PHYSICAL REVIEW VOLUME 72, NUMBER 11 DECEMBER 1, 1947

Theory of the Propagation of Shock Waves from Infinite Cylinders of Explosive*

STUART R. BRINKLEY, JR.** AND JOHN G. KIRKWOOD***
Department of Chemistry, Cornell University, Ithaca, New York

(Received August 15, 1947)

The assumption of energy dissipation at a single shock, formulated in an earlier paper, is employed in the formulation of a pair of ordinary differential equations for peak pressure and shock-wave energy as functions of radial distance from the source for the shock wave produced by an infinite cylinder of explosive along which a stationary detonation wave is traveling with finite velocity. The profile of the wave may be determined by means of an auxiliary integration. The theory takes proper account of the finite entropy increment of the fluid produced by the passage of the shock, and permits the use of the exact Hugoniot curve of the fluid in the numerical integration of the basic equations.

1. INTRODUCTION

IN a recently published paper,[1] hereafter designated [*I*], a theory of propagation of one-dimensional, that is, plane, cylindrical, and spherical, shock waves is described. The partial differential equations of hydrodynamics and the Hugoniot relation between pressure and particle velocity are used to provide three relations between the four partial derivatives of pressure and particle velocity, with respect to time and distance from the source, at the shock front. An approximate fourth relation is set up by imposing a similarity restraint on the shape of the energy-time curve and by utilizing the second law of thermodynamics to determine, at an arbitrary distance, the distribution of the initial energy input between dissipated energy residual in the fluid already traversed by the shock front and energy available for further propagation. The four relations are used to formulate a pair of ordinary differential equations for peak pressure and shock-wave energy as functions of distance from the source. The theory takes proper account of the finite entropy increment of the fluid produced by the passage of the shock and permits the use of the exact Hugoniot curve of the fluid in the numerical integration of the basic equations.

In the present communication, we apply the methods of [*I*] to a description of the shock wave produced by an infinite cylinder of explosive along which a detonation wave is traveling with finite velocity. The results of the theory should be applicable to the shock wave produced by changes of finite length up to distances from the charge which are of the order of magnitude of its length.

2. THE PROPAGATION EQUATIONS

For the present purpose, it is convenient to write the equations of hydrodynamics in the Eulerian form,

$$\frac{1}{\rho c^2}\frac{Dp}{Dt}+\frac{1}{r}\frac{\partial}{\partial r}ru_r+\frac{\partial u_z}{\partial z}=0,$$

$$Du_r/Dt=-\partial p/\rho\partial r,\quad Du_z/Dt=-\partial p/\rho\partial z,\qquad(1)$$

where r and z are the Euler radial and axial cylindrical coordinates, relative to an origin fixed in the detonation wave with z-axis coincident with the axis of the cylinder, u_r and u_z the radial and axial components of particle velocity, measured relative to the moving coordinate system, p the pressure in excess of the pressure p_0 of the undisturbed fluid, ρ the density, D the detonation velocity in the negative z-direction, and D/Dt is the Euler total-time derivative which follows the fluid. The Euler sound velocity c is equal to $[(\partial p/\partial\rho)_S]^{\frac{1}{2}}$. Equations (1) are supplemented by the equation of state of the fluid and the entropy transport equation, $DS/Dt=0$, the latter of which we shall not use explicitly. They are to be solved subject to initial conditions specified on a curve in the r, z, t-space and to the

* This paper is based upon work done at Cornell University under Contract OEMsr-121 with the Office of Scientific Research and Development.
** Present address: Central Experiment Station, U. S. Bureau of Mines, Pittsburgh 13, Pennsylvania.
*** Present address: Department of Chemistry, California Institute of Technology, Pasadena, California.
[1] S. R. Brinkley, Jr. and J. G. Kirkwood, Phys. Rev. **71**, 606 (1947).

Rankine-Hugoniot[2] conditions at the shock front,

$$\begin{aligned} p &= \rho_0 u_n U, \\ \rho(U-u_n) &= \rho_0 U, \\ \Delta H &= (p/2)(1/\rho_0+1/\rho), \end{aligned} \tag{2}$$

where ΔH is the specific enthalpy increment experienced by the fluid in traversing the shock front, U is the velocity of the shock front, and u_n is the component of particle velocity normal to the shock front. Equations (2) constitute supernumerary boundary conditions which are compatible with Eq. (1) and the specified boundary conditions only if the shock front follows an implicitly prescribed curve $r(t)$, $z(t)$ in the r, z, t-space.

