Fluctuations of the Number of Neutrons in a Pile[1]

E. D. COURANT[2] AND P. R. WALLACE[3]
Chalk River Laboratory, Chalk River, Ontario, Canada

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Statistical fluctuations in the number of neutrons in a pile which is just under critical and contains distributed neutron sources such as spontaneous fissions, $\alpha-n$ reactions, etc., are considered for the following two cases: (a) The delayed neutrons are taken into account, but the pile is assumed to be operating at a steady power, and (b), the power at which the pile operates is assumed to change periodically, but the delayed neutrons are neglected.

For both cases the expression for the standard deviation of the number of neutrons is derived, and in case (a) the expression for the coefficient of correlation between the number of neutrons and the number of excited nuclei of each particular type, which lead to the formation of the delayed neutrons, is also given.

The method used throughout the calculations is that of the probability generating function.

For the case of a steadily operating pile, the effect of a finite resolving time of the recording instrument upon the observed value of the standard deviation of the number of neutrons is also determined through a study of the spectrum of the fluctuations.

I. INTRODUCTION

We consider a chain-reacting pile in which there is a source of neutrons (spontaneous fissions, $\alpha-n$ reactions, cosmic rays, etc.). Let N be the number of neutrons present in the pile, and let $Q_1, Q_2, Q_3 \cdots Q_m$ be the number of radioactive fission-product nuclei of decay constants $a_1, a_2 \cdots a_m$, respectively, capable of giving off delayed neutrons. The time-dependent equations for N and Q_i are

$$dN/dt = S-(1-k'/\tau)N+\sum_{i=1}^{m} a_iQ_i, \quad (1.1)$$

$$(dQ_i/dt) = (c_i/\tau)N - a_iQ_i. \quad (1.2)$$

Here S = the number of neutrons emitted by the source per unit time,

k' = the mean number of neutrons formed instantaneously per neutron lost in the pile through absorption or escape,

c_i = the mean number of radioactive nuclei of type i formed per neutron lost,

τ = the mean lifetime of a neutron in the pile.

It is also convenient to define

$c=\sum_{i=1}^{m} c_i$ = the mean number of delayed neutrons formed per neutron lost,

$k = k'+c$ = the effective multiplication factor of the system.

Equations (1.1) and (1.2) would be satisfied exactly if neutron capture and production were continuous processes. Since they are, in fact, discrete elementary processes occurring at random, the rates dN/dt and dQ_i/dt given by (1.1) and (1.2) are only *average* rates, and the true rates will fluctuate about these average rates. As a result, the actual values of N and Q_i at any given time will not be given exactly by the solution of (1.1) and (1.2), but will fluctuate at random about these solutions. It is the object of this paper to estimate the magnitude of these fluctuations in a number of cases.

Let $P(N, Q_1, Q_2, \cdots; t)$ be the probability that N neutrons, Q_1 nuclei of type 1, Q_2 nuclei of type 2, etc., are present in the system at time t. The average number of neutrons is

$$\langle N\rangle_{\text{Av}} = \sum_N \sum_{Q_1} \sum_{Q_2} \cdots NP(N, Q_1, Q_2 \cdots; t), \quad (1.3)$$

and the average number of nuclei of type i is

$$\langle Q_i\rangle_{\text{Av}} = \sum_N \sum_{Q_1} \sum_{Q_2} \cdots Q_iP(N, Q_1, Q_2, \cdots; t). \quad (1.4)$$

A measure of the fluctuations in N is given by the standard deviation

$$\varphi_{NN} = \langle N^2\rangle_{\text{Av}} - \langle N\rangle_{\text{Av}}^2 = \sum_{N, Q_i} N^2P(N, Q_i; t) - \Big[\sum_{N, Q_i} NP(N, Q_i; t)\Big]^2. \quad (1.5)$$

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[2] Now at Cornell University, Ithaca, New York.

[3] Now at McGill University, Montreal, Quebec.

We also define

$$\varphi_{Nj}=\langle NQ_j\rangle_{Av}-\langle N\rangle_{Av}\langle Q_j\rangle_{Av}$$
$$=\sum_{N,Q_i} NQ_jP(N,Q_i;t)$$
$$-[\sum_{N,Q_i} NP(N,Q_i;t)]$$
$$\times[\sum_{N,Q_i} Q_jP(N,Q_i;t)], \quad (1.6)$$

$$\varphi_{jk}=\langle Q_jQ_k\rangle_{Av}-\langle Q_j\rangle_{Av}\langle Q_k\rangle_{Av}$$
$$=\sum_{N,Q_i} Q_jQ_kP(N,Q_i;t)$$
$$-[\sum_{N,Q_i} Q_jP(N,Q_i;t)][\sum_{N,Q_i} Q_kP(N,Q_i;t)]. \quad (1.7)$$

The problem is to obtain values for $\langle N\rangle_{Av}$, $\langle Q_j\rangle_{Av}$, φ_{NN}, φ_{Nj}, and φ_{jk}. To do this we define the "probability generating function"

$$F(x,y_1,y_2,\cdots;t)=\sum_N\sum_{Q_1}\sum_{Q_2}\cdots$$
$$\times x^Ny_1^{Q_1}y_2^{Q_2}\cdots P(N,Q_1,Q_2,\cdots;t). \quad (1.8)$$

This function has the property that

$$F|_{x=y_i=1}=1, \quad (1.9)$$

$$\langle N\rangle_{Av}=x\partial F/\partial x|_{x=y_i=1}, \quad (1.10)$$

$$\langle Q_j\rangle_{Av}=y_j\partial F/\partial y_j|_{x=y_i=1}, \quad (1.11)$$

$$\varphi_{NN}=[(x\partial/\partial x)^2F-(x\partial F/\partial x)^2]_{x=y_i=1}, \quad (1.12)$$

$$\varphi_{Nj}=[x(\partial/\partial x)y_j(\partial/\partial y_j)F$$
$$-xy_j\partial F/\partial x\cdot\partial F/\partial y_j]_{x=y_i=1}, \quad (1.13)$$

$$\varphi_{jk}=[y_j(\partial/\partial y_j)y_k(\partial/\partial y_k)F$$
$$-y_jy_k\partial F/\partial y_j\cdot\partial F/\partial y_k]_{x=y_i=1}, \quad (1.14)$$

as is easily verified by differentiation of (1.8), since

$$\sum_{N,Q_i} P(N,Q_i;t)=1.$$

In the next section, a partial differential equation for F will be derived. From this equation it is possible to obtain expressions for the derivatives of F when x and all the y_i are equal to unity, and from Eqs. (1.9) to (1.14) we can then obtain the mean values and mean deviation of N and Q_i.

