Relativistic Correction to the Magnetic Moment of the Deuteron*

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The corrections referred to in the title are considered for a proton and neutron moving in each other's field. Two alternative equations are used as starting points for the calculations. One of these is an extension of the electrodynamic laws of interaction between charged particles. The other is typical of Hamiltonians giving rise to inverted (Thomas term like) vector spin orbit doublets and is an extension of the scalar field one-body equation. The calculations reported here show about the same degree of sensitivity of the relativistic correction of the deuteron's moment to the type of interaction assumed between particles as has been previously obtained in simplified considerations with a single particle model. It is concluded, therefore, that one cannot be sure of estimates of relativistic corrections to approximately their whole magnitude and that even the sign of the correction cannot be considered as certain. Considerations regarding additivity of nuclear moments involving an accuracy of much better than 0.01 nuclear Bohr magneton appear to be obscured not only by the presence of relativistic corrections but also by the remoteness of sufficient knowledge concerning the interactions between particles which is essential for the determination of the effects of relativity.

1. INTRODUCTION

It has been pointed[1] out that relativistic effects for the magnetic moment of a nuclear proton depend not only on its kinetic energy but also on the nature of forces binding the proton to the other nuclear particles. In the work just quoted, the proton was considered to be in a central field, and it was found that different results were obtained depending on whether the field was of the four-vector or of the scalar type. For a discussion of the principle of the matter it is satisfactory to make the simplified assumption of the central field. It is, nevertheless, desirable to know whether the effects of relativity are seriously affected by the introduction of the somewhat more realistic appearing view of interactions between pairs of nuclear particles. In the present note, calculations are described in which the interactions are of the latter type. Since the true interaction is not known and since it is probably impossible to describe it without the explicit introduction of a meson field or its equivalent, and since the meson theory of nuclear forces is still in a very rudimentary and unsatisfactory stage, it appears to be premature to try to give a unique or correct answer even to the deuteron problem at this stage. It may also be recalled that the existing treatments of nuclear forces by meson fields are not worked out to the point of obtaining interactions correctly even to order of the square of the ratio of the velocity of the particle to the velocity of light $(v/c)^2$ and that the field-theoretic divergence troubles usually set in as soon as this order of accuracy is attempted.

The calculations described below are, therefore, of necessity of a provisional character and they are intended in the spirit of finding out what kind of effects might exist and where they can reasonably be looked for. The starting point of the calculations is a set of wave equations[2] which has been set up in such a way as to have invariance to order v^2/c^2. One of the wave equations is an extension of the laws of interaction between charged particles according to electrodynamics. While the field theoretic difficulties are formally present in electrodynamics as well as in meson theories, the understanding of the physical applicability of approximations is a much better one in the electrodynamic case and it is consequently possible to have some confidence in the validity of this equation at least in the special case of a meson field degenerating into a field of photons.

Another equation which is tried out below for the deuteron has been set up[2] in such a way as to give the inverted order of the fine structure of nuclear levels for which there has been some experimental evidence.[3] The fine structure of

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[1] G. Breit, Phys. Rev. **71**, 400 (1947).

[2] G. Breit, Phys. Rev. **51**, 248 (1937).

[3] L. H. Rumbaugh and L. R. Hafstad, Phys. Rev. **50**, 681 (1936); D. R. Inglis, *ibid.* 783; W. H. Furry, *ibid.* 784; G. Breit, reference 2; G. Breit and J. R. Stehn, Phys. Rev. **53**, 459 (1938).

nuclear levels appears from the point of view of this equation as a relativistic effect and arises from the same set of terms in the equation which give rise to some of the relativistic effects on the magnetic moment which are studied below. Thus in this case, also, there is a partial tie-up with experiment through the supposedly observed fine structure.

Neither of the equations tried satisfies the requirements of saturation of nuclear forces. Equations invariant to order v^2/c^2 have been devised[4] for exchange forces and could, of course, be tried. This has not been done for the present note because it appears that at this time the qualitative rather than the quantitative aspect is important and because it is believed that the variability of the answer to the assumptions made has become sufficiently established through the combined evidence of calculations for central fields[1] with those reported on now.