Taylor[3] has shown that the Rankine-Hugoniot and Chapman-Jouget[4] conditions can be satisfied at the front of a stationary detonation wave by solutions of the equations of hydrodynamics which depend only on $\mathbf{r}/t$, where $\mathbf{r}$ is the Euler position vector of a point relative to an origin traveling with the detonation front. For solutions of the Taylor type,

$$D/Dt=(u_r-r/t)\partial/\partial r+(u_z-z/t)\partial/\partial t, \tag{3}$$

and Eqs. (1) become

$$\begin{aligned} \left(u_r-\frac{r}{t}\right)\frac{\partial p}{\partial r}+\left(u_z-\frac{z}{t}\right)\frac{\partial p}{\partial z} &= -\frac{\rho c^2}{r}\frac{\partial}{\partial r}ru_r-\rho c^2\frac{\partial u_z}{\partial z}, \\ \left(u_r-\frac{r}{t}\right)\frac{\partial u_r}{\partial r}+\left(u_z-\frac{z}{t}\right)\frac{\partial u_r}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}, \\ \left(u_r-\frac{r}{t}\right)\frac{\partial u_z}{\partial r}+\left(u_z-\frac{z}{t}\right)\frac{\partial u_z}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}. \end{aligned} \tag{4}$$

The solutions of Eqs. (4) are compatible with the boundary conditions of continuity of pressure and normal component of particle velocity at the boundary between the explosion products and the exterior medium if solutions of the Taylor type, dependent only on r/t and z/t, are valid in the explosion products behind the detonation wave.

The shock front in the exterior medium will be a surface of revolution with a profile

$$z=\zeta(r) \tag{5}$$

in any r, z-plane. The profile of the shock front has the differential equation

$$d\zeta/dr=\tan\vartheta, \tag{6}$$

where ϑ is the angle between the tangent to the profile and the r-axis. Since the distance traveled by the shock front in the direction of its normal in time dt is Udt and the origin of the coordinate system travels a distance Ddt in the negative z-direction in time dt, we have

$$\cos\vartheta=U/D. \tag{7}$$

The normal component of the particle velocity is given by

$$u_n=u_r\sin\vartheta+(D-u_z)\cos\vartheta. \tag{8}$$

For continuity of the tangential component of the particle velocity,

$$u_r\cos\vartheta=(D-u_z)\sin\vartheta. \tag{9}$$

If the second of Eqs. (2) is combined with Eqs. (7), (8), and (9), there result the relations,

$$\begin{aligned} u_r &= (p/\rho_0 D)\tan\vartheta, \\ D-u_z &= p/\rho_0 D. \end{aligned} \tag{10}$$

A derivative along the shock front, for which we use the notation d/dR, is

$$d/dR=(\partial/\partial r)_\sigma+(\partial/\partial z)_\sigma\tan\vartheta, \tag{11}$$

where the subscript σ implies that the partial derivatives are to be evaluated at the shock front $r=R$. When this derivative is applied to Eqs. (10), two new relations are obtained,

$$\begin{aligned} \beta\left(\frac{\partial p}{\partial r}\right)_\sigma\cot\vartheta+\beta\left(\frac{\partial p}{\partial z}\right)_\sigma &= -\rho_0 D\left[\left(\frac{\partial u_r}{\partial r}\right)_\sigma+\left(\frac{\partial u_r}{\partial z}\right)_\sigma\tan\vartheta\right], \\ \left(\frac{\partial p}{\partial r}\right)_\sigma+\left(\frac{\partial p}{\partial z}\right)_\sigma\tan\vartheta &= -\rho_0 D\left[\left(\frac{\partial u_z}{\partial r}\right)_\sigma+\left(\frac{\partial u_z}{\partial z}\right)_\sigma\tan\vartheta\right], \end{aligned} \tag{12}$$

where $\beta=1-g\sec^2\vartheta$, $g=1-d(\log U)/d(\log p)$.