II. DIFFERENTIAL EQUATION FOR THE GENERATING FUNCTION

We wish to establish equations for the rates of variation of the probabilities $P(N,Q_1,Q_2,\cdots;t)$ and, consequently, for the probability generating function F defined by (1.8). To do this, let us suppose that we know the probabilities at time t, and let us try to determine what they become at time $t+dt$, i.e., after an infinitesimal time interval.

The probability that in such an interval *two* fundamental processes occur is $O(dt^2)$; we need only consider, therefore, the effect of a single process (emission by the source, capture with possible fission, or emission by excited fission-product nuclei).

Let us introduce the function

$$f(x,y_1,y_2\cdots)=\sum_{n,m_i} p_{nm_1m_2}\cdots x^ny_1^{m_1},y_2^{m_2}\cdots, \quad (2.1)$$

in which $p_{nm_1m_2}\cdots$ is the probability that a capture process will lead to formation of n instantaneous fission neutrons, and m_i radioactive fission-product nuclei of type i. $p_{nm_1m_2}\cdots$ will certainly be zero for $\sum m_i>2$, and perhaps even for $\sum m_i>1$, i.e., at most one or two delayed neutrons are emitted in one fission. We note the identity

$$f|_{x=y_i=1}=1.$$

The multiplication factor (i.e., the average number of neutrons released in the system per neutron lost through escape or absorption) is

$$k=\sum_{nm}(n+\sum_i m_i)p_{nm_1m_2}\cdots$$
$$=(x\partial/\partial x+\sum y_i\partial/\partial y_i)f|_{x,y_i=1}. \quad (2.2)$$

The mean square of the number of neutrons formed per neutron lost is

$$k_2=\sum_{mn}[n+\sum_i m_i]^2p_{nm_1m_2}\cdots$$
$$=[x/\partial x+\sum y_i\partial/\partial y_i]^2f|_{x,y_i=1}. \quad (2.3)$$

We may then write

$$k_2-k=\frac{\partial^2f}{\partial x^2}+2\sum_i\frac{\partial^2f}{\partial x\partial y_i}+\sum_{i,j}\frac{\partial^2f}{\partial y_i\partial y_j}. \quad (2.4)$$

In the notation of paragraph 1, we may break k down into its component parts:

$$k'=\partial f/\partial x|_{x,y_i=1} \quad (2.5)$$

due to instantaneous neutrons, and

$$c_i=\partial f/\partial y_i|_{x,y_i=1} \quad (2.6)$$

due to delayed neutrons of class i, where

$$k=k'+\sum c_i=k'+c. \qquad (2.7)$$

Let us also introduce the abbreviation

$$c'=[2\sum \partial^2 f/\partial x\partial y_i+\sum_{ij} \partial^2 f/\partial y_i\partial y_j]_{x=y_i=1}, \quad (2.8)$$

so that

$$k_2-k=\partial^2 f/\partial x^2+c'. \qquad (2.9)$$

For the source emission, we may introduce a similar function,

$$g(x, y_1, y_2\cdots)=\sum q_{nm_1m_2\cdots}x^n y_1^{m_1}y_2^{m_2}\cdots, \quad (2.10)$$

where the $q_{nm_1m_2\cdots}$ are source emission probabilities similar to the $p_{nm_1m_2\cdots}$. We shall, however, for simplicity, assume that the source produces only one neutron at a time, i.e., that $g(x, y_i\cdots)=x$. It can be shown that this assumption does not seriously affect any of the results of this paper.

Given the probabilities for different numbers of neutrons and radioactive fission-product nuclei at time t, what is the probability that there are N neutrons, Q_i nuclei of type i, at time $t+dt$? It is made up of four terms:

(i) the probability that there are $(N-1)$ neutrons, Q_i nuclei of type i at time t, multiplied by the probability that a source emission gives rise to one neutron in time dt. This is

$$P(N-1, Q_1, Q_2, \cdots; t)Sdt.$$

(ii) the probability that there are $(N-n+1)$ neutrons, (Q_i-m_i) nuclei at time t, multiplied by the probability that a neutron loss gives rise to the production of n neutrons and m_i nuclei in time dt. This is

$$\sum\nolimits_{n,m_i} P(N-n+1, Q_1-m_1, Q_2-m_2, \cdots; t) \cdot\frac{(N-n+1)dt}{\tau}\cdot p_{nm_1m_2\cdots}$$

(iii) the probability that there are $(N-1)$ neutrons, (Q_I+1) nuclei of a particular type I and $Q_i(i\neq I)$ of all others, multiplied by the probability that a nucleus of type I decays in time dt. This is

$$\sum\nolimits_I P(N-1, Q_1, \cdots Q_I+1, \cdots; t)(Q_I+1)a_I dt,$$

where a_I is the decay exponent of radioactive nuclei of type I.

(iv) the probability that no source emission, neutron loss, or radioactive decay takes place in time dt, multiplied by the probability that there were at the beginning of this time N neutrons, Q_i nuclei. This is

$$\left(1-Sdt-N\frac{dt}{\tau}-\sum a_I Q_I dt\right)P(N, Q_1, Q_2\cdots; t).$$

Combining the above results we get

$$\begin{aligned}P(N, Q_1, Q_2\cdots; t+dt)&=P(N-1, Q_1, Q_2, \cdots; t)Sdt\\&+\sum_{nm_i} P(N-n+1, Q_1-m_1, Q_2-m_2, \cdots; t)\\&\times((N-n+1)dt/\tau)p_{nm_1m_2\cdots}\\&+\sum_I P(N-1, Q_1, \cdots, Q_I+1, \cdots; t)a_I(Q_I+1)dt\\&+[1-Sdt-N(dt/\tau)-\sum_I a_I Q_I dt]\\&\times P(N, Q_1, Q_2, \cdots; t).\end{aligned}$$

Let us now multiply both sides of the above by

$$x^N y_1^{Q_1} y_2^{Q_2}\cdots,$$

and sum over N, Q_1, Q_2, $\cdots$. The left-hand side becomes $F(t+dt)$, and the terms on the right become, respectively,

$$SdtxF, \quad \frac{dt}{\tau}f(x, y_i)\partial F/\partial x, \quad x\sum_i a_i\partial F/\partial y_i,$$

and

$$\left[F(t)-SFdt-\frac{dt}{\tau}x\frac{\partial F}{\partial x}-\sum_i a_i y_i\frac{\partial F}{\partial y_i}dt\right].$$

By equating the two sides, dividing by dt, and letting $dt\to 0$, we get

$$\partial F/\partial t=(x-1)SF+\tau^{-1}(f(x, y_i)-x)\partial F/\partial x+\sum_i a_i(x-y_i)\partial F/\partial y_i. \quad (2.11)$$

It should be noted that the derivation of this equation does not demand the assumption of the constancy in time of S, $f(x, y_i)$, or τ. We shall, in fact, in later sections, deal with cases in which some of these quantities vary with time.