2. INTERACTIONS OF THE FOUR-VECTOR TYPE

For obvious reasons it is logical to refer to an interaction having transformation properties similar to those of the electrodynamic case as an interaction of the four-vector type. This equation has been discussed as Eq. (16.1) in reference 2 and has the form

$$\{E+c(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})+c(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})+(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})Mc^2+J-Q\}\psi=0; \quad (1)$$

$$Q=(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})J/2-(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})dJ/2rdr. \quad (1.1)$$

Here the wave function ψ has 16 components, 4 for each particle. The vectors $\boldsymbol{\alpha}_{\mathrm{I}}$ have for components Dirac's α matrices for particle I multiplied by the unit matrix for particle II. The matrix β is the fourth Dirac α matrix in accordance with the usual notation. The energy, including that of the rest mass, is E, c is the velocity of light, the $\mathbf{p}$ are vector momentum operators, and the mass of either particle is M. The quantity J depends only on the distance r between the particles and is the negative of the potential energy in the non-relativistic limit. The vector $\mathbf{r}$ is the displacement from II to I. The quantity Q, defined by Eq. (1.1), is such as to make Eq. (1) invariant to order v^2/c^2. Dirac's original choice of matrices $\boldsymbol{\alpha}$, β will be used. For the single particle problem the components ψ_a $(a=1, 2)$ are small while the ψ_b $(b=3, 4)$ are relatively large at low velocities. For the two particle problem the wave function $\psi_{\mu\nu}$ has sixteen components which can be labeled by two indices, the first index referring to particle I and the second to particle II. It is convenient to break up the sixteen components into four groups of four as follows:

Ψ for both μ and ν equal 3 or 4;
χ_{I} for $\mu=1$ or 2, $\nu=3$ or 4;
χ_{II} for $\mu=3$ or 4, $\nu=1$ or 2;
φ for both μ and ν equal 1 or 2.

Each of the four quantities just listed has two components for the first particle and two for the second. It is convenient to write equations between these quantities with the following convention. The distinction between indices 1 and 3 and indices 2 and 4 will be dropped in linear equations connecting any of the four quantities Ψ, χ_{I}, χ_{II}, and φ with each other. In other words each of the four quantities is considered as a column matrix with 4 rows arranged in order for the four $\psi_{\mu\nu}$ in such a way that in the first row $\mu, \nu=\mu', \nu'$, in the second $\mu, \nu=\mu', \nu''$, in the third $\mu, \nu=\mu'', \nu'$, in the fourth $\mu, \nu=\mu'', \nu''$ with $\mu''>\mu'$, $\nu''>\nu'$. For Ψ it will be understood that $\mu'=\nu'=3$, $\mu''=\nu''=4$; for χ_{I} one takes $\mu'=1$, $\mu''=2$, $\nu'=3$, $\nu''=4$; for χ_{II} similarly $\mu'=3$, $\mu''=4$, $\nu'=1$, $\nu''=2$; for φ, on the other hand, $\mu'=\nu'=1$, $\mu''=\nu''=2$. With this notation one rewrites Eq. (1) as

$$(E-2Mc^2+J)\Psi+c(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})\chi_{\mathrm{I}}+c(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})\chi_{\mathrm{II}}+X\varphi=0; \quad (1.2)$$

$$c(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})\Psi+(E+J)\chi_{\mathrm{I}}+X\chi_{\mathrm{II}}+c(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})\varphi=0; \quad (1.3)$$

$$c(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})\Psi+X\chi_{\mathrm{I}}+(E+J)\chi_{\mathrm{II}}+c(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})\varphi=0; \quad (1.4)$$

$$X\Psi+c(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})\chi_{\mathrm{I}}+c(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})\chi_{\mathrm{II}}+(E+2Mc^2+J)\varphi=0; \quad (1.5)$$

with

$$X=-(\boldsymbol{\sigma}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})J/2+(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r})dJ/2rdr. \quad (1.6)$$