---

[2] W. J. M. Rankine, Phil. Trans. **A160**, 277 (1870). H. Hugoniot, J. de l'ecole polyt. **57**, 3 (1887); **58**, 1 (1888).

[3] G. I. Taylor (Reference is to informal memorandum; more complete citation is not possible (1941).)

[4] D. L. Chapman, Phil. Mag. [5] **47**, 90 (1889); E. Jouget, Comptes rendus **132**, 573 (1901). See also H. L. Dryden, F. D. Murnaghan, and H. Bateman, Bull. Nat. Res. Council, No. 84, 551 (1932).

As in [I], we proceed to set up a supplementary relation involving the reduced energy-time integral, a slowly varying function of distance from the charge which can be estimated without integration of the partial differential equations of hydrodynamics. If $K(R)/a_0$ is the energy transmitted per unit area of the initial generating surface along the z-axis by the excess pressure of the shock wave to the fluid initially exterior to a cylinder of radius a_0, the assumption of dissipation at a single shock formulated in [I] yields

$$K(R)=\int_R^\infty \rho_0 r_0 h[p(r_0)]dr_0, \tag{13}$$

where a_0 is the initial radius of the cylinder and where $h(p)$ is the specific enthalpy increment imparted to an element of fluid traversed by a shock wave of peak pressure p after the excess pressure has returned to the value zero. The energy of explosion released by unit length of the explosive is, of course, $2\pi K(a_0)$. Equating $K(R)$ to the work due to excess pressure per unit area of initial generating surface performed by the fluid initially contained in the cylinder of radius R on the fluid exterior to the cylinder, we obtain

$$2\pi K(R)dz_0=\int_{t_0(R)}^\infty p'\mathbf{u}'\cdot d\mathbf{A}'dt, \tag{14}$$

where $d\mathbf{A}$ is the Euler area element into which the Lagrange area element $2\pi r_0 dz_0 \mathbf{1}_r$ is transformed by the passage of the shock wave, and where $\mathbf{u}$ is the vector particle velocity and $\mathbf{1}_r$ the unit radial vector. The primed symbols denote quantities behind the shock front, unprimed symbols being reserved henceforth for quantities on the shock front. The integral is taken along a path of constant Lagrange coordinates r_0, z_0. Now $d\mathbf{A}=2\pi r ds\mathbf{n}$, where $\mathbf{n}$ is the normal to the vector $d\mathbf{s}$ into which $dz_0\mathbf{1}_z$ is transformed by the passage of the shock wave and $\mathbf{1}_z$ is the unit axial vector. Since

$$\mathbf{n}=\frac{\partial z}{\partial z_0}\frac{dz_0}{ds}\mathbf{1}_r-\frac{\partial r}{\partial z_0}\frac{dz_0}{ds}\mathbf{1}_z,$$

Eq. (14) can be written in the form,

$$K(R)=\int_{t_0(R)}^\infty [u_r'\partial z/\partial z_0 + (D-u_z')\partial r/\partial z_0]r'p'dt. \tag{15}$$

The energy-time integral can be expressed in reduced form

$$K(R)=Fp\nu,$$
$$F=[u_r\partial z/\partial z_0+(D-u_z)\partial r/\partial z_0]rp,$$
$$\frac{1}{\mu}=-(D\log F/Dt)_\sigma,$$
$$\nu=\int_0^\infty f(R,\tau)d\tau, \quad \tau=[t-t_0(R)]/\mu,$$
$$f(R,\tau)=F'/F. \tag{16}$$

The function $f(R, \tau)$ is the energy-time integrand, normalized by its peak value at the shock front, expressed as a function of R and a reduced time τ which normalizes its initial slope to -1 if μ does not vanish. Elimination of μ from Eq. (16) yields an additional relation, supplementing Eqs. (4) and (12) between the partial derivatives at the shock front. This set of equations is exact, involving integrals of Eqs. (4) for a knowledge of the reduced energy-time function $f(R, \tau)$. As in [I], we impose a similarity restraint on the energy-time curve by the assignment of a value independent of R to ν. The peak approximation, appropriate for an initial estimate of $f(\tau)$, leads to the value $\nu=1$. For the asymptotic quadratic energy-time curve, $\nu=2/3$. As a convenient empirical interpolation formula between the two extreme values, we have employed the relation

$$\nu=1-\tfrac{1}{3}\exp(-p/p_0). \tag{17}$$

Reference is made to [I] for a more detailed discussion of the similarity restraint on the energy-time curve.

