The Double Current Sheet in Diffraction

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(Received August 1, 1947)

The electromagnetic radiation from an aperture in a plane-conducting screen is identical with that which would exist if screen and aperture were replaced by a double current sheet fitting the aperture. The current density i is equal and opposite in the two layers of the sheet and is determined by the original tangential electric field E over the opening so that E is $-\frac{1}{2}j\omega\mu\delta i$. The distance δ between layers is infinitesimal. A rigorous expression for the vector potential of the emergent radiation in terms of E is given. This is the expression used by Bethe in finding the diffraction from small holes. It is shown to provide a simpler method of getting the results of Stratton and Chu for rectangular openings. It is then applied to the calculation of the excitation of a rectangular wave guide by a coaxial line termination in one side. An expression for the output resistance is given.

THERE is a simple but apparently little used method of setting up certain electromagnetic boundary value problems. It applies to the radiation from apertures in plane-conducting surfaces such as the wall of a rectangular wave guide. The uniqueness theorem[1a] for electromagnetic waves, when modified to fit the steady-state condition, asserts that an electromagnetic field is uniquely determined within an empty bounded region by the values of the tangential component of the electric vector (or of the magnetic vector) over the boundary. Evidently with conducting boundaries the electric field, whose tangential component over the screen is zero, must be used because the tangential magnetic field, which is not in general zero, is unknown. To use this theorem we need a source giving a tangential electric field that has any desired arbitrary value over an area S of an infinite plane and is zero over the remainder. Consider a thin plane, double current sheet in which the distance δ between layers is very small and the current densities in the two layers at any point of the sheet are equal and opposite. Such a sheet, with the thickness exaggerated, and with the dotted portion cut away along a line of flow is shown in Fig. 1(a). If the sheet is uniform in the direction of flow all the current passes around the edge, but if it is stronger in the center, as shown in Fig. 1(b), part will turn back before reaching the edge. Because the sheet is very thin the external magnetic induction is negligible compared with its value B_i between layers. Hence an application of the magnetomotance law to the rectangle $abcd$, which is normal to i and fits closely a section of the upper layer, shows[1b] B_i to be μi. The changing flux $N=B_i\delta dl$ through the rectangle $a'b'c'd'$ produces an electromotance $-dN/dt=\int \boldsymbol{E}\cdot d\boldsymbol{s}$ around it. As $\delta\rightarrow 0$ half of this appears along the side $a'b'$ so that the electric field strength E just above the sheet is $-\frac{1}{2}j\omega\mu\delta i$. Clearly a double current sheet giving any desired values of E can be built up out of infinitesimal solenoids of cross-sectional area δdl and length dc whose magnetic moment is $\boldsymbol{n}\times \boldsymbol{i}\delta dldc$ or $-2(j\omega\mu)^{-1}\boldsymbol{n}\times \boldsymbol{E}dS$. It is evident from symmetry that $\boldsymbol{E}$ is normal to the plane of the sheet outside the hole's boundaries. The vector potential[2] at P of the radiation from a small, oscillating current loop is normal to the loop axis and proportional to the sine of the angle between this axis and the radius vector $\boldsymbol{r}$ from the loop to P. Thus for the diffracted field, in m.K.S. units, the vector potential is

$$\boldsymbol{A}=\frac{1}{2\pi\omega}e^{j\omega t}\int_S \frac{(\beta r-j)(\boldsymbol{n}\times\boldsymbol{E})\times\boldsymbol{r}_1}{r^2}e^{-j\beta r}dS \quad (1)$$

where $\boldsymbol{r}_1$ is a unit vector in the direction of $\boldsymbol{r}$, β is the wave number and ω the angular frequency. The electric and magnetic fields are given by

$$\boldsymbol{E}=-j\omega\boldsymbol{A},\quad \boldsymbol{B}=\nabla\times\boldsymbol{A}. \quad (2)$$

[1a] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 487.

[1b] Throughout this paper m.K.S. units are used and $j=(-1)^{\frac{1}{2}}$.

[2] W. R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill Book Company, Inc., New York, 1939), p. 477.

If Φ is written for $r^{-1}e^{-j\beta r}$ then the factors multiplying $\boldsymbol{n}\times\boldsymbol{E}$ in the integrand can be replaced by $\nabla\Phi$. In this form the integral is seen to be twice one of those obtained by Stratton (reference 1, p. 466) by direct integration of the field equations. It is the only one used by Bethe[3] in his rigorous calculation of the diffraction from a small hole. When the correct value of $\boldsymbol{E}$ is used the expression is exact. This double current shell replaces the fictitious magnetic currents and charges used by Stratton.

