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Disintegration of the Deuteron in Flight

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A calculation is made of the cross section for the disintegration of high velocity deuterons by nuclei in their paths. In this paper attention is centered on those processes in which there is no direct nuclear collision, and where the disintegration is due to electric forces. The cross section for 200-Mev deuterons proves to be of the order $2\times10^{-29}Z^2$ cm^2 where Z is the atomic number of the bombarded element. Large yields may be expected in typical cases. The two ejected particles need not share the energy equally; neutrons of energy 75 Mev to 125 Mev may be expected. They emerge in a fairly well collimated beam whose half-width at half-maximum is about 4°.

Quantitatively this process proves usually to be less important than the disintegration involving a direct nuclear collision in which one of the two particles escapes.

1. INTRODUCTION

THE possibility that projected deuterons of sufficiently high energy will be disintegrated by collision with nuclei in their paths has been pointed out by Oppenheimer.[1] His calculations are supplemented in this paper by methods suitable for the case of very high energy deuterons, such as are produced in the Berkeley 200-Mev synchro-cyclotron. A distinction must be made between direct nuclear collisions, in which one or both of the particles in the deuteron collide with the nuclear matter, and electric collisions, in which only the long-range electric forces play a part. A detailed treatment is given here for the latter process.

2. THEORY

We calculate the probability of the disintegration of the deuteron by the electric forces encountered on its collision with a heavy nucleus of charge Ze. We carry out the calculation in the frame in which the center of gravity of the deuteron is initially at rest. The heavy nucleus is taken to be incident along the x axis with velocity v. The electrostatic potential energy of the deuteron is

$$V=Ze^2/[(r_{p\perp})^2+(r_{px}-vt)^2]^{\frac{1}{2}} \qquad (1)$$

where r_{px} and $r_{p\perp}$ are the components of the protons coordinate along the x axis and transverse to it. We write

$$\mathbf{r}_p=\mathbf{R}+\tfrac{1}{2}\boldsymbol{\rho}$$

where $\mathbf{R}$ is the position vector of the deuteron's center of gravity and $\boldsymbol{\rho}$ is the vector from neutron to proton. Transitions are induced from an initial state

$$\psi_0=L^{-\frac{3}{2}}\exp i(\mathbf{k}_0\cdot\mathbf{R})\cdot U_0(\rho)e^{iE_0t/\hbar}$$

to a final state

$$\psi_1=L^{-\frac{3}{2}}\exp i(\mathbf{k}_1\cdot\mathbf{R})\,U_1(\rho)e^{iE_1t/\hbar}.$$

E_0 and E_1 are the total energies of the respective states. For the energies involved here, the Born approximation should give reasonably good results except for the heaviest elements. Accordingly, plane waves are taken for the center of gravity wave functions and they are normalized to unit integral in a cube of side L. The initial propagation vector, k_0, will henceforth be set equal to zero. $U_0(\rho)$ is the ^3S ground state wave function of the deuteron and $U_1(\rho)$ is a state in the continuum which will prove to be a ^3P, as in the usual photoelectric disintegration of the deuteron. U_0 is normalized to unit quadratic integral and U_1 to unit energy.

After the collision has taken place the probability amplitude of ψ_1 has grown (in first order) to a value

$$A_1=(i/\hbar)\int_{-\infty}^{\infty}dt\,V_{01}e^{i\omega t}, \qquad (2)$$

where $\hbar\omega=E_1-E_0$ and V_{01} is the spatial part

[1] J. R. Oppenheimer, Phys. Rev. **47**, 845 (1945).

of the matrix element. This is

$$A=2i(Ze^2/\hbar v)\int\int d\mathbf{R}d\boldsymbol{\varrho}$$
$$\times L^3\exp(i\mathbf{k}\cdot\mathbf{R})U_1^*(\rho)U_0(\rho)$$
$$\times\exp(-i\omega r_{p\perp}/v)K_0(\omega r_{p\perp}/v)\quad(3)$$

where K_0 is the Bessel function of imaginary argument. Now the volume element may as well be written $d\mathbf{r}_p d\boldsymbol{\varrho}$ and, accordingly

$$A_1=(2i/L^3)(Ze^2/\hbar v)I_pI_\rho,$$

with

$$I_p=\int d\mathbf{r}_p\exp[i(\mathbf{k}\cdot\mathbf{r}_p-\omega r_{px}/v)]K_0(\omega r_{p\perp}/v),$$
$$I_\rho=\int d\rho\exp[-i\tfrac{1}{2}(\mathbf{k}\cdot\boldsymbol{\varrho})]U_1^*(\rho)U_0(\rho).\quad(4)$$