In order to express $1/\mu$ in terms of partial derivatives at the shock front, use is made of Eq. (3) and of the identities,

$$\frac{D}{Dt}\left(\frac{\partial z}{\partial z_0}\right)=\frac{\partial u_z}{\partial z_0}=\frac{\partial u_z}{\partial z}\frac{\partial z}{\partial z_0}+\frac{\partial u_z}{\partial r}\frac{\partial r}{\partial z_0},$$
$$\frac{D}{Dt}\left(\frac{\partial r}{\partial z_0}\right)=\frac{\partial u_r}{\partial z_0}=\frac{\partial u_r}{\partial z}\frac{\partial z}{\partial z_0}+\frac{\partial u_r}{\partial r}\frac{\partial r}{\partial z_0}. \tag{18}$$

The components of the deformation-rotation tensor at the shock front are determined by the fact that an element of fluid experiences a pure strain of magnitude $\rho_0/\rho-1$ in a direction normal to the shock front as the result of the passage of the shock wave. Then

$$(\partial r/\partial r_0)_\sigma = 1 + (\rho_0/\rho - 1)\sin^2\vartheta,$$
$$(\partial z/\partial z_0)_\sigma = 1 + (\rho_0/\rho - 1)\cos^2\vartheta,$$
$$(\partial r/\partial z_0)_\sigma = (\partial z/\partial r_0)_\sigma = (1 - \rho_0/\rho)\sin\vartheta\cos\vartheta. \quad (19)$$

Equations (3), (16), (18), and (19), together with the Hugoniot equations, finally yield

$$\left(\frac{\partial p}{\partial r}\right)_\sigma + \frac{1-\lambda}{\lambda}\left(\frac{\partial p}{\partial z}\right)_\sigma \cot\vartheta + \rho_0 D\left(\frac{\partial u_r}{\partial r}\right)_\sigma \cot\vartheta + \frac{1-\lambda}{\lambda}\rho_0 D\left(\frac{\partial u_r}{\partial z}\right)_\sigma \cot^2\vartheta = -\frac{p}{R} - \frac{\nu R p^2}{K(R)}, \quad (20)$$

where $\lambda = p/\rho_0 D^2$, for the case $r/t = z/t = 0$. (r/t and z/t vanish at any finite distance from an infinite cylinder of explosive.)

Equations (4), specialized to the shock front and to the infinite cylinder, and Eqs. (12) and (20) constitute six relations between six partial derivatives, evaluated at the shock front. They may be solved for the partial derivatives, and an ordinary differential equation for the peak pressure p as a function of the radial distance R may be formulated with the aid of Eq. (11). A second ordinary differential equation relating K to R may be obtained by differentiation of Eq. (13). The resulting expressions may be written in the form,

$$dK/dR = -RL(p),$$
$$dp/dR = -\nu(Rp^3/K)M(p)\Phi(p, D) - (p/2R)N(p)\Phi(p, D), \quad (21)$$

where the functions $L(p)$, $M(p)$, $N(p)$ are identical with the expressions given in [I],

$$L(p) = \rho_0 h(p),$$
$$M(p) = \frac{1}{\rho_0 U^2}\frac{G}{2(1+g)-G},$$
$$N(p) = \frac{4(\rho_0/\rho) + 2(1-\rho_0/\rho)G}{2(1+g)-G},$$
$$G = 1 - (\rho_0 U/\rho c)^2, \quad g = 1 - p\,dU/U\,dp,$$

and where

$$\Phi(p, D) = \left[1 - \frac{\rho_0}{\rho}\frac{2(1-g)+G}{2(1-g)-G}\frac{U^2}{D^2 - U^2}\right]^{-1}.$$

The functions $L(p)$, $M(p)$, and $N(p)$ can be evaluated as functions of the pressure and $\Phi(p, D)$ can be evaluated as a function of pressure and detonation velocity by means of an equation of state of the fluid and the Hugoniot relations, Eqs. (2). We remark that Eqs. (21) are independent of any assumption regarding the equation of state of the fluid, that they take proper account of the finite entropy increment of the fluid produced by the passage of the shock, and that they permit the use of the exact Hugoniot curves of the fluid in their numerical integration.