The few cases in which the value of $\boldsymbol{E}$ over the aperture can be calculated rigorously are those in which at least one dimension of the aperture and the thickness of the screen are small compared with a wave-length. These include small circular holes in thin sheets and narrow slits. The first of these has been worked out by Bethe[3] who assumes that near the hole phase differences are negligible so static field values, multiplied by $e^{j\omega t}$, can be used. He shows that any incident wave can be broken up into an electric field E_z, normal to the sheet, and a magnetic induction B_x, tangent to it. Both static problems are very simple in the spheroidal coordinate system (reference 2, p. 156) ξ, ζ, ϕ in which the sheet is $\xi=0$ and the hole of radius c is $\zeta=0$. Positive values of ζ cover the space on the incident side and negative values that on the emergent side. The electric field component in the hole in the plane of the sheet due to E_z is radial, and is found from the potential function (reference 2, p. 160, Eq. 6) for a sheet bounding a uniform static field, thus

$$E_\rho=-\frac{1}{h_1}\frac{\partial V}{\partial \xi}\bigg|_{\zeta=0}=-\frac{Ec}{\pi h_1}\bigg|_{\zeta=0}=\frac{-E\rho}{\pi(c^2-\rho^2)^{\frac{1}{2}}}. \quad (3)$$

It is easily verified that a scalar magnetic potential Ω that gives a uniform induction B_x parallel to the sheet when ζ is $+\infty$, zero induction when ζ is $-\infty$, and makes $\boldsymbol{B}$ everywhere tangential to the sheet is

$$\begin{aligned}\mu\Omega&=B_xc[jP_1^1(j\zeta)+\pi^{-1}Q_1^1(j\zeta)]P_1^1(\xi)\cos\phi\\&=-(B_xx/\pi)[\tfrac{1}{2}\pi+\tan^{-1}\zeta+\zeta/(1+\zeta^2)].\end{aligned}$$

The normal component of $\boldsymbol{B}$ on the emergent side of the hole is

$$\frac{\mu\partial\Omega}{h_2\partial\zeta}\bigg|_{\zeta=0}=\frac{-2B_xx}{\pi(c^2-\rho^2)^{\frac{1}{2}}}=\frac{1}{j\omega}\frac{\partial E_y}{\partial x}.$$

This relation between B_x and E_y on the emergent face of the hole follows because $\partial E_y/\partial x$ equals the time derivative of the flux leakage $\partial N/\partial x$ or $-B_z$ in the double current sheet. Integration gives

$$E_y=-(2j\omega B_x/\pi)(c^2-\rho^2)^{\frac{1}{2}}. \quad (4)$$

Insertion of (3) and (4), suitably weighted according to polarization and incident angle, in (1) gives Bethe's results.

When the tangential electric field over the aperture is unknown, distant radiation patterns can usually be found with considerable accuracy by using its unperturbed value. This is surprising in view of Andrews'[4] measurements on circular apertures. Unlike Kirchhoff's method this makes no assumption that the magnetic and normal electric fields are unperturbed. Neither does it neglect the currents and charges on the emergent face of the screen. For these reasons it gives good results where Kirchhoff's formula breaks down at wave-lengths comparable with the aperture dimensions. An example proving this was worked out by Stratton and Chu[5] whose results compare favorably with such rigorous solutions as are known. Their results for the distant radiation from a rectangular aperture in a thin plane-conducting sheet, upon which a plane wave impinges at any angle, can be derived

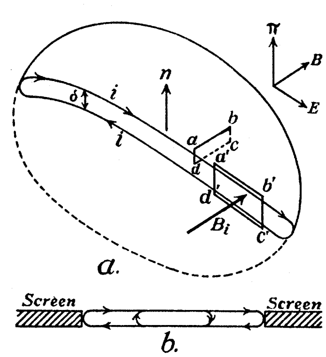

FIG. 1. Schematic diagram of the double current sheet.

[3] H. A. Bethe, Phys. Rev. 66, 163 (1944).

[4] C. L. Andrews, Phys. Rev. 71, 777 (1947).

[5] J. A. Stratton and L. J. Chu, Phys. Rev. 56, 106 (1939).