From the equation for F, it is possible, by differentiating successively and putting x, $y_i=1$, to obtain equations for $\langle N\rangle_{\text{Av}}$, $\langle Q_i\rangle_{\text{Av}}$, φ_{NN}, φ_{Nj}, φ_{ij}, and also, if desired, the higher moments of the probability distribution. In simple particular cases it may even be possible to solve Eq. (2.11) completely for F.

III. STEADILY OPERATING PILE

In the case of a pile operating at a constant mean intensity, all the probabilities $P(N, Q_i; t)$ are independent of time, and therefore

$$\partial F/\partial t=0.$$

The differential equation (2.11) thus becomes

$$S(x-1)F+\tau^{-1}[f(x, y_i)-x]F_x+\sum_{i=1}^{m} a_i(x-y_i)F_i=0, \quad (3.1)$$

where the subscripts x, i indicate differentiation with respect to x, y_i, respectively.

Differentiating (3.1) in turn by x and each of the y_i, adding the resulting equations, and setting $x=y_i=1$, we obtain, with the help of (1.10), (2.2), and (2.6),

$$\begin{aligned}\langle N\rangle_{\text{Av}}&=F_x=S\tau/(1-k)\,;\\ \langle Q_i\rangle_{\text{Av}}&=F_i=c_iS\tau/\epsilon_i(1-k),\end{aligned} \quad (3.2)$$

where $\epsilon_i=a_i\tau$.

Differentiating (3.1), in turn, by all sets of two of the variables x and y_i and setting $x=y_i=1$, we obtain the following set of equations for the moments φ_{NN}, φ_{Ni}, φ_{ik}:

$$(1-k+c)\varphi_{NN}-\sum \epsilon_i\varphi_{Ni}=[\tfrac{1}{2}f_{xx}+(1-k+c)]\langle N\rangle_{\text{Av}}, \quad (3.3)$$

$$-c_i\varphi_{NN}+(1-k+c+\epsilon_i)\varphi_{Ni}-\sum_j \epsilon_j\varphi_{ij}=(f_{xi}-2c_i)\langle N\rangle_{\text{Av}}, \quad (3.4)$$

$$-c_i\varphi_{Nk}-c_k\varphi_{Ni}+(\epsilon_i+\epsilon_k)\varphi_{ik}=(f_{ik}+2c_i\delta_{ik})\langle N\rangle_{\text{Av}}. \quad (3.5)$$

We shall solve these equations for the case when there are no delayed neutrons, for one delayed-neutron period, and for many delayed-neutron periods.

A. No Delayed Neutrons

When the delayed neutrons are neglected, Eqs. (3.3) to (3.5) reduce to the single equation

$$(1-k)\varphi_{NN}=[\tfrac{1}{2}(k_2-k)+(1-k)]\langle N\rangle_{\text{Av}}, \quad (3.6)$$

so that

$$\begin{aligned}\varphi_{NN}&=[1+(k_2-k)/2(1-k)]\langle N\rangle_{\text{Av}}\\ &=[(k_2-k)/2S\tau]\langle N\rangle_{\text{Av}}^2+\langle N_{\text{Av}}\rangle. \end{aligned}\quad (3.7)$$

Thus when the power level is changed by varying k, the mean square fluctuations are essentially proportional to the square of the number of neutrons; in other words the mean fractional deviation of the number of neutrons from the mean is independent of the number. We shall see in the following sections how this conclusion is modified when the delayed neutrons are taken into account.

B. One Delayed-Neutron Period

Let us assume that only one type of fission-product nucleus gives rise to delayed neutrons. Equations (3.3) to (3.5) then become

$$(1-k+c)\varphi_{NN}-\epsilon\varphi_{NQ}=[\tfrac{1}{2}f_{xx}+(1-k+c)]\langle N\rangle_{\text{Av}}, \quad (3.8)$$

$$-c\varphi_{NN}+(1-k+c+\epsilon)\varphi_{NQ}-\epsilon\varphi_{QQ}=(f_{xy}-2c)\langle N\rangle_{\text{Av}}, \quad (3.9)$$

$$-c\varphi_{NQ}+\epsilon\varphi_{QQ}=(c+\tfrac{1}{2}f_{yy})\langle N\rangle_{\text{Av}}. \quad (3.10)$$

Adding (3.8), (3.9), and (3.10), we obtain

$$(1-k)(\varphi_{NN}+\varphi_{NQ})=[\tfrac{1}{2}(k_2-k)+(1-k)]\langle N\rangle_{\text{Av}}. \quad (3.11)$$

Solving (3.8), (3.10), and (3.11), we find

$$\varphi_{NN}=\langle N\rangle_{\text{Av}}\left[1+\frac{(1-k+\epsilon)(k_2-k)}{2(1-k)(1-k+\epsilon+c)}-\frac{\tfrac{1}{2}c'}{1-k+\epsilon+c}\right], \quad (3.12)$$

$$\varphi_{NQ}=\langle N\rangle_{\text{Av}}\left[\frac{c(k_2-k)}{2(1-k)(1-k+\epsilon+c)}+\frac{\tfrac{1}{2}c'}{1-k+\epsilon+c}\right], \quad (3.13)$$

$$\varphi_{QQ}=(\langle N\rangle_{\text{Av}}/\epsilon)\left[\frac{c^2(k_2-k)}{2(1-k)(1-k+\epsilon+c)}+\frac{\tfrac{1}{2}cc'}{1-k+\epsilon+c}+c^2+\tfrac{1}{2}cf_{yy}\right]. \quad (3.14)$$

In practice the second term in the bracket of (3.12) will be large compared to the first and third, so that we have, approximately,

$$\varphi_{NN}=\langle N\rangle_{\text{Av}}\frac{(1-k+\epsilon)(k_2-k)}{2(1-k)(1-k+\epsilon+c)}$$
$$=\langle N\rangle_{\text{Av}}{}^2\frac{(k_2-k)(1-k+\epsilon)}{2S_T(1-k+\epsilon+c)}. \quad (3.15)$$

When $1-k\gg c$ and $1-k\gg\epsilon$, this is identical with the expression (3.7) for the case in which there are no delayed neutrons. However, when the number of neutrons is increased by bringing k closer to unity, (3.15) increases more slowly than (3.7), i.e., the delayed neutrons tend to smooth out the fluctuations.

