymbol{\Gamma}=[\boldsymbol{\sigma}+\boldsymbol{\beta}\times(\boldsymbol{\beta}\times\boldsymbol{\sigma})]/\lambda(1-\beta^2), \tag{16}$$

with

$$\boldsymbol{\sigma}=[\boldsymbol{\gamma}(1-\beta^2)+\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})]/(1-\beta^2)^{5/2}.$$

It is apparent from these equations that the uniformly accelerated motion of a particle, in the sense of our definition, is characterized by the vanishing of the Abraham 4-vector.

4. THE MECHANICS OF A PARTICLE

The complete form-invariance of the electromagnetic field equations under C_4 leads one naturally to seek a formulation of the equations of motion of a particle having the same symmetry group. It is at once apparent from the form of the acceleration 4-vector that this is not possible without admitting a mass term which depends

[18] In this section and in certain equations of the following section, we have employed the symbols β, α, γ in a double meaning, using them for both ordinary 3-dimensional vectors and for 4-vectors, but the context makes the meaning clear.

[19] The finite transformations of C_4 are most readily computed by means of hexaspherical coordinates, which are described in H. Weyl, *Space-Time-Matter* (Methuen, London, 1922), p. 286, and in F. Flein, *Höhere Geometrie* (Springer, Berlin, 1926), p. 247. The function in Eq. (14) is not directly comparable with the results of Infeld and Schild, reference 12, without a special investigation of the subgroups of C_4 under which λ is invariant in form.

[20] H. Bateman, reference 8, p. 253, and W. Pauli, reference 13, p. 654. The factor $1/\lambda$ is paralleled by a similar factor in the form of the Lorentz electromagnetic force on a particle.

explicitly on the space-time coordinates. On writing out the form of the acceleration from Eq. (10), using the line-element of Eq. (9) we find

$$\alpha^i=\frac{1}{\lambda^2}\left[\frac{d\bar{\beta}^i}{d\varphi}+(\bar{\beta}^i\bar{\beta}^j-\eta^{ij})\frac{\partial \ln\lambda}{\partial x^j}\right]$$

and there is no apparent way of eliminating the explicit term in $\ln\lambda$. In the discussion of Infeld and Schild this circumstance is accepted, but while such a conception may be suitable in the cosmological applications, we are not prepared to adopt it in general physical theory. There is, of course, no mystery in this difficulty in terms of ordinary mechanical conceptions, since the presence of a material particle performing a uniformly accelerated motion would be *prima facie* evidence of the existence of an external field. The existence of a problem at all arises from the association with electrodynamical theory in the treatment of the motions of charged particles.

We make contact at this point with the formulation of the classical equations of motion of an electron by Dirac,[21] which has been studied in a number of papers by Eliezer.[22] The radiation reaction terms are obtained in this theory by a particular method of evaluating the electromagnetic field along the world-line of the particle, taking into consideration both retarded and advanced potentials. Since Dirac's principal arguments refer only to the properties of the electromagnetic field, it would appear that the analysis of this paper would have some bearing on his results. We can, in fact, write down two possible sets of equations of motion which are form-invariant under the Lorentz group, and which might be considered as the generalization of Dirac's equations. The first of these is obtained by introducing the Abraham radiation reaction directly into the Minkowski form of the equations of motion of a particle in the special theory of relativity, which yields the equation

$$\frac{d}{d\tau}\left[\frac{m_0c^2\boldsymbol{\beta}}{(1-\beta^2)^{\frac{1}{2}}}\right]-\frac{2e^2}{3(1-\beta^2)}\left\{\boldsymbol{\gamma}+\frac{\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})}{1-\beta^2}+\frac{\boldsymbol{\beta}}{1-\beta^2}\left[(\boldsymbol{\gamma}\cdot\boldsymbol{\beta})+\frac{3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{1-\beta^2}\right]\right\}=e[\mathbf{E}+\boldsymbol{\beta}\times\mathbf{H}]. \quad (17)$$