Here $\boldsymbol{\sigma}_{\mathrm{I}}$, $\boldsymbol{\sigma}_{\mathrm{II}}$ are, respectively, the Pauli spin matrices for particles I and II. Equation (1.2) is obtained from Eq. (1) by setting $\mu=b=3, 4$; $\nu=b=3, 4$. Equation (1.3) is obtained for $\mu=a=1, 2$; $\nu=b=3, 4$. Similarly, Eqs. (1.4), (1.5) are obtained from Eq. (1) for $\mu=b$, $\nu=a$ and $\mu=a$, $\nu=a$, respectively.

From Eq. (1.3), neglecting φ one finds a first approximation to χ_{I} expressed in terms of Ψ. Similarly Eq. (1.4) gives a first approximation to χ_{II}. These first approximations, when introduced

[4] G. Breit, Phys. Rev. **53**, 153 (1938).

into Eq. (1.5) give φ, correct to order v^2/c^2. The formulas for φ, χ_{I}, χ_{II} so obtained when introduced into Eqs. (1.3), (1.4) yield expressions for χ_{I}, χ_{II} also correct to order v^2/c^2. The results are

$$\varphi=[(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})-MX]\Psi/(4M^2c^2)\,; \tag{2}$$

$$\chi_{\mathrm{I}}=\{-[1-(W+J)/2Mc^2]\times(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})/2Mc-p_{\mathrm{II}}^2(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})/8M^3c^3+[(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})X+2X(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})]/8M^2c^3\}\Psi. \tag{2.1}$$

The contribution to the magnetic moment caused by the Dirac current of particle I, which is assumed here to have a charge e, is the expectation value of

$$\boldsymbol{\mu}_e=e[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]/2. \tag{2.2}$$

The computation of the expectation value can be carried out by substitution of Eqs. (2), (2.1) into Eq. (2.2). The calculation is described in Appendix I. The work is straightforward since by well-known formulas one can always linearize an expression in Pauli sigmas. One finds for the expectation value of a ^3S state

$$\mu_e/\mu_0=[1-\langle T\rangle/3Mc^2]+\langle T\rangle/6Mc^2=1-\langle T\rangle/6Mc^2 \tag{2.3}$$

where μ_0 is the nuclear Bohr magneton, and the quantity in square brackets is the value obtained for μ_e/μ_0 if one neglects the term Q in Eq. (1). The quantity $\langle T\rangle$ is the mean of the non-relativistic kinetic energy of the deuteron in its center-of-mass system.

The calculation would have the appearance of being more accurate if one computed the value of μ_e for the exact mixture of the ^3S and ^3D states which corresponds to the bound state of the deuteron. The correction for the presence of the ^3D state has, however, been made by Arnold and Roberts[5] making use of calculations of Rarita and Schwinger.[6] The correction is of the order of one percent of the moment of the deuteron, and the relativistic corrections to this correction are probably negligible. It may appear, on the other hand, to be inconsistent to neglect the admixture of the ^3D state which is demanded by Eq. (1) for the ground state of the deuteron. The four-component form of Eq. (1) indeed contains terms in $(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r})$ where $\mathbf{r}$ is the displacement vector from II to I and because of these it is impossible to have a pure ^3S state. Estimates show, however, that the admixture of the ^3D state corresponding to these tensor force terms is much smaller than that used by Rarita and Schwinger. The coefficient of the ^3D state is about 1/10 or 1/20 of that of Rarita and Schwinger and it appears justifiable to assume, therefore, for purposes of the calculation of the magnetic moment, that the ground state of Eq. (1) is a pure S state.