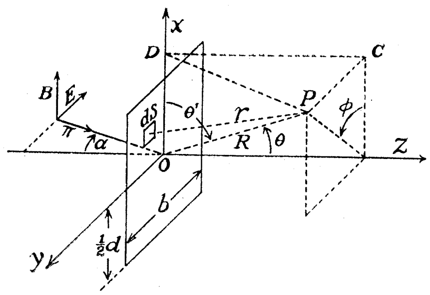

FIG. 2. Relations for an incident plane wave. (The notation $\frac{1}{2}d$ should read $\frac{1}{2}a$.)

much more simply by using the double current sheet.

Consider a plane wave whose electric vector lies in the plane of incidence and which is incident at any angle α on an aperture in the $z=0$ plane bounded by $x=\pm\frac{1}{2}a$ and $y=\pm\frac{1}{2}b$, as shown in Fig. 2. Let x_1 and y_1 be the coordinates of an area element dS in the aperture, and let x and y be those of the field point P at a distance z from the screen. Let $\boldsymbol{r}$ be the radius vector from dS to P and $\boldsymbol{R}$ that from O to P. Then if $R\gg a$ and $R\gg b$, we may take $\boldsymbol{r}$ parallel to $\boldsymbol{R}$ and

$$r\approx R-x_1\cos\phi\,\sin\theta-y_1\sin\phi\,\sin\theta.$$

The tangential component of $\boldsymbol{E}$ varies in phase at $z=0$, so that

$$E_t=E\cos\alpha\,\exp j(\omega t-y_1\sin\alpha).$$

Since $\boldsymbol{n}\times\boldsymbol{E}_t$ parallels the x axis, $(\boldsymbol{n}\times\boldsymbol{E}_t)\times\boldsymbol{R}$ is normal to DP and $\boldsymbol{R}$, parallel to the yz plane and proportional to $\sin\theta'$. Thus

$$(\boldsymbol{n}\times\boldsymbol{E}_t)\times\boldsymbol{r}_1=E_t(-\boldsymbol{j}\cos\theta+\boldsymbol{k}\sin\theta\,\sin\phi).$$

Substitution of these values in (1) gives A_y and A_z. The spherical components are given by the equations

$$A_\theta=A_y\sin\phi\,\cos\theta-A_z\sin\theta,\quad A_\phi=A_y\cos\phi,\quad A_r=0.$$

If we write

$$P=\beta\cos\phi\,\sin\theta,\quad Q=\beta(\sin\phi\,\sin\theta-\sin\alpha),$$

the resultant expression for A_θ is

$$\frac{-\beta E\cos\alpha\,\sin\phi}{2\pi\omega R}e^{j(\omega t-\beta R)}\int_{-\frac{1}{2}a}^{+\frac{1}{2}a}e^{jPx_1}dx_1\int_{-\frac{1}{2}b}^{+\frac{1}{2}b}e^{jQy_1}dy_1.$$

A straightforward integration gives

$$A_\theta=\frac{2\beta E\cos\alpha\,\sin\phi\,\sin(\frac{1}{2}Pa)\,\sin(\frac{1}{2}Qb)}{\pi\omega RPQ}e^{j(\omega t-\beta R)},\qquad(5)$$

$$A_\phi=-A_\theta\cot\phi\,\cos\theta.\qquad(6)$$

Stratton and Chu[5] obtained this formula by quite a different method which involved the superposition or "reflection" of two solutions of Maxwell's equations, each of which assumed unperturbed electric and magnetic fields over the openings. This was necessary to eliminate the tangential electric field over the screen, and is exactly equivalent to discarding the three integrals derived from the tangential magnetic field and doubling the remaining integral. Six figures[5] comparing their formulas with the rigorous ones of Morse and Rubenstein[6] for a slit

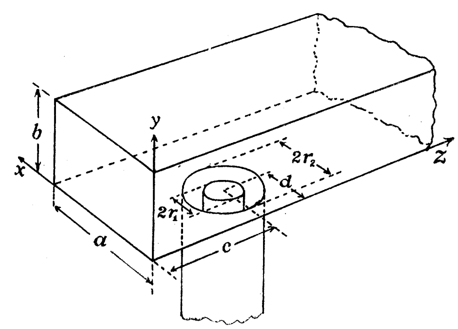

FIG. 3. Coaxial line attached to a rectangular wave guide.

show that the initial assumption of an unperturbed tangential electric field gives surprisingly accurate results. They also show the total failure of Kirchhoff's formula, which ignores polarization when the slit width is much less than a wave-length.