Now for disintegrations of deuterons of energy up to 200 Mev the center of gravity recoil, k, is quite small and the product $(k\rho/2)$ is well under 0.1 for the important range of the variables. Consequently we expand the exponential and keep only the second term $(-i\mathbf{k}\cdot\boldsymbol{\varrho}/2)$. It follows that U_1 must be a ^3P state with $(\cos\theta)$ angular dependence about k as an axis. We have

$$I_\rho=-\tfrac{1}{2}ik(\rho_x)_{01}.\quad(5)$$

The matrix element is that of the dipole moment and is well known from the literature.[2] We obtain

$$I_\rho=-\tfrac{1}{2}ik\hbar(8/3\pi M)^{\frac{1}{2}}(\epsilon_0\epsilon_1{}^3)^{\frac{1}{4}}/(\epsilon_0+\epsilon_1)^2.\quad(6)$$

M is the proton mass, ϵ_0 the binding energy of the deuteron, and ϵ_1 the disintegration energy in state ψ_1.

To get I_p, we note first that the two-dimensional integral

$$\int d\mathbf{r}_{p\perp}K_0(\omega r_{p\perp}/v)\exp i(\mathbf{k}\cdot\mathbf{r}_{p\perp})$$
$$=2\pi/(k_\perp{}^2+(\omega/v)^2),\quad(7)$$

$\mathbf{k}_\perp$ and $\mathbf{r}_{p\perp}$ being the transverse components of the respective vectors. Also

$$\int_{-L/2}^{L/2}dr_{px}\exp[ir_{px}(k_x-\omega/v)]$$
$$=2\sin[(k_x-\omega/v)L/2]/(k_x-\omega/v).\quad(8)$$

As we let $L\rightarrow\infty$, the integral (8) will result in the restriction $k_x-\omega/v=0$, which means that (7) may be replaced by $(2\pi/k^2)$. Finally

$$I_p=4\pi\sin[(k_x-\omega/v)L/2]/k^2(k_x-\omega/v).\quad(9)$$

Now the probability that a transition has taken place is $|A_1|^2$ and the cross section is

$$\sigma=|A_1|^2L^2,$$
$$\sigma d\epsilon_1 d\mathbf{k}d\Omega_1=(16\cdot 8\pi/3L^4)(Ze^2/\hbar v)^2$$
$$\times(\hbar^2/Mk^2)(\epsilon_0\epsilon_1{}^3)^{\frac{1}{2}}/(\epsilon_0+\epsilon_1)^4$$
$$\times\{\sin^2[(k_x-\omega/v)L/2]/(k_x-\omega/v)^2\}$$
$$\times(L/2\pi)^3d\mathbf{k}d\epsilon_1\{(3/4\pi)\cos^2\gamma\}d\Omega_1.$$

This gives the differential cross section for transition to a state for which the center of gravity propagation vector is in $d\mathbf{k}$, the disintegration energy is in $d\epsilon_1$, and the proton's direction is in $d\Omega_1$. The factor $(L/2\pi)^3$ is necessary to convert the normalization of the wave functions to unit $\mathbf{k}$. The factor $(3/4\pi)\cos^2\gamma$ takes account of the angular distribution of the ejected protons, γ being the angle between $\mathbf{r}_p$ and $\mathbf{k}$.

We are not interested in the $\mathbf{k}$ distribution so we first integrate over that variable. If θ_k is the polar angle of $\mathbf{k}$ and θ_1 is the polar angle of the proton direction (with the x axis as pole), then

$$\int d\mathbf{k}\{\cos^2\gamma\cdot\sin^2[(k_x-\omega/v)L/2]/k^2(k_x-\omega/v)^2\}$$
$$=\tfrac{1}{2}\pi^2L[\tfrac{1}{2}(3\cos^2\theta_1-1)\cos^2\theta_k$$
$$+\sin^2\theta_1\cdot\ln\cos\theta_k]^1_{(\cos\theta_k)_{\min}}.\quad(11)$$

Here the integral over k has been carried out first and the circumstance that $L\rightarrow\infty$ guarantees that contributions to this integral are restricted to the point

$$k=\omega/v\cos\theta_k.\quad(12)$$

Thus, when the integral over $\cos\theta_k$ is performed, the variable has a lower limit corresponding to the largest value that k may take; and this lower limit is indicated in (11).

[2] H. Bethe and R. Peierls, Proc. Roy. Soc. **A148**, 146 (1935).