3. INTERACTIONS OF THE SCALAR TYPE

These interactions are discussed by means of Eq. (18.1) of reference[2] which has the form

$$\{E+c(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})+c(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})+(\beta_{\mathrm{I}}+\beta_{\mathrm{II}})Mc^2+\beta_{\mathrm{I}}\beta_{\mathrm{II}}J+(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})J/2+(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})dJ/2rdr\}\psi=0. \tag{3}$$

Equation (18.2) of reference 2 shows that there is no tensor force in the four component Pauli form and that its ground state is a pure ^3S term. The calculation is just like that for the four-vector equation. The approximations to order v^2/c^2 are

$$\varphi=[(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})-MY]\Psi/(4M^2c^2)\,; \tag{3.1}$$

$$\chi_{\mathrm{I}}=\{-[1-(W-J)/2Mc^2]\times(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})/2Mc-p_{\mathrm{II}}^2(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})/8M^3c^3+[(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})Y+2Y(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{p}_{\mathrm{II}})]/8M^2c^3\}\Psi, \tag{3.2}$$

where

$$Y=J(\boldsymbol{\sigma}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})/2+(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\sigma}_{\mathrm{II}}\mathbf{r})dJ/2rdr. \tag{3.3}$$

Substitution into Eq. (2.2) gives

$$\mu_e/\mu_0=[1+(-W+\langle T\rangle/3)/Mc^2]-\langle T\rangle/6Mc^2=1+(-W+\langle T\rangle/6)/Mc^2. \tag{3.4}$$

In the above formula the expression in square brackets is the result of neglecting Y which is equivalent to neglecting the terms in $(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})$ and in $(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})$ in Eq. (3). The quantity $W=-2.17$ Mev in Eq. (3.4) is the non-relativistic energy of the deuteron.

The calculation leading to Eq. (3.4) is very similar to that which gave Eq. (2.3). Many of the same quantities occur in both. The evaluation of the expectation value of the operator on the right side of Eq. (2.2) is made this time also for a ^3S state, and Eq. (3.4) is supposed to be right only for this state.

As has been mentioned in the introduction, Eq. (3) corresponds to spin orbit interactions of the simplest Thomas effect type as may be seen by inspection of Eq. (18.2) of reference 2. The term $-\langle T\rangle/6Mc^2$ which is added to the bracket in the expression for the relativistic correction factor

[5] W. R. Arnold and A. Roberts, Phys. Rev. **70**, 766 (1946).

[6] W. Rarita and J. Schwinger, Phys. Rev. **59**, 436 (1941).

given by Eq. (3.4) arose through the presence in Eq. (3) of the terms in $(\boldsymbol{\alpha}_{\mathrm{I}}\boldsymbol{\alpha}_{\mathrm{II}})$, $(\boldsymbol{\alpha}_{\mathrm{I}}\mathbf{r})(\boldsymbol{\alpha}_{\mathrm{II}}\mathbf{r})$. If these terms were omitted in Eq. (3), the spin orbit interaction would not be affected, but the invariance to order v^2/c^2 would be destroyed.

4. NUMERICAL APPLICATIONS AND DISCUSSION

The magnetic moment of the proton $\mu_p=2.79$ and the magnetic moment of the neutron $\mu_n=-1.93$ in units of a nuclear Bohr magneton. The proton moment can be broken up into $1+1.79$. The first part (1) will be considered as caused by the Dirac current associated with the proton's charge and the second (1.79) will be attributed to an intrinsic moment of the type first introduced by Pauli[7] and referred to below as the "Pauli part." A calculation of relativistic effects for this moment has been made by Caldirola[8] for the special case of a Diracian particle in a four-vector field. Caldirola's formulas as they appear in print are not consistent with each other, and it appears from his arithmetic that an incorrect sign of the correction term has been arrived at. It has been shown[1] that for the Pauli part the relativistic correction factor for S terms is

$$1-\langle T_p\rangle/3Mc^2, \quad 1-\langle T_n\rangle/3Mc^2, \tag{4}$$

for the proton and neutron, respectively, independent of the type of field to which the particle is exposed, and it has also been shown there that this correction factor is a purely kinematic effect which is expected quite apart from quantum theory as a consequence of the Lorentz contraction of a magnetic doublet.