It has been shown that the double current sheet yields correct known solutions; it can now be applied to a new problem. Suppose that the propagation space of a coaxial line, which has internal and external radii r_1 and r_2, terminates in an annular opening whose center is at $x=d$, $y=0$, $z=c$ in the bottom of a rectangular wave guide closed at $z=0$. The walls of the guide are at $x=0$, $x=a$, $y=0$, and $y=b$ and the frequency is such that only the TE_{10} mode is transmitted. The arrangement is shown in Fig. 3. At some

[6] P. M. Morse and Pearl J. Rubenstein, Phys. Rev. **54**, 898 (1938).

positive value of z the transmitted wave is completely absorbed and we desire to find the output resistance of the coaxial line. To make E_y zero when z is zero requires two double current sheets of opposite sign, one at $z=c$ and one at $z=-c$. Each consists of a flat torus in the xz plane of a guide bounded by $x=0$, $x=a$, $y=-b$, and $y=b$. To calculate the excitation of the TE_{10} mode, whose electric field is everywhere parallel to the y axis, the thickness δ of the torus is exaggerated as shown in Fig. 4. The exact value of E over the annular opening is unknown but the field of the principal coaxial mode, which carries the energy and is the only field far from the opening, is inversely proportional to r. If the potential of the center conductor is V and the current density in the double layer in amperes per radian is i, then

$$E=\tfrac{1}{2}j\omega\mu i\delta/r=V/[r\ln(r_2/r_1)],$$

so that

$$i=2V/[j\omega\mu\delta\ln(r_2/r_1)]. \qquad (7)$$

Thus all currents encircle the entire section of the torus. The only current elements that excite the TE_{10} mode are parallel to the y axis and so are those of length δ that form the curved edges of the torus. The TE_{10} term, A_{10}, in the vector potential of the waves excited in an a by $2b$ guide, extending from $z=-\infty$ to $z=+\infty$, by a current element Idy_0 at $x=x_0$, $z=z_0$, is in the y direction.

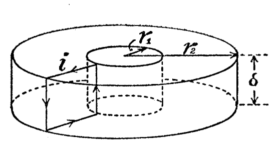

FIG. 4. Form of the toroidal current sheet for the wave guide.

$$A_{10}=(\mu Idy_0/2\beta' ab)\sin(\pi x_0/a)\times\sin(\pi x/a)\exp j[\omega t-\beta'(z-z_0)],$$

where

$$\beta'^2=\omega^2\mu\epsilon-\pi^2/a^2=4\pi^2/\lambda_{10}{}^2,$$

and λ_{10} is the guide wave-length. This is easily verified by multiplication of B_x by $\sin(\pi x/a)dxdy$ and integration over one face of the z_0 plane. The integral vanishes except over the element where its value is $\tfrac{1}{2}\mu Idy_0\sin(\pi x_0/a)$ by the magnetomotance law. We shall calculate first the contribution from four similarly directed current elements of width $d\theta$, whose x and z coordinates are $d\pm r_2\sin\theta$ and $c\pm r_2\cos\theta$, as shown by crosses in Fig. 5. The resultant dA_{10} is

$$[2\mu i\delta/(\beta' ab)]\sin(\pi x/a)\,\{\exp j[\omega t-\beta'(z-c)]\}\times\sin(\pi d/a)f(\theta)d\theta,$$

where

$$f(\theta)=\cos[(\pi r_2/a)\sin\theta]\cos(\beta' r_2\cos\theta).$$

If we write βr_2 for $(\pi^2/a^2+\beta'^2)^{\frac{1}{2}}r_2$, where $\beta^2=\omega^2\mu\epsilon$, and $\tan\phi$ for $\pi/(\beta' a)$, the integral of the θ terms takes the form

$$\tfrac{1}{2}\int_0^{\frac{1}{2}\pi}\{\cos[\beta r_2\cos(\theta-\phi)]+\cos[\beta r_2\cos(\theta+\phi)]\}d\theta.$$

Both terms in the integrand have a period of $\tfrac{1}{2}\pi$ so that the value of the integral is independent of ϕ. By choosing $\phi=0$ the integral takes the standard Bessel integral form

$$\tfrac{1}{2}\int_0^{\pi}\cos(\beta r_2\cos\theta)d\theta=\tfrac{1}{2}\pi J_0(\beta r_2)=\tfrac{1}{2}\pi J_0(2\pi r_2/\lambda),$$

where λ is the free-space wave-length. Addition of a similar expression for the currents at the inner edge of the torus and for the edges of the image sheet and substitution for i from (7) give

$$A_{10}=C\sin(\pi x/a)\exp j(\omega t-\beta' z), \qquad (8)$$

where

$$C=\frac{4\pi V\sin(\pi d/a)\sin(\beta' c)}{j\omega\beta' ab\ln(r_2/r_1)}\times[J_0(2\pi r_2/\lambda)-J_0(2\pi r_1/\lambda)].$$