The fact that k is really bounded may be verified by going back to (7). For large k, the integral comes mainly from small values of $r_{p\perp}$, i.e., from small impact parameters. Now in fact, for impact parameters smaller than a certain distance, R_0, the collision has the character of a direct nuclear collision. It is not our purpose to consider such collisions in this paper. We have therefore to restrict the values of $r_{p\perp}$ included to those equal to or greater than R_0. This can be approximately achieved by setting $k_{\max}=(1/R_0)$; this is certainly the right order of magnitude, and any better specification of $k_{\max}$ is made difficult by the somewhat hazy character of the separation between "nuclear" and "electric" collisions. The sensitiveness of the results to the details of the cut-off will be discussed in a later section. Now $\hbar\omega$ is the energy exchange:

$$\hbar\omega=\epsilon_1+\epsilon_0+\hbar^2k^2/4M. \qquad (13)$$

The last term, which is the energy in the center of gravity motion, is very small and may be dropped from (13) as far as the determination of the lower limit is concerned. We have

$$(\cos\theta_k)_{\min}=(\epsilon_0+\epsilon_1)R_0/\hbar v=1/\Gamma, \qquad (14)$$

R_0 may be taken as the radius of the nucleus plus the radius of the deuteron. The integral (11) becomes

$$\tfrac{1}{2}\pi^2L[\tfrac{1}{2}(3\cos^2\theta_1-1)(1-\Gamma^{-2})+\sin^2\theta_1\cdot\ln\Gamma] \qquad (15)$$

and the differential cross section becomes

$$\sigma d\epsilon_1 d\Omega_1=(2/\pi)(Ze^2/\hbar v)^2(\hbar^2/M)\times[(\epsilon_0\epsilon_1{}^3)^{\frac{1}{2}}/(\epsilon_0+\epsilon_1)^4]d\epsilon_1 d\Omega_1\cdot[\tfrac{1}{2}(3\cos^2\theta_1-1)(1-\Gamma^{-2})+\sin^2\theta_1\cdot\ln\Gamma]. \qquad (16)$$

It is instructive to compare this formula with the corresponding one obtained on calculating this process by the method of virtual quanta. In this picture the field of the passing heavy nucleus is replaced by a superposition of quanta; each travelling parallel to the path of the nucleus, and each capable of producing the photoelectric effect in the deuteron if its frequency is sufficiently high.

The calculation results in the formula:

$$\sigma d\epsilon_1 d\Omega_1=(2/\pi)(Ze^2/\hbar v)^2(\hbar^2/M)\times[(\epsilon_0\epsilon_1{}^3)^{\frac{1}{2}}/(\epsilon_0+\epsilon_1)^4]d\epsilon_1 d\Omega_1\sin^2\theta_1\cdot\ln\Gamma'. \qquad (17)$$

The quantity Γ' is identical with Γ above except for a factor $(1-v^2/c^2)^{\frac{1}{2}}$ which appears in the denominator of the former and which makes a negligible difference in the present connection. Otherwise (17) is just the part of (16) that comes from the second term in the bracket. This is to be understood as follows. For the method of virtual quanta to be valid, the velocity must be sufficiently high so that the electric field is essentially transverse and the Poynting flux longitudinal. Now if the Poynting flux in the field of the passing particle is separated into a longitudinal and a transverse part, it can be shown that the total momentum transfer associated with the longitudinal flux is greater than that of the transverse flux by just a factor $\ln\Gamma'$, providing Γ' is large compared to unity. Thus the first term in the bracket of (16), which will be negligible in the limit of large values of Γ, is to be identified with the effect of longitudinal electric fields (transverse Poynting vector). In the case under consideration the two terms are of the same order of magnitude, the important values of Γ being in the range 2 to 3.

The first term gives no contribution to the total cross section, but only affects the angular distribution. We account for this by remarking that a longitudinal electric field cannot by itself break up the deuteron, since it can supply no net impulse. However, it can change the direction of emergence of the ejected protons. In particular it can cause particles to be ejected in the forward direction. Whereas there are no particles in the forward direction for a transverse electric field alone, in the present case of 200 Mev deuterons, the distribution has almost a maximum in the forward direction.

3. TOTAL CROSS SECTION

Integration of (16) over the angles gives for the distribution in ϵ_1:

$$d\epsilon_1=(16/3)(Ze^2/\hbar v)^2(\hbar^2/M)\times[(\epsilon_0\epsilon_1{}^3)^{\frac{1}{2}}/(\epsilon_0+\epsilon_1)^4](d\epsilon_1)\ln\Gamma. \qquad (18)$$

The integration over ϵ_1 is carried out numerically. A value of R_0 of 1.1×10^{-12} cm is chosen, corresponding to the sum of deuteron and nuclear radii for about atomic weight 100. This implies a maximum disintegration energy, $(\epsilon_1)_{\max}$, of about