The second set is obtained by dropping the term in the velocity 4-vector in the definition of the Abraham 4-vector of Eq. (11), which leads to the result

$$\frac{d}{d\tau}\left[\frac{m_0c^2\boldsymbol{\beta}}{(1-\beta^2)^{\frac{1}{2}}}\right]-\frac{2e^2}{3(1-\beta^2)}\left\{\boldsymbol{\gamma}+\frac{\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})}{1-\beta^2}+\frac{\boldsymbol{\beta}}{1-\beta^2}\left[(\boldsymbol{\gamma}\cdot\boldsymbol{\beta})+\alpha^2+\frac{4(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2}{1-\beta^2}\right]\right\}=e[\mathbf{E}+\boldsymbol{\beta}\times\mathbf{H}]. \quad (17a)$$

The association of the Abraham vector with uniformly accelerated motions gives one at once a qualitative idea of the origin of the "non-physical" solutions of the Dirac equations which have been discussed in some detail by Eliezer and others.[23]

One might suppose that since Dirac's theory is based primarily on the field equations, it should be possible to generalize it to have the symmetry group C_4. A direct attempt to do this is fraught with difficulties, but the discussion given here makes it seem probable that such a treatment would lead to a formulation in terms of the Abraham 4-vector, since this is the simplest kinematical vector which is form-invariant under C_4 and has the appropriate properties for expressing radiation reaction. However, this leaves us with precisely the same unsolved difficulty so far as the inertial term is concerned. It appears then that the generality of Dirac's theory is limited by the unexplained nature of the mass of a particle to invariance under the Lorentz group.

By the omission of the inertial term on the left-hand side of Eq. (17) we can obtain an equation which is invariant under the full group C_4, in which we can introduce any invariant scalar in place of the Abraham factor $2e^2/3$. Considered as an equation of motion of an ordinary charged particle this is quite inacceptable, but it still has a certain suggestiveness. In the first place, it is known that for the linear uniformly accelerated motion of a point-charge no radiation field is generated.[24] It appears probable that the uniformly accelerated motions, as defined here, constitute a unique class of radiationless orbits, although the detailed proof is not easy to give

[21] P. A. M. Dirac, Proc. Roy. Soc. **A167**, 148 (1938).

[22] C. J. Eliezer, Proc. Camb. Phil. Soc. **42**, 40 (1946); Phys. Rev. **71**, 49 (1947). See also a forthcoming review of the theory by Eliezer in the Rev. Mod. Phys.

[23] H. J. Bhabha, Phys. Rev. **70**, 759 (1946). N. Arley, Phys. Rev. **71**, 272 (1947).

[24] W. Pauli, reference 13, p. 648.

owing to the complex nature of the general transformation of C_4; the discussion of section 7, reference 4, shows this to be the case for motions differing infinitesimally from rest. Secondly, it is worthy of note that in the detailed calculations of Hassé[9] on the self-force of certain model electrons, making use of the transformation of the self-field under C_4, no terms corresponding to radiation reaction were found.

It has been remarked to the writer by Professor E. Fermi that there may be an essential difficulty in defining the radiation from orbits of the uniformly accelerated type in which the particle comes from, and proceeds to, infinity with a speed asymptotically approaching the speed of light. Similar questions have been raised in conversations with Professor R. Feynman. The problem deserves a new investigation. It can be shown that the coulomb field of a point charge transforms exactly into the standard solution for linear uniformly accelerated motion as obtained by retarded potential methods, under the transformation given in section 6, reference 4; this suggests that a more complete analysis of the behavior of retarded and advanced solutions of the field equations under C_4 would be valuable.[25]

From these remarks we get the suggestion that Eq. (17), without the inertial term, might be interpreted as the equation of motion of a type of charged particle of zero rest mass, the particle behaving merely as a singularity in the electromagnetic field. By including only the *external* field in the right hand side of Eq. (17), we would obtain the trajectories for uniformly accelerated motion as solutions of the equations of free motion of the particles. Singularities moving with the speed of light are included under the special case (Appendix *A*) $\gamma=0$, $\alpha=0$. This does not lead to a separate expression for the "particle" energy, so that all of the energy must be associated with the field; owing to the singularity this is infinitely great.