In practice ϵ will be much smaller than c: ($c\approx 0.01$; $\epsilon=\tau/$(lifetime of delayed-neutron emitters) $\approx 10^{-3}$ sec./10 sec. $=10^{-4}$). There will, therefore, be a range of values of k for which $\epsilon\ll(1-k)\ll c$, so that

$$\varphi_{NN}\approx((k_2-k)/2c)\langle N\rangle_{\text{Av}}, \quad (3.16)$$

i.e., in this range the fluctuations increase with the square root of $\langle N\rangle_{\text{Av}}$. Finally, if $1-k\gg\epsilon$, (3.15) becomes

$$\varphi_{NN}=\langle N\rangle_{\text{Av}}{}^2\frac{(k_2-k)\epsilon}{2S_T(c+\epsilon)}, \quad (3.17)$$

i.e., the fluctuations are again proportional to $\langle N\rangle_{\text{Av}}$, but with a smaller constant of proportionality than in the case of no delayed neutrons.

C. Several Delayed-Neutron Groups

When there are m distinct groups of delayed neutrons, the equations to be solved are (3.3), the m equations (3.4), and the $\frac{1}{2}m(m+1)$ equations (3.5). These equations determine the $\frac{1}{2}(m+1)(m+2)$ quantities φ_{NN}, φ_{Ni}, and φ_{ik}. Adding (3.3), all Eq. (3.4), and all Eq. (3.5) multiplied by $\frac{1}{2}$, we obtain

$$(1-k)(\varphi_{NN}+\sum_i\varphi_{Ni})=\langle N\rangle_{\text{Av}}[\tfrac{1}{2}(k_2-k)+(1-k)]. \quad (3.18)$$

Let us define ϵ^* by

$$\sum_i\epsilon_i\varphi_{Ni}=\epsilon^*\sum_i\varphi_{Ni}. \quad (3.19)$$

Equations (3.3) and (3.18) for φ_{NN} and $\sum\varphi_{Ni}$ are now exactly similar to Eqs. (3.8) and (3.11) for φ_{NN} and φ_{NQ}, with ϵ^* taking the place of ϵ. The solutions are, therefore,

$$\varphi_{NN}=\langle N\rangle_{\text{Av}}\left[1+\frac{(1-k+\epsilon^*)(k_2-k)}{2(1-k)(1-k+c+\epsilon^*)}-\frac{c'}{2(1-k+c+\epsilon^*)}\right], \quad (3.20)$$

$$\sum\varphi_{Ni}=\langle N\rangle_{\text{Av}}\left[\frac{c(k_2-k)}{2(1-k)(1-k+c+\epsilon^*)}+\frac{c'}{2(1-k+c+\epsilon^*)}\right]. \quad (3.21)$$

The problem is to find ϵ^*. To do this, in other words to determine the proper weight function to be used in averaging the ϵ_i, we have to solve Eqs. (3.3) to (3.5) numerically. This can be done if the number of delayed-neutron periods is not too great.

In the special case where $1-k$ is small enough, it is possible to obtain approximate explicit expressions for the φ's.

We first solve (3.5) for φ_{ik}, obtaining

$$\varphi_{ik}=[c_i\varphi_{Nk}+c_k\varphi_{Ni}+\langle N\rangle_{\text{Av}}(f_{ik}+2c_i\delta_{ik})]/(\epsilon_i+\epsilon_k). \quad (3.22)$$

Equation (3.4) becomes

$$(1-k+\epsilon_i)\varphi_{Ni}-\sum_k\frac{\epsilon_k c_i\varphi_{Nk}-\epsilon_i c_k\varphi_{Ni}}{\epsilon_i+\epsilon_k}$$
$$=c_i\varphi_{NN}+\langle N\rangle_{\text{Av}}[f_{xi}-c_i+\sum_i\epsilon_k f_{ik}/(\epsilon_i+\epsilon_k)]. \quad (3.23)$$

Let us now assume

$$(1-k)/\epsilon_i=\mu_i\ll 1. \quad (3.24)$$

Since $\varphi_{NN}\gg\langle N\rangle_{\text{Av}}$ and $\varphi_{NN}\gg S_T$, the right-hand side of (3.23) is approximately equal to $c_i\varphi_{NN}$. Let us assume for a moment that the summation on the left-hand side may be neglected. Equation (3.23) then becomes, approximately,

$$\varphi_{Ni}\approx[c_i/\epsilon_i(1+\mu_i)]\varphi_{NN}\approx(c_i/\epsilon_i)\varphi_{NN}(1-\mu_i). \quad (3.25)$$

We now substitute this into the summation to see whether the assumption that the summation may be neglected is justified. We find

$$\sum_j (\epsilon_j c_i \varphi_{Nj} - \epsilon_i c_j \varphi_{Ni})/(\epsilon_i+\epsilon_j) = c_i \varphi_{NN} \sum_j (\mu_i-\mu_j) c_j/(\epsilon_i+\epsilon_j). \quad (3.26)$$

Our assumption is justified if

$$|\sum_j (\mu_i-\mu_j) c_j/(\epsilon_i+\epsilon_j) \ll 1 \quad (3.27)$$

for all i. This will be true if

$$\mu_i \ll 1/\sum (c_j/\epsilon_j) \text{ for all } i.$$

Neglecting μ_i in (3.25), we have

$$\varphi_{Ni} \approx c_i \varphi_{NN}/\epsilon_i; \quad (3.28)$$

substituting in (3.19), we find

$$\epsilon^* = \sum c_i / \sum (c_i/\epsilon_i). \quad (3.29)$$

Since we have assumed $1-k \ll \epsilon_i$, we must have $1-k \ll \epsilon^*$, so that (3.20) becomes

$$\varphi_{NN} \approx \frac{\epsilon^*(k_2-k)}{2(c+\epsilon^*)(1-k)} \langle N \rangle_{Av} = \frac{\epsilon^*(k_2-k)}{2(c+\epsilon^*)S\tau} \langle N \rangle_{Av}{}^2 \quad (3.30)$$

for the limiting case $1-k \ll \epsilon_i / \sum (c_j/\epsilon_j) = \epsilon_i \epsilon^*/c$, i.e., for sufficiently high power levels.

IV. CASE OF VARIABLE k IN THE ABSENCE OF DELAYED NEUTRONS

Let us consider the problem of fluctuations in intensity, in the absence of delayed neutrons, in a pile in which the relative probabilities of loss and reproduction of neutrons varies with time. Such a time variation will be met in a "pulse generator," or in the application of cyclical variations for determining the operating characteristics of any pile.

It will be justifiable in a first approximation to neglect the delayed-neutron term, and include the contribution of delayed neutrons in the source, if the period of time variation of the function f is short compared to the delayed-neutron periods. This will of course give a constant source only if the *mean* level of operation of the pile is held constant.