From the formal point of view of calculation the independence of this correction factor on the special form of the interaction between particles is caused by the fact that the small components of the wave function, ψ, have to be known in terms of the large ones only to the first rather than to the second order in v/c. This circumstance has its origin in the fact that the intrinsic moment is a multiple of $\langle -\rho_3\sigma_z\rangle$ in Dirac's original notation and that $\rho_3\sigma_z$ is diagonal, so that only squares of absolute values of wave function components enter the calculation. The same circumstance makes it unnecessary to bring the second order quantity φ into the calculation for the two body problem, and the quantities χ_{I}, χ_{II} are needed only to the first order. To this order they are expressed in terms of Ψ by operating on Ψ by the same operators which give for the one body case the small Dirac wave function components in terms of the large ones *viz.*,

$$\chi_{\mathrm{I}}\cong-(\boldsymbol{\sigma}_{\mathrm{I}}\mathbf{p}_{\mathrm{I}})\Psi/(2Mc) \tag{4.1}$$

and similarly for χ_{II}. In the calculation of the correction factor to the Pauli part of μ_p one needs only χ_{I}, and for μ_n one needs only χ_{II}. The relativistic correction factors to the Pauli parts of the moments are thus verified to be the same functions of the momentum operators of the individual particles as for the one-body problem.

Since

$$\langle T_p\rangle=\langle T_n\rangle=\langle T\rangle/2 \tag{4.2}$$

the combined contribution to the relativistic correction of the deuteron caused by the Pauli parts of the moments of both proton and neutron is

$$-(1.79-1.92)(\langle T\rangle/6Mc^2)=0.022\langle T\rangle/Mc^2. \tag{4.3}$$

Because of the approximate cancellation of the Pauli parts the contribution attributable to (4.3) is so small that it could be neglected and is carried here only for the sake of completeness.

Combining Eq. (2.3) with Eq. (4.3) the relativistic correction becomes for the four-vector field

$$\Delta_1\mu=(0.022-1/6)\langle T\rangle/Mc^2=-0.145\langle T\rangle/Mc^2 \tag{5}$$

which gives for[1]

$$\langle T\rangle/Mc^2=0.0078 \tag{5.1}$$

the value

$$\Delta_1\mu=-0.0011. \tag{5.2}$$

If, on the other hand, one omits the effect of Q in Eq. (2.3) then one obtains

$$\Delta_1'\mu=(0.022-1/3)\langle T\rangle/Mc^2 = -0.311\langle T\rangle/Mc^2 \tag{5.3}$$

so that for the numerical value of $\langle T\rangle$ in Eq. (5.1) one obtains

$$\Delta_1'\mu=-0.0024. \tag{5.3'}$$

Equations (5.3) and (5.3′) have been given here only for comparison with the more logically derived Eqs. (5) and (5.1) so as to show the magnitude of the effect of the terms arising from

[7] W. Pauli, *Handbuch der Physik* (Verlagsbuchhandlung, Julius Springer, Berlin, 1933), Vol. 24/1, p. 221.

[8] P. Caldirola, Phys. Rev. **69**, 567 (1946); H. Margenau, Phys. Rev. **57**, 383 (1940); G. Breit, Nature **122**, 649 (1928).

Q. The quantity Q, it will be remembered, has been introduced into the four-vector equation so as to correct for lack of invariance in its absence. It is the extension of the combined effect of magnetic interaction and the effect of electrostatic interaction of electrodynamics. The sensitivity of the relativistic correction to assumptions regarding the nature of interactions between particles is especially striking if one notes that the right sides of Eqs. (5.2) and (5.3′) differ by more than a factor 2 and that this difference is produced entirely by terms which give effects of order v^2/c^2 in the energy.