It is to be hoped that experiments with the new high energy betatrons and other devices will prove sufficiently accurate to give us our first real measurements on the radiation from particles, not arising directly from collisions with other particles.

APPENDIX A

The symmetry group of Eq. (6) can be studied by standard procedures of group theory, but it is more instructive to show its relation to the group C_4 by an indirect method. We pose the argument in two stages:

(a) Among the trajectories of particles obeying Eq. (6) are those of particles moving with the speed of light; for brevity we shall refer to these as *c-particles*. Any transformation which preserves Eq. (6) must leave the family of orbits of *c*-particles invariant. For *c*-particles we have the conditions, following directly from the properties of Lorentz transformations,[26]

$$\beta=1, \quad \boldsymbol{\alpha}=0, \quad \boldsymbol{\gamma}=0.$$

The first of these conditions yields the differential equation for the trajectories of *c*-particles

$$(d\tau)^2-(dx)^2-(dy)^2-(dz)^2=0. \qquad (18)$$

The second condition implies in addition that these trajectories are straight lines. With the third condition, which is a consequence of the second, we see that Eq. (6) is automatically satisfied for *c*-particles. But the work of Lie[3, 6] shows in fact that the invariance of Eq. (18) defines the group C_4. This argument was used by Engstrom and Zorn[3] in discussing Page's original theory,[1] and was employed also in the writer's earlier paper.[4] However, since the trajectories of *c*-particles form only a subset of all of the trajectories defined by Eq. (6), it proves in reality only that the symmetry group of Eq. (6) must be a subgroup of C_4.

(b) The second part of our argument consists in the direct verification that Eq. (6) is actually

[25] *Note added in proof.* Since this paragraph was written a more complete study has been made of the transformation of the coulomb field of a point charge for the case of uniformly accelerated motion along a line. The formulas obtained give the analytic continuation of the field into the region in which the retarded field itself vanishes. The resultant field is strongly influenced by the singularity in the transformation. The absence of a radiation field is related to the appearance of a second singularity in the field, other than that of the particle itself, corresponding to the "ideal charge at infinity"; owing to the nature of the transformation this singularity is brought into the finite region of the laboratory system, so that the complete transformed solution corresponds to the motion of *two* moving charges, of opposite signs, in the laboratory system. This emphasizes the importance of the singularities of the transformation, as well as of those of the field itself, in both the finite and the infinite regions of space-time.

[26] *Note added in proof.* This procedure is now seen to attach too great importance to the class of trajectories of *c*-particles which are straight lines. The most general condition on a *c*-particle is that $\boldsymbol{\alpha}\cdot\boldsymbol{\beta}=0$ when $\beta=1$, which is sufficient to guarantee Eq. (6).

invariant under *all* of the transformations of C_4, thus showing that C_4 must be a subgroup of the symmetry group of Eq. (6). These two results taken together imply that C_4 actually is the symmetry group of Eq. (6). For this purpose we take the formula for the general infinitesimal transformation of C_4 from reference 4, with an obvious re-definition of the parameters to conform to our present choice of variables, and also define

$$\delta\mathbf{s}=\mathbf{i}\delta\mu_1+\mathbf{j}\delta\mu_2+\mathbf{k}\delta\mu_3,\qquad \delta\boldsymbol{\lambda}=\mathbf{i}\delta\mu_8+\mathbf{j}\delta\mu_9+\mathbf{k}\delta\mu_{10},$$
$$\delta\boldsymbol{\omega}=\mathbf{i}\delta\mu_5+\mathbf{j}\delta\mu_6+\mathbf{k}\delta\mu_7,\qquad \delta\mathbf{v}=\mathbf{i}\delta\mu_{11}+\mathbf{j}\delta\mu_{12}+\mathbf{k}\delta\mu_{13}.$$