The derivation of the differential equations for $\langle N \rangle_{Av}$ and φ_{NN} does not differ from that when the function f has constant coefficients. The equations are

$$d\langle N \rangle_{Av}/dt = S - \kappa \langle N \rangle_{Av} \quad (\kappa = (1-k)/\tau), \quad (4.1)$$

$$d\varphi_{NN}/dt = -2\kappa \varphi_{NN} + \kappa_2 \langle N \rangle_{Av} + S, \quad (4.2)$$

where $\kappa_2 = (k_2 - 2k + 1)/\tau \approx (k_2 - 1)/\tau$. This is very nearly constant if k remains near unity.

The solution of (4.1) is

$$\langle N \rangle_{Av} = \exp\left(-\int_0^t \kappa dt'\right) \times \left\{ S \int_0^t \exp\left(\int_0^{t'} \kappa dt''\right) dt' + \langle N(0) \rangle_{Av} \right\}. \quad (4.3)$$

Inserting this into (4.2) we get

$$\varphi_{NN} = \exp\left(-2\int_0^t \kappa dt'\right) \times \left\{ \int_0^t [\langle N(t') \rangle_{Av} \kappa_2 + S] \times \exp\left(2\int_0^{t'} \kappa dt''\right) dt' + \varphi_{NN}(0) \right\}. \quad (4.4)$$

Let us consider the case in which k is constant. We find then that

$$\langle N(t) \rangle_{Av} - S/\kappa = e^{-\kappa t} [\langle N(0) \rangle_{Av} - S/\kappa], \quad (4.5)$$

i.e., any excess of $\langle N(0) \rangle_{Av}$ over the steady-state value will decay or grow exponentially, according to whether $k<1$ or $k>1$.

Substituting (4.5) into (4.2) and simplifying, we obtain

$$\varphi_{NN}(t) - (S/2\kappa)(1+\kappa_2/\kappa) = [\varphi_{NN}(0) - (S/2\kappa)(1+\kappa_2/\kappa)] e^{-2\kappa t} + (\kappa_2/\kappa)[\langle N(0) \rangle_{Av} - S/\kappa](e^{-\kappa t} - e^{-2\kappa t}). \quad (4.6)$$

Thus, in addition to a decay or growth of the excess of the initial value of φ_{NN} over the steady-state value, there is another term which is due to the deviation of $\langle N(0) \rangle_{Av}$ from its steady-state value.

Case of Periodic "k"

Let us consider now the case in which k, and consequently κ, is periodic with period T, i.e.,

$$\kappa(t+T)=\kappa(t). \tag{4.7}$$

Let us introduce the following notations:

$$\int_0^t \kappa dt'=\sigma(t), \tag{4.8}$$

$$\sigma(T)=I. \tag{4.9}$$

Now suppose $mT\leq t'\leq (m+1)T$. Then

$$\int_0^{t'} \kappa dt''=mI+\sigma(t'-mT). \tag{4.10}$$

Now put

$$\int_0^t e^{\sigma(t')}dt'=\chi(t), \tag{4.11}$$

$$\chi(T)=A. \tag{4.12}$$

We then find that, for $nT\leq t\leq (n+1)T$,

$$\int_0^t \exp\left(-\int_0^{t'}\kappa dt''\right)$$
$$=A(1+e^I+e^{2I}+\cdots e^{(n-1)I})+e^{nI}\chi(t-nT)$$
$$=A\frac{e^{nI}-1}{e^I-1}+e^{nI}\chi(t-nT).$$

It follows that, if we put $\eta=t-nT$ (that is, if η is time elapsed since the end of the last completed period)

$$\langle N\rangle_{\mathrm{Av}}(t)=e^{-nI-\sigma(\eta)}$$
$$\times\left\{\langle N\rangle_{\mathrm{Av}}(0)+S\left[A\frac{e^{nI}-1}{e^I-1}+e^{nI}\chi(\eta)\right]\right\}.$$

As $n\rightarrow\infty$, that is, when the oscillation has been in effect for a sufficient length of time so that the effect of starting has disappeared, this reduces to

$$\langle N\rangle_{\mathrm{Av}}(T)=Se^{-\sigma(\eta)}\left\{\frac{A}{e^I-1}+\chi(\eta)\right\}. \tag{4.13}$$

If we now substitute into Eq. (4.4), and, as above, make use of the periodicity of κ, we get

$$\varphi_{NN}=e^{-(2nI+2\sigma(\eta))}\left\{(1+e^{2I}+e^{4I}+\cdots+e^{2(n-1)I})S\right.$$
$$\times\int_0^T\left[\kappa_2e^{-\sigma(\eta')}\left(\frac{A}{e^I-1}+\chi(\eta')\right)+1\right]e^{2\sigma(\eta')}d\eta'$$
$$+Se^{2nI}\int_0^\eta\left[\kappa_2e^{-\sigma(\eta')}\left(\frac{A}{e^I-1}+\chi(\eta')\right)+1\right]$$
$$\left.\times e^{2\sigma(\eta')}d\eta'+\varphi_{NN}(0)\right\}.$$

We must again let $n\rightarrow\infty$ to neglect the transient effects. We then obtain

$$\varphi_{NN}(t)=Se^{-2\sigma(\eta)}\left\{\frac{1}{e^{2I}-1}\left[\kappa_2A^2\left(\frac{1}{e^I-1}+\tfrac{1}{2}\right)+A_2\right]\right.$$
$$\left.+\kappa_2\chi(\eta)\left[\frac{A}{e^I-1}+\tfrac{1}{2}\chi(\eta)\right]+\chi_2(\eta)\right\}, \tag{4.14}$$

where

$$\chi_2(\eta)=\int_0^\eta e^{2\sigma(\eta')}d\eta', \tag{4.15}$$

$$\chi_2(T)=A_2. \tag{4.16}$$

V. EFFECT OF RESOLVING TIME OF THE MEASURING INSTRUMENT ON THE MEASUREMENT OF STATISTICAL FLUCTUATIONS IN A PILE

Any practical measurement of fluctuations will be subject to limitations imposed by the resolving time of the measuring instrument and by the fact that the measurements will extend over only a finite time interval.

The effect of a resolving time of the measuring instrument is that the quantity measured is not an instantaneous value of N (or of the neutron intensity), but rather a value averaged over an interval of length Δt, where Δt is the resolving time.

Thus fluctuations whose period is appreciably shorter than Δt will not be observed. Similarly, if the measurements extend over a time T, fluctuations whose period are appreciably longer than T will not be observed. In order to interpret experimental measurements, it is therefore necessary to know the frequency distribution of the fluctuations.