The TE_{10} electric field and magnetic induction are

$$B_x=-\partial A_{10}/\partial z=j\beta' A_{10},\quad E_y=-j\omega A_{10}.$$

The mean Poynting vector $\langle\Pi_z\rangle_{\text{Av}}$ is

$$\langle\Pi_z\rangle_{\text{Av}}=\tfrac{1}{2}EB^*/\mu=(\tfrac{1}{2}\omega\beta'/\mu)\,|C|^2\sin^2(\pi x/a). \qquad (9)$$

The upper half of the tube, from $y=0$ to $y=b$,

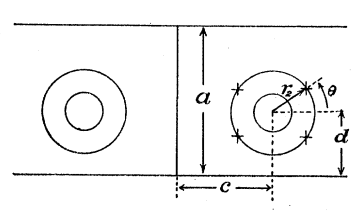

FIG. 5. Arrangement of associated current elements.

satisfies the boundary conditions of the original guide of Fig. 3, so that integration of $\langle \Pi_z \rangle_{Av}$ over its area gives the rate of power dissipation or $\frac{1}{2}V^2/R$ where R is the radiation resistance. Thus we find

$$R = \frac{\mu\omega\beta' ab \ln^2(r_2/r_1)}{8\pi^2 \sin^2(\pi d/a)\sin^2\beta' c} \times [J_0(2\pi r_2/\lambda) - J_0(2\pi r_1/\lambda)]^{-2}. \quad (10)$$

The same result is obtained by the integration of Poynting's vector over the coaxial opening, using $r_1 V[r \ln(r_2/r_1)]^{-1}$ for $\mathbf{E}$ and $\nabla \times \mathbf{A}_{10}$ for $\mathbf{B}$, but the integration is a little more difficult. The integration of Poynting's vector for the other non-transmitted modes over the annular opening should give the output reactance of the line. Unfortunately it does not seem possible to evaluate the resultant integrals analytically.

PHYSICAL REVIEW VOLUME 72, NUMBER 11 DECEMBER 1, 1947

X-Ray Emitting Isotopes of Radioactive Sb and Sn

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(Received August 8, 1947)

Three new x-ray emitting activities in Sb with half-lives of 2.8 hours, 5.1 hours and 39 hours have been found by the use of the curved crystal camera in conjunction with the x-ray decay curves. The assignments are Sb^{117}, Sb^{118}, and Sb^{119} respectively. All three decay by K-electron capture giving characteristic Sn x-rays. The 2.8-hour Sb^{117} also emits 0.46-Mev electrons while the 5.1-hour Sb^{118} gives 0.20-Mev electrons and 1.5-Mev gamma-rays.

X-ray periods of 1.25 days and 9 days have been found in Sn. The reported decay scheme for Sn^{113} was verified but the 0.085-Mev gamma-ray was not found.

1. INTRODUCTION

Of the several activities which have been attributed to radioactive Sb isotopes, those of half-lives of 2.8 days,[1] 60 days[2] and 17 minutes[1] have been assigned to Sb^{122}, Sb^{124} and Sb^{120}, respectively. The first two activities emit beta-rays and gamma-rays, while the third emits positrons. These three periods have been made by four different reactions.[3]

A 2.7-year activity, emitting low energy electrons and gamma-rays, has been obtained from fission,[4] and tentatively assigned to Sb^{125}. A 3.6-minute positron activity obtained from In$+\alpha$ activation[5] has been assigned to either Sb^{116} or Sb^{118}. In addition, several Sb activities are found in fission fragments[4] with masses greater than 125.

Approximately fifteen activities have been reported as belonging to Sn isotopes.[4,6] Because of the large number of stable isotopes and because of the limited possibilities for making these activities by use of different bombarding particles and target materials, the only one which has been assigned with certainty is the 105-day[7] activity of Sn^{113}.

The present investigation was undertaken to study the application of the Cauchois curved-crystal camera, to the x-ray activities induced in Sn when activated with 10-Mev deuterons. Bombardments were also made using 5-Mev protons on Sn and 20-Mev alpha particles on In.

2. EXPERIMENTAL PROCEDURE

Characteristic x-ray lines were photographed by using a pair of Cauchois cameras[8] equipped

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