The general infinitesimal transformation of the thrice-extended group C_4 now becomes

$$\delta\mathbf{r}=-\delta\mathbf{s}-(\delta\boldsymbol{\omega}\times\mathbf{r})-\tau\delta\boldsymbol{\lambda}+\tfrac{1}{2}(r^2-\tau^2)\delta\mathbf{v}-\mathbf{r}(\mathbf{r}\cdot\delta\mathbf{v})-\tau\mathbf{r}\delta\mu_{14}+\mathbf{r}\delta\mu_{15}, \tag{19a}$$

$$\delta\tau=-\mu_4-(\mathbf{r}\cdot\delta\boldsymbol{\lambda})-\tau(\mathbf{r}\cdot\delta\mathbf{v})-\tfrac{1}{2}(r^2+\tau^2)\delta\mu_{14}+\tau\delta\mu_{15}, \tag{19b}$$

$$\delta\boldsymbol{\beta}=-(\delta\boldsymbol{\omega}\times\boldsymbol{\beta})-\delta\boldsymbol{\lambda}+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\delta\boldsymbol{\lambda})+(\mathbf{r}\cdot\boldsymbol{\beta}-\tau)\delta\mathbf{v}-(\mathbf{r}-\tau\boldsymbol{\beta})(\boldsymbol{\beta}\cdot\delta\mathbf{v})+[\boldsymbol{\beta}(\mathbf{r}\cdot\boldsymbol{\beta})-\mathbf{r}]\delta\mu_{14}, \tag{19c}$$

$$\begin{aligned}\delta\boldsymbol{\alpha}=&-(\delta\boldsymbol{\omega}\times\boldsymbol{\alpha})+\boldsymbol{\alpha}2(\boldsymbol{\beta}\cdot\delta\boldsymbol{\lambda})+\boldsymbol{\beta}(\boldsymbol{\alpha}\cdot\delta\boldsymbol{\lambda})+[(\mathbf{r}\cdot\boldsymbol{\alpha})+\beta^2-1]\delta\mathbf{v}-(\mathbf{r}-\tau\boldsymbol{\beta})(\boldsymbol{\alpha}\cdot\delta\mathbf{v})+\boldsymbol{\alpha}(\mathbf{r}\cdot\delta\mathbf{v})\\&+\boldsymbol{\alpha}2\tau(\boldsymbol{\beta}\cdot\delta\mathbf{v})+[\boldsymbol{\alpha}2(\mathbf{r}\cdot\boldsymbol{\beta})+\boldsymbol{\beta}(\beta^2-1)+\boldsymbol{\beta}(\mathbf{r}\cdot\boldsymbol{\alpha})+\boldsymbol{\alpha}\tau]\delta\mu_{14}-\boldsymbol{\alpha}\delta\mu_{15},\end{aligned} \tag{19d}$$

$$\begin{aligned}\delta\boldsymbol{\gamma}=&-(\delta\boldsymbol{\omega}\times\boldsymbol{\gamma})+\boldsymbol{\gamma}3(\boldsymbol{\beta}\cdot\delta\boldsymbol{\lambda})+\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\delta\boldsymbol{\lambda})+\boldsymbol{\beta}(\boldsymbol{\gamma}\cdot\delta\boldsymbol{\lambda})+[(\mathbf{r}\cdot\boldsymbol{\gamma})+3(\boldsymbol{\beta}\cdot\boldsymbol{\alpha})]\delta\mathbf{v}+\boldsymbol{\gamma}2(\mathbf{r}\cdot\delta\mathbf{v})\\&+3(\boldsymbol{\alpha}+\tau\boldsymbol{\gamma})(\boldsymbol{\beta}\cdot\delta\mathbf{v})+\boldsymbol{\alpha}3\tau(\boldsymbol{\alpha}\cdot\delta\mathbf{v})-(\mathbf{r}-\tau\boldsymbol{\beta})(\boldsymbol{\gamma}\cdot\delta\mathbf{v})\\&+[\boldsymbol{\gamma}3(\mathbf{r}\cdot\boldsymbol{\beta})+\boldsymbol{\gamma}2\tau+\boldsymbol{\alpha}3(\mathbf{r}\cdot\boldsymbol{\alpha})+\boldsymbol{\alpha}3\beta^2+\boldsymbol{\beta}3(\boldsymbol{\beta}\cdot\boldsymbol{\alpha})+\boldsymbol{\beta}(\mathbf{r}\cdot\boldsymbol{\gamma})]\delta\mu_{14}-2\boldsymbol{\gamma}\delta\mu_{15}.\end{aligned} \tag{19e}$$