Let $\varphi_{NN}^{(\Delta t)}$ be the standard deviation of the number of neutrons averaged over an interval Δt, i.e., the standard deviation observed when measurements with an instrument of resolving time Δt extend over an infinite time. If sharp measurements (resolving time zero) could be taken over a finite interval, T, the standard deviation observed would be then, on the average,

$$\varphi_{NN(T)}=\varphi_{NN}-\varphi_{NN}^{(T)}.$$

For

$$\varphi_{NN}=\langle N^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2;$$

$$\varphi_{NN}^{(T)}=\langle\langle N\rangle_{\text{Av}T}^2\rangle_{\text{Av}}-\langle N\rangle_{\text{Av}}^2,$$

where "AvT" denotes averaging over the time interval, T, and

$$\varphi_{NN(T)}=\langle\langle N^2\rangle_{\text{Av}T}-\langle N\rangle_{\text{Av}T}^2\rangle_{\text{Av}}=\langle N^2\rangle_{\text{Av}}-\langle\langle N\rangle_{\text{Av}T}^2\rangle_{\text{Av}}.$$

The standard deviation observed when measurements with resolving time Δt extend over an interval T is then

$$\varphi_{NN}\,|\,_T{}^{\Delta t}=\varphi_{NN}^{(\Delta t)}-\varphi_{NN}^{(T)}.$$

We now determine the function $\varphi_{NN}^{(\Delta t)}$. The quantity $\langle N^2\rangle_{\text{Av}}$, which was used in Section II for the calculation of fluctuations in the case of sharp measurements, must be replaced by

$$\langle N^2\rangle_{\text{Av}\Delta t}=\frac{1}{(\Delta t)^2}\sum_{N',N'',Q_i',Q_i''}N'N''\int_{t-\Delta t}^{t}dt' \times\int_{t-\Delta t}^{t}dt''R(N',N'';Q_i',Q_i'';t',t''),\quad(5.1)$$

where $R(N',N'';Q_i',Q_i'';t',t'')$ is the probability that there are N' neutrons, Q_i' excited nuclei at time t', and N'' neutrons, Q_i'' excited nuclei at time t''. Equation (5.1) may be transformed to

$$\langle N^2\rangle_{\text{Av}\Delta t}=\frac{2}{(\Delta t)^2}\sum_{N',N'',Q_i',Q_i''}N'N''\int_{t-\Delta t}^{t}dt' \times\int_{0}^{t-t'}d\xi\cdot R^*(N',N'';Q_i',Q_i'';t',\xi),\quad(5.2)$$

where R^* is the probability that there are N' neutrons, Q_i' excited nuclei at time t', and N'' neutrons, Q_i'' excited nuclei at a time ξ later (i.e., at the time $t'+\xi$). We may write

$$\begin{aligned}R^*&(N',N'';Q_i',Q_i'';t',\xi)\\&=(\text{the probability }P(N',Q_i';t')\text{ of }N'\text{ neutrons, }Q_i'\text{ excited nuclei at }t')\\&\quad\times(\text{the probability }P_{N'Q_i'}{}^{N''Q_i''}(\xi)\text{ that }N',Q_i'\text{ give rise to }N'',Q_i''\text{ in time }\xi).\end{aligned}\quad(5.3)$$

We may introduce the probability-generating function

$$G(x,x';y_i,y_i';t,\xi)=\sum_{N,N',Q_i,Q_i'}R^*(N,N';Q_i,Q_i';t,\xi)\times x^Nx'^{N'}\Pi_i\{y_i^{Q_i}y_i'^{Q_i'}\},\quad(5.4)$$

where the $\Pi_i\{\ \}$ indicates a product over the y's corresponding to the various delayed-neutron periods. Then

$$\langle N^2\rangle_{\text{Av}\Delta t}=(2/(\Delta t)^2)\int_{t-\Delta t}^{t}dt' \times\int_0^{t-t'}d\xi\partial^2G/\partial x\partial x'\,|_{x=x'=y_i=y_i'=1}.\quad(5.5)$$

This may be written alternatively, on interchanging the order of integrations,

$$\langle N^2\rangle_{\text{Av}\Delta t}=(2/(\Delta t)^2)\int_0^{\Delta t}d\xi \times\int_{t-\Delta t}^{t-\xi}dt'\partial^2G/\partial x\partial x'\,|_{x=x'=y_i=y_i'=1}.\quad(5.6)$$

Our problem is now to find the function G.

Let us now introduce the following probability-generating functions:

(a) Let

$$\varphi(x,y_i,t)=\sum_{N'Q_i'}\varphi_{N'Q_i'}(t)x^{N'}\Pi_iy_i^{Q_i'}\quad(5.7)$$

be the function whose coefficients are the probabilities that, in the absence of sources, 1 neutron at time 0 will give rise to N' neutrons, Q_i' excited nuclei at time t.

(b) Let

$$\psi^{(i)}(x,y_i,t)=\sum_{N'Q_i'}\psi_{N'Q_i'}{}^{(i)}(t)x^{N'}\Pi_iy_i^{Q_i'}\quad(5.8)$$

be the function whose coefficients are the probabilities that, in the absence of sources, one excited nucleus of type j at time 0 will give rise to N' neutrons, Q_i' excited nuclei at time t.

(c) Let

$$F_s(x, y_i, t) = \sum_{N', Q_i'} P_{N'Q_i'}{}^{(s)}(t) x^{N'} \Pi_i y_i^{Q_i'} \quad (5.9)$$

be the function whose coefficients are the probabilities that N' neutrons, Q_i' excited nuclei are produced by a source of strength S in time t, there being no neutrons or excited nuclei present at $t=0$. It is then easily demonstrated that in the absence of sources, φ^N is the probability-generating function for the initial condition of N neutrons alone present at $t=0$; $\psi^{(j)Q_j}$ is that for the initial condition of Q_j excited nuclei of type j alone present at $t=0$. The function $\varphi^N \Pi_j \psi^{(j)Q_j}$ is the probability-generating function corresponding to N neutrons, Q_1 excited nuclei of type 1, etc., at $t=0$. Finally, the function

$$\Omega_{NQ_1Q_2}\cdots(x, y_i; t) = F_s \varphi^N \Pi_j \psi^{(j)Q_j} \quad (5.10)$$

is the probability-generating function for the same problem when sources are present.

It is easily verified that Ω satisfies the appropriate differential equation (2.11).