It is evident from a consideration of the Hugoniot equations as applied to the detonation wave that the assumption of adiabatic isometric conversion of the explosive to its products corresponds to infinite detonation velocity. With instantaneous conversion, the dependence of the properties of the fluid on the axial coordinate vanishes and the shock wave becomes one-dimensional. We note that

$$\lim(D \to \infty)\Phi(p, D) = 1,$$

and in this limit Eqs. (21) become identical with the one-dimensional equations of [I] when the latter are written for the cylindrical case. Furthermore,

$$\lim(p \to 0)\Phi(p, D) = 1,$$

and the asymptotic solutions of Eqs. (21) are identical with those for the one-dimensional cylindrical wave, given in [I].

When $p(R)$ is known from the integration of Eqs. (21), the profile $\zeta(R)$ of the shock wave may be obtained by an auxiliary integration,

$$\zeta(R) = \int_{a_0}^{R}\left\{\frac{U[p(r_0)]^2}{D^2} - 1\right\}^{\frac{1}{2}} dr_0. \quad (22)$$

3. THE IMPULSE

The impulse I delivered by the shock wave at a point of fixed Euler coordinate r is

$$I = \int_{t_0(R)}^{\infty} p' dt, \quad (23)$$

along a path of constant r and constant $z-Dt$. If the excess pressure p' has a negative phase, the positive impulse is obtained if the time integral is extended not to infinity, but to the time at which the excess pressure vanishes. The pressure-time integral can be expressed in reduced form in a manner analogous to the reduction of Eq. (15).

$$I=\nu^* p\theta,$$

$$1/\theta=-(\partial \log p'/\partial t)_{r,\, z-Dt,\, t=t_0(R)},$$

$$\nu^*=\int_0^\infty (p'/p)d\tau^*,$$

with

$$\tau^*=(t-t_0(R))/\theta. \qquad (24)$$

The partial derivatives at the shock front were obtained as functions of the peak pressure by the solutions of Eqs. (4), (12), and (20). When these results are employed with the definition of θ, one obtains

$$-1/\theta=\left(\frac{D^2-U^2}{D^2}\right)\frac{U}{G}\times\left\{\frac{1}{R}+\left[\frac{\rho}{\rho_0}(1+g)+\left(1-\frac{\rho_0}{\rho}\right)G-(1-g)\left(\frac{U^2}{D^2-U^2}\right)\right]dp/pdR\right\}. \qquad (25)$$

We note that in the limit of infinite detonation velocity, Eq. (25) reduces to the expression for the one-dimensional cylindrical wave, given in [I]. For an exponential pressure-time curve, the reduced pressure-time integral ν^* is equal to unity, and for the asymptotic linear pressure-time curve, consistent with the asymptotic quadratic energy-time curve, $\nu^*=\frac{1}{2}$ for the positive phase of the wave. As in [I], we employ

$$\nu^*=1-\tfrac{1}{2}\exp(-p/p_0)^{\frac{1}{2}} \qquad (26)$$

as an empirical interpolation formula between the two values.

4. INITIAL CONDITIONS

The two constants of integration may be determined from the considerations given in [I]. The initial pressure p_1, on the generating surface, is determined by the fact that the Riemann r-function, computed normal to the generating surface, initially vanishes in the receding rarefaction wave. In the development of the theory, the rate of energy delivery has been approximated by an exponential function of time. For shock waves from explosive sources in air, it is assumed that the integral of this expression is equal to the total energy of explosion, since experimental evidence suggests that there is little energy available for second shocks. For shock waves in water, experimental evidence suggests that approximately one-half the energy of explosion is delivered to the first shock. The initial value of K is readily calculated from these considerations, and the disadvantages of the approximate nature of this procedure are minimized by the circumstance that except in the immediate vicinity of the explosive charge, the shock wave parameters are not very sensitive to the initial energy.