For interactions of the scalar type one obtains, combining Eqs. (3.4) and (4.3),

$$\Delta_2\mu = (-W + 0.189\langle T\rangle)/Mc^2 \tag{6}$$

so that with $W = -2.17$ Mev and the value of $\langle T\rangle$ following from Eq. (5.1) one obtains

$$\Delta_2\mu = 0.0023 + 0.0015 = 0.0038. \tag{6.1}$$

The number 0.0023 is the contribution of the term in W in Eq. (6). If the terms attributable to Y are neglected in Eq. (3.4) then one obtains

$$\Delta_2'\mu = (-W + 0.355\langle T\rangle)/Mc^2 \tag{6.2}$$

which gives, on making use of Eq. (5.1),

$$\Delta_2'\mu = 0.0023 + 0.0028 = 0.0051. \tag{6.3}$$

In this case, just as in the equation of the four-vector type, one has an appreciable effect of the terms analogous to Q of Eq. (1) as evidenced by the fact that the difference of the right sides of Eqs. (6.1) and (6.3) is of the order of magnitude of themselves.

It may be appropriate to discuss briefly the degree of certainty in believing the differences $\Delta_1\mu - \Delta_1'\mu$ and $\Delta_2\mu - \Delta_2'\mu$ to be physical consequences of the equations which have been examined. Reasonable doubt can be entertained regarding this point because in the electromagnetic case the order v^2/c^2 is believed to be the last one in which the Hamiltonian function can be consistently used for the description of two particles with a complete elimination of field quantities. There is some possibility, therefore, that in higher orders there will be additional effects which cannot be foreseen through the Hamiltonian. This possibility cannot be categorically denied. It appears to be an unlikely one for the electromagnetic case but for the deuteron the situation is so full of unknown circumstances that it would be hazardous to claim too much for the value of a systematic expansion in powers of v/c. But if one makes a development of quantities in such powers, then one cannot avoid the introduction of some such term as Q in Eq. (1). One of the functions of this term is to make the equations of motion invariant to order v^2/c^2. It would be wrong, therefore, to leave this term out of consideration.

It should be remarked, however, that considerations of invariance do not suffice to fix such a term as Q. This aspect of the situation has been discussed at length in reference 2. It has to be brought up here only for the purpose of emphasizing the lack of uniqueness in the answers. It is possible to vary the values of the correction not only by making the function J occur in a place characteristic of purely four-vector or scalar types of interaction as in Eqs. (1) and (3), but one can vary even such differences as $\Delta_1\mu - \Delta_1'\mu$ by a suitable change in the choice of the quantity Q.

In addition to the factors which have just been mentioned it is necessary to consider that there is the further flexibility of choice caused by the unknown exchange character of nuclear forces. The terms analogous to Q are different for exchange forces from what they are for the ordinary ones.

It should finally be mentioned that the electric quadrupole of the deuteron speaks for the presence of the ^3D to an extent of about 4 percent on the basis of probability which corresponds to a coefficient of about 0.2 for the ^3D wave function. In the non-relativistic single body problem the magnetic moment operators are diagonal in the orbital angular momentum L and the whole magnetic moment is in this approximation a weighted mean of magnetic moments in the ^3S and ^3D states in the proportions 0.96 and 0.04. The fact that the deuteron has two, rather than one, particles makes no difference in this connection because for any wave function for which

$$(\mathbf{p}_{\mathrm{I}} + \mathbf{p}_{\mathrm{II}})\Psi = 0$$

one has

$$\begin{aligned} L_{\mathrm{I}x}\Psi &= (y_{\mathrm{I}}p_{\mathrm{I}z} - z_{\mathrm{I}}p_{\mathrm{I}y})\Psi = -(y_{\mathrm{I}}p_{\mathrm{II}z} - z_{\mathrm{I}}p_{\mathrm{II}y})\Psi \\ &= (y_{\mathrm{I}}p_z - z_{\mathrm{I}}p_y)\Psi \end{aligned}$$

where $\mathbf{p} = (\mathbf{p}_{\mathrm{I}} - \mathbf{p}_{\mathrm{II}})/2$ is the relative momentum.