The condition that Eq. (6) shall be invariant under an infinitesimal transformation becomes

$$\delta\boldsymbol{\gamma}(1-\beta^2)+\delta\boldsymbol{\alpha}3(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})+\boldsymbol{\alpha}\{3\delta(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})+6(\boldsymbol{\beta}\cdot\delta\boldsymbol{\beta})(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})/(1-\beta^2)\}=0. \tag{20}$$

The required proof of the invariance under C_4 is now obtained by substitution of Eqs. (19a–19e) into Eq. (20), taken together with Eq. (6) itself. The details of the reduction will be left to the reader.

APPENDIX B

It will be convenient to establish here the general integral of Eq. (6) for the sake of later reference. Equation (6) can be written in the form

$$(d/d\tau)[\boldsymbol{\alpha}/(1-\beta^2)^{\frac{3}{2}}]=0, \tag{21}$$

of which we have the first integral

$$\boldsymbol{\alpha}/(1-\beta^2)^{\frac{3}{2}}=\boldsymbol{\varepsilon}, \tag{22}$$

in which $\boldsymbol{\varepsilon}$ is a constant vector and $|\boldsymbol{\varepsilon}|=\epsilon$.

By dot-multiplication with $\boldsymbol{\beta}$ we obtain the relation

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})/(1-\beta^2)^{\frac{3}{2}}=d(1-\beta^2)^{-\frac{1}{2}}/d\tau=\boldsymbol{\varepsilon}\cdot\boldsymbol{\beta},$$

and by a second differentiation we get the scalar equation

$$(d^2/d\tau^2)(1-\beta^2)^{-\frac{1}{2}}=\epsilon^2(1-\beta^2)^{\frac{3}{2}},$$

which has the general solution

$$1-\beta^2=k_1/[k_1^2(\tau-\tau_1+k_2)^2+\epsilon^2], \tag{23}$$

k_1 and k_2 being integration constants.

On inserting Eq. (23) into Eq. (22) the integration for the velocity vector leads to the result

$$\boldsymbol{\beta}(\tau)=\boldsymbol{\beta}_1+\boldsymbol{\varepsilon}(k_1^{\frac{1}{2}}/\epsilon^2)(d\psi/d\tau)$$

with

$$\psi(\tau)=[(\tau-\tau_1+k_2)^2+\epsilon^2/k_1^2]^{\frac{1}{2}}-k_1k_2(\tau-\tau_1)/(k_1^2k_2^2+\epsilon^2)^{\frac{1}{2}}.$$

A final integration gives the formula for the trajectory

$$\mathbf{r}(\tau)=\mathbf{r}_1+\boldsymbol{\beta}_1(\tau-\tau_1)+\boldsymbol{\varepsilon}(k_1^{\frac{1}{2}}/\epsilon^2)[\psi(\tau)-(k_1^2k_2^2+\epsilon^2)^{\frac{1}{2}}/k_1], \tag{24}$$

$\mathbf{r}_1$ and $\boldsymbol{\beta}_1$ being the displacement and velocity vectors of the particle at the initial instant τ_1.

The constants of integration can be expressed as

$$\begin{aligned}k_1&=(\boldsymbol{\beta}_1\cdot\boldsymbol{\varepsilon})^2+\epsilon^2(1-\beta_1^2),\\k_2&=(\boldsymbol{\beta}_1\cdot\boldsymbol{\varepsilon})/k_1(1-\beta_1^2)^{\frac{1}{2}}.\end{aligned} \tag{25}$$

It will be convenient also to give the reduction of this equation for the case of 1-dimensional motion, say along the x-axis

$$[x-x_1\pm(1-\beta_1^2)^{-\frac{1}{2}}/\epsilon]^2-[\tau-\tau_1\pm\beta_1(1-\beta_1^2)^{-\frac{1}{2}}/\epsilon]^2=1/\epsilon^2. \tag{26}$$

It is readily verified that in the limit $c\to\infty$ these results degenerate into those for the uniformly accelerated motion of a particle in Newtonian mechanics.