We may now write down the function

$$G(x, x'; y_i, y_i'; t, \xi).$$

It is, in fact, equal to

$$F_s(x', y_i', \xi) F[x\varphi(x', y_i', \xi) \cdot y_1 \psi^{(1)}(x', y_i', \xi), \cdots t], \quad (5.11)$$

where $F(x, y_1, y_2, \cdots; t)$ is the probability-generating function for the number of neutrons and excited nuclei in the system at time t, as defined in Section II. For this is equal to

$$F_s \sum P(N, Q_i; t)(x\varphi)^N (y_1 \psi^{(1)})^{Q_1} \cdots = \sum P(N, Q_i; t) \Omega_{NQ_i}(x', y_i'; \xi) x^N y_i^{Q_i}. \quad (5.12)$$

Comparison with (5.4) enables us to identify this function with G.

Differentiating (5.11) with respect to x and x', and setting all the x, x', y_i, y_i' equal to unity, we find

$$\partial^2 G/\partial x \partial x' |_{x=x'=y_i=y_i'=1} = \langle N_s(\xi)\rangle_{\mathrm{Av}} \langle N(t)\rangle_{\mathrm{Av}} + \langle N_n(\xi)\rangle_{\mathrm{Av}} \langle N^2(t)\rangle_{\mathrm{Av}} + \sum \langle N_j(\xi)\rangle_{\mathrm{Av}} \langle NQ_j(t)\rangle_{\mathrm{Av}}, \quad (5.13)$$

where $\langle N_s(\xi)\rangle_{\mathrm{Av}}$ is the mean number of neutrons present at time ξ arising from the action of the source if $N(0) = Q_i(0) = 0$; $\langle N_n(\xi)\rangle_{\mathrm{Av}}$ is the mean number of neutrons at time ξ if $S=0$, $N(0)=1$, $Q_i(0)=0$; $\langle N_j(\xi)\rangle_{\mathrm{Av}}$ is the mean number of neutrons at time ξ if $S=0$, $N(0)=0$, $Q_i(0)=0$ for $i \neq j$, and $Q_j(0)=1$.

In the steady state $G(x, y_i, \cdots; t, \xi)$ does not depend on t, and (5.6) becomes

$$\varphi_{NN}{}^{(\Delta t)} = (2/(\Delta t)^2) \int_0^{\Delta t} (\Delta t - \xi) \times [\langle N_n(\xi)\rangle_{\mathrm{Av}} \langle N^2\rangle_{\mathrm{Av}} + \sum \langle N_j(\xi)\rangle_{\mathrm{Av}} \langle NQ_j\rangle_{\mathrm{Av}} - \langle N\rangle_{\mathrm{Av}} (\langle N\rangle_{\mathrm{Av}} - \langle N_s(\xi)\rangle_{\mathrm{Av}})] d\xi. \quad (5.14)$$

But $\langle N\rangle_{\mathrm{Av}} - \langle N_s(\xi)\rangle_{\mathrm{Av}}$ is the average number of neutrons which would exist at time t if $S=0$, $N(0) = \langle N\rangle_{\mathrm{Av}}$, $Q_j(0) = \langle Q_j\rangle_{\mathrm{Av}}$:

$$\langle N\rangle_{\mathrm{Av}} - \langle N_s(\xi)\rangle_{\mathrm{Av}} = \langle N\rangle_{\mathrm{Av}} \langle N_n(\xi)\rangle_{\mathrm{Av}} + \sum_j \langle Q_j\rangle_{\mathrm{Av}} \langle N_j(\xi)\rangle_{\mathrm{Av}}. \quad (5.15)$$

Substituting in (5.14) we get

$$\varphi_{NN}{}^{(\Delta t)} = (2/(\Delta t)^2) \int_0^{\Delta t} (\Delta t - \xi) \times [\varphi_{NN} \langle N_n(\xi)\rangle_{\mathrm{Av}} + \sum_j \varphi_{Nj} \langle N_j(\xi)\rangle_{\mathrm{Av}}] d\xi. \quad (5.16)$$

$\langle N_n\rangle_{\mathrm{Av}}(t)$ is determined by Eqs. (1.1) and (1.2), with $S=0$ and the initial conditions $\langle N\rangle_{\mathrm{Av}}(0)=1$, $\langle Q_i\rangle_{\mathrm{Av}}(0)=0$. Let us look for solutions of the form

$$\langle N\rangle_{\mathrm{Av}} = \langle N\rangle_{\mathrm{Av}}(0) e^{-\gamma t},$$

$$\langle Q_i\rangle_{\mathrm{Av}} = \langle Q_i\rangle_{\mathrm{Av}}(0) e^{-\gamma t}.$$

We find that γ must satisfy the equation

$$\kappa' - \gamma = (1/\tau) \sum_{i=1}^{n} a_i c_i / (a_i - \gamma), \quad (5.17)$$

which has $(n+1)$ solutions γ_s. Here $\kappa' = (1-k')/\tau$. The general solution is then

$$\langle N\rangle_{\mathrm{Av}} = \sum_{s=0}^{n} \rho_s e^{-\gamma_s t}, \quad (5.18)$$

$$\langle Q_i\rangle_{\mathrm{Av}} = (c_i/\tau) \sum_{s=0}^{n} \rho_s e^{-\gamma_s t} / (a_i - \gamma_s). \quad (5.19)$$

With the initial condition $N(0)=1$, $Q_i(0)=0$, we get the equations for ρ_s:

$$\sum_{s=0}^{n} \rho_s = 1, \tag{5.20}$$

$$\sum_{s=0}^{n} \rho_s/(a_i-\gamma_s)=0, \quad (i=1, 2, \cdots, n). \tag{5.21}$$

Alternatively, the ρ_s's may be determined by solving (1.1) and (1.2) above, for the case $S=0$, by Laplace transforms. It is then easily verified that

$$\rho_s=\gamma_s\tau/[\kappa\tau+\sum_i c_i/(1-a_i/\gamma_s)^2]. \tag{5.22}$$

Since if there is only an excited nucleus present at $\xi=0$, it must first decay, and this is followed by the decay of the resulting neutron, we have

$$\langle N_j(\xi)\rangle_{\text{Av}}=a_j\int_0^{\xi} e^{-a_j\xi'}\langle N_n(\xi-\xi')\rangle_{\text{Av}}d\xi'. \tag{5.23}$$

Explicitly,

$$\begin{aligned}\langle N_j(\xi)\rangle_{\text{Av}}&=a_j\sum_s [\rho_s/(a_j-\gamma_s)](e^{-\gamma_s\xi}-e^{-a_j\xi})\\ &=a_j\sum_s \rho_s e^{-\gamma_s\xi}/(a_j-\gamma_s),\end{aligned} \tag{5.24}$$

because of (5.21). Substituting from (5.17) and (5.20) into (5.16), we get

$$\varphi_{NN}{}^{(\Delta t)}=\sum_{s=0}^{n}\frac{2\rho_s}{(\gamma_s\Delta t)^2}(\gamma_s\Delta t-1+e^{-\gamma_s\Delta t})\times\left[\varphi_{NN}{}^{(0)}+\sum_{j=1}^{n}\frac{a_j\varphi_{Nj}{}^{(0)}}{a_j-\gamma_s}\right]. \tag{5.25}$$