Since Ψ has a definite parity, an interchange of I and II leaves $\Psi^*\Psi$ unchanged and one has, therefore, for expectation values

$$\langle L_{\mathrm{I}x}\rangle = \langle L_{\mathrm{II}x}\rangle = \langle L_x\rangle/2$$

as for two particles moving around each other according to Newtonian dynamics. From this point of view it might appear that the 4 percent admixture of the ^3D state could be left out of account since 4 percent of the corrections could be surmised to be negligible. Such a conclusion would not be a safe one, however, for the Dirac part of the moment, because the operator entering the calculation of expectation value of the magnetic moment with Ψ contains e.g., a term in

$$(\mathbf{r}_{\mathrm{I}}\boldsymbol{\sigma}_{\mathrm{II}})\boldsymbol{\nabla}_{\mathrm{II}}J$$

which has non-vanishing matrix elements between ^3S and ^3D states. Because of this part of the operator one has cross terms between these states which enter, therefore, with the coefficient 0.2 rather than $(0.2)^2 = 0.04$. A quantitative examination of these terms at this stage appears pointless on account of the many other uncertainties, but their existence is nevertheless of interest.

The value of $\langle T\rangle$ which was used in the numerical estimates is the same as in reference 1. It corresponds to a "square well" interaction potential with radius e^2/mc^2. In a final calculation of the effect, one would have to take into account the presence of the ^3D state in some such way as has been done by Rarita and Schwinger.[6] The change in the effective interaction potential found by them is large, and the value of $\langle T\rangle$ used here is only provisional. Since the equations which have been used above do not have as a consequence, however, any strong admixture of the ^3D state, it is a self-consistent procedure to neglect in the present connection the effects which will enter on account of the Rarita-Schwinger calculations.

The considerations which have just been presented give the same general picture as those in reference 1. The relativistic correction to the magnetic moment is sensitive to the assumptions made regarding the forces between the particles. The sensitivity is so great and there are so many contributing effects that a close evaluation on any particular model would be pointless, and even the possibility of a close compensation of the effects cannot be excluded since different signs for the effect have been obtained for different assumed interactions.

The fact that the corrections attributable to relativistic effects are subject to so many uncertainties makes it somewhat doubtful that one will be able to test hypotheses of additivity in their finer features also in other nuclei such as H^3 or He^3. Presumably, however, the effects will not exceed .01 of a nuclear Bohr magneton and to this precision it should be possible to make use of measured values with the calculations of Sachs and Schwinger[9] and of Sachs.[10]

APPENDIX

Calculation of the Dirac Part of Moment

The Dirac part of the proton's magnetic moment is given by Eq. (2.2). The expectation value of $[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]$ is first expressed as the three-dimensional integral of

$$\psi^+[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\alpha}_{\mathrm{I}}]\psi = \chi_{\mathrm{I}}^+[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]\Psi + \Psi^+[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]\chi_{\mathrm{I}} + \chi_{\mathrm{II}}^+[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]\varphi + \varphi^+[\mathbf{r}_{\mathrm{I}}\times\boldsymbol{\sigma}_{\mathrm{I}}]\chi_{\mathrm{II}}$$

and on the right side of the above equation the values of χ_{I}, χ_{II}, and φ in terms of Ψ are substituted by means of Eqs. (2), (2.1), (3.1), and (3.2). The row matrices φ^+, Ψ^+, $\cdots$ are, respectively, the conjugates transposed of the column matrices φ, Ψ, $\cdots$. The quantity Ψ is substituted for in terms of the closely related quantity $\Psi^{(1)}$ which should be[2,4] normalized to unity. Somewhat lengthy but otherwise straightforward manipulations yield expressions involving expectation values of W, J, and rdJ/dr. These quantities are present, of course, only in the correction terms to the moment, and the evaluation of their expectation values may be carried out, therefore, non-relativistically. For the non-relativistic wave equation the virial theorem holds for expectation values so that

$$\langle rdJ/dr\rangle = -2\langle T\rangle$$

holds with sufficient accuracy. By means of this relation all of the correction terms can be expressed by means of W and $\langle J\rangle$ or else by means of W and $\langle T\rangle$. The correction factors in the text have been obtained by the process described above.

[9] R. G. Sachs and J. Schwinger, Phys. Rev. **70**, 41 (1946).
[10] R. G. Sachs, Phys. Rev. **69**, 611 (1946).