Expression (5.25) may be simplified for the case of very small or very large values of Δt. If $\gamma_s\Delta t\ll 1$ for all γ_s, we may write

$$2(\gamma_s\Delta t-1+e^{-\gamma_s\Delta t})/(\gamma_s\Delta t)^2=1-\tfrac{1}{3}\gamma_s\Delta t. \tag{5.26}$$

With the help of (5.20) and (5.21) and the relation

$$\sum_{s=0}^{m}\rho_s\gamma_s=\kappa' \tag{5.27}$$

(which follows from (5.17)), we can transform (5.25) to

$$\varphi_{NN}{}^{(\Delta t)}=\varphi_{NN}(1-\tfrac{1}{3}\kappa'\Delta t)+(\Delta t/3\tau)\sum_l \epsilon_l\varphi_{Nl}. \tag{5.28}$$

From (3.19), (3.20), and (3.21) we have

$$\varphi_{NN}=(1-k+\epsilon^*)\Phi/(1-k+\epsilon^*+c);$$
$$\sum \epsilon_l\varphi_{Nl}=\epsilon^*c\Phi/(1-k+\epsilon^*+c),$$

where

$$\Phi=\tfrac{1}{2}(k_2-k)\langle N\rangle_{\text{Av}}/(1-k),$$

so that

$$\varphi_{NN}{}^{(\Delta t)}/\varphi_{NN}=1-\tfrac{1}{3}\kappa\Delta t(1-k+c+\epsilon^*)/(1-k+\epsilon^*). \tag{5.29}$$

Similarly, it can be shown that, if $\gamma_s\Delta t\gg 1$ for all γ_s, i.e., if Δt is longer than the longest relaxation period of the pile,

$$\begin{aligned}\varphi_{NN}{}^{(\Delta t)}/\varphi_{NN}&=2\Phi/\kappa\Delta t\varphi_{NN}\\ &=2(1-k+\epsilon^*+c)/\kappa\Delta t(1-k+\epsilon^*).\end{aligned} \tag{5.30}$$

In practice Δt will often be longer than the shortest relaxation period and shorter than the longest. Let us assume that the various relaxation periods are well separated, and that

$$\gamma_{r+1}\Delta t\ll 1\ll\gamma_r\Delta t.$$

Then, by writing

$$(2/(\gamma_s\Delta t)^2)(\gamma_s\Delta t-1+e^{-\gamma_s\Delta t})=\begin{cases}1, & s\geqslant r+1\\ 2/\gamma_s\Delta t, & s\leqslant r\end{cases},$$

we obtain

$$\varphi_{NN}{}^{(\Delta t)}/\varphi_{NN}=\sum_{s=r+1}^{m}\rho_s[1+\sum_{l=1}^{m}a_l\varphi_{Nl}/(a_l-\gamma_s)\varphi_{NN}]. \tag{5.31}$$

In particular, if $r=0$, i.e., Δt is large compared to the shortest period but small compared to all others, (5.31) can be transformed to

$$\varphi_{NN}{}^{(\Delta t)}/\varphi_{NN}=1-(1-k)\epsilon^*/(1-k+c)(1-k+c+\epsilon^*), \tag{5.32}$$

and for $r=m-1$, i.e., for Δt large compared to all except the longest relaxation period of the pile,

$$\varphi_{NN}{}^{(\Delta t)}/\varphi_{NN}=\rho_m(1-k+\epsilon^*+c)/(1-k+\epsilon^*). \tag{5.33}$$

We may further deduce the "normalized spectrum" of the fluctuations. For, if $S(\sigma)d\sigma$ is the proportion of fluctuations corresponding to the frequency σ, it follows from a theorem of Wang and Uhlenbeck,[4] that $S(\sigma)$ is the Fourier cosine transform of

$$[\langle N(t)N(t+\xi)\rangle_{Av}-\langle N\rangle_{Av}^2]/\varphi_{NN};$$

in fact, according to Wang and Uhlenbeck,

$$S(\sigma)=4\int_0^\infty \frac{\langle N(t)N(t+\xi)\rangle_{Av}-\langle N\rangle_{Av}^2}{\varphi_{NN}} \times\cos 2\pi\sigma\xi d\xi. \quad (5.34)$$

Using the relation

$$\langle N(t)N(t+\xi)\rangle_{Av}-\langle N\rangle_{Av}^2 =\varphi_{NN}\langle N_n(\xi)\rangle_{Av}+\sum_l \varphi_{Nl}\langle N_l(\xi)\rangle_{Av},$$

we get

$$S(\sigma)=\frac{4}{\varphi_{NN}}\left\{\varphi_{NN}\sum_{s=0}^{n}\frac{\rho_s\gamma_s}{\gamma_s^2+4\pi^2\sigma^2}+\sum_{l=1}^{n}a_l\varphi_{Nl}\frac{\rho_s}{a_l-\gamma_s}\times\left[\frac{\gamma_s}{\gamma_s^2+4\pi^2\sigma^2}-\frac{a_l}{a_l^2+4\pi^2\sigma^2}\right]\right\}. \quad (5.35)$$

[4] M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. 17, 326 (1945).

One can easily calculate the proportion of fluctuations of period greater than T; it is

$$\int_0^{1/T} S(\sigma)d\sigma=\frac{2}{\varphi_{NN}\pi}\left\{\varphi_{NN}\sum_{s=0}^{n}\rho_s\tan^{-1}\frac{2\pi}{\gamma_sT}+\sum_{l=1}^{n}a_l\varphi_{Nl}\sum_{s=0}^{n}\frac{\rho_s}{a_l-\gamma_s}\times\left[\tan^{-1}\frac{2\pi}{\gamma_sT}-\tan^{-1}\frac{2\pi}{a_lT}\right]\right\}. \quad (5.36)$$

The terms for which $\gamma_sT\ll 1$ are approximately equal to the corresponding term in the expression (5.25) for $\varphi_{NN}^{(\Delta t)}/\varphi_{NN}$ if

$$T=\pi^2\Delta t/3;$$

the terms with $\gamma_sT\gg 1$ are approximately equal to the corresponding terms in (5.25) if

$$T=2\Delta t.$$

In other words, one can get the order of magnitude of the ratio by which the resolving time cuts down the fluctuations by omitting those fluctuations whose period is more than two or three times the resolving time.

Such a result is to be expected, since the effect of resolving time and finite measuring time is, approximately, to eliminate from observation those fluctuations of periods shorter than the resolving time or longer than the measuring time.