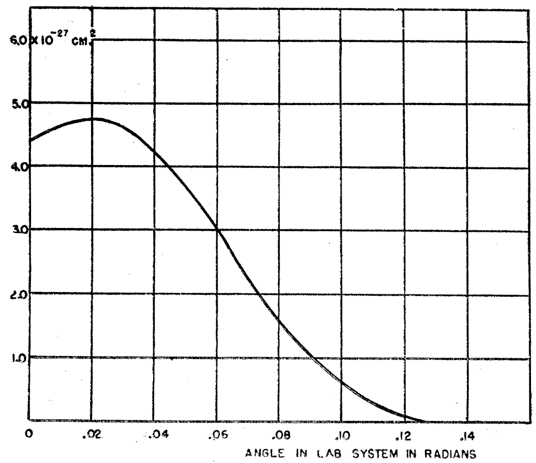

FIG. 1. $N(\theta)$: Angular distribution of emergent neutrons [Number in $d\theta$ is $Z^2N(\theta)d(\cos\theta)$].

5.6 Mev. At this energy, both proton and neutron have very low velocities and the neglect, implied in (1), of all interaction except the instantaneous coulomb interaction, is here justified. The result of the integration for 200 Mev deuterons is

$$\sigma = 2.01 \times 10^{-29} Z^2 \text{ cm}^2. \quad (19)$$

The same calculation has been carried out for 185-Mev deuterons on U^{238}. For this case the Born approximation is more questionable. It is nevertheless included, since only for large values of Z will the effect discussed here be experimentally prominent. The total cross section becomes

$$\sigma = 1.35 \times 10^{-29} Z^2 \text{ cm}^2. \quad (19')$$

The yield is most conveniently expressed by the method of the first reference. For 200-Mev deuterons, $A = 100$, the total cross section may also be written

$$\sigma = 0.338 Z^2 e^4 / M v^2 \epsilon_0.$$

This formula gives fairly accurately the variation with energy in the neighborhood of 200 Mev.

If the deuteron traverses material with N atoms/cm^3 and loses energy dE by ionization then the yield of the above process in that thickness is

$$\text{Yield in } dE = N\sigma dE/(dE/dx)$$

where

$$dE/dx = (4\pi Z e^4 N / m v^2) \ln(2mv^2/I)$$

and m is the electronic mass. I is an average ionization potential of about 80 ev. If we neglect the variation of the logarithm over a small range of velocities, then

$$\text{Yield in } dE = 1.90 \times 10^{-6} Z \cdot dE/\epsilon_0.$$

Thus the yield per unit energy interval is approximately independent of the energy. If the deuteron loses by ionization an amount ΔE (in Mev), then the yield is

$$\text{Yield } (\Delta E) = 8.72 \times 10^{-7} Z (\Delta E). \quad (20)$$

The corresponding formula for 185-Mev deuterons, $A = 238$ is

$$\text{Yield } (\Delta E) = 5.85 \times 10^{-7} Z (\Delta E). \quad (20')$$

4. ANGULAR DISTRIBUTION

We now wish to transform the differential cross section into the laboratory system. We propose to use as independent variables, instead of the ϵ_1, θ_1 of (16), the total kinetic energy and angle of ejection of the proton (or neutron) measured in the laboratory system of coordinates. The Lorentz equations of transformation are

$$\begin{aligned} p\cos\theta &= \gamma^{-1}(p'\cos\theta' - vW/c^2), \\ p\sin\theta &= p'\sin\theta', \\ W &= \gamma^{-1}(-vp'\cos\theta' + W'). \end{aligned} \quad (21)$$

The unprimed system is the laboratory system, the primed system is one moving in the $-x$ direction with the initial velocity v of the deuteron. $\gamma = (1 - v^2/c^2)^{\frac{1}{2}}$. The momenta of the ejected proton in the two systems are p and p'. W and W' are the respective total energies. The polar angles of the momentum vector in the two systems (the $+x$ axis being pole) are θ and θ'.

Since the motions are always non-relativistic in the primed system, we may set

$$W' = Mc^2 + (p')^2/2M.$$

We may also write

$$\mathbf{p}' = \pm \mathbf{p}_1 + \hbar\mathbf{k}/2$$

the first term being the momentum of the proton (or neutron) in the deuteron system, the second being the contribution of the center of gravity momentum. Now

$$p'\cos\theta' = \pm p_1 \cos\theta_1 + \tfrac{1}{2}\hbar k \cos\theta_k$$

the angles having been defined in (11). If we use the relation obtained from (12) and (13),

$$\hbar kv\cos\theta_k=\epsilon_1+\epsilon_0+\hbar^2k^2/4M$$
$$=\epsilon_0+(\hbar^2k^2/4M)+(p_1^2/M). \quad (22)$$

We arrive at the following:

$$W=\gamma^{-1}[Mc^2-\tfrac{1}{2}\epsilon_0 \pm p_1v\cos\theta_1\mp\hbar(\mathbf{p}_1\cdot\mathbf{k})/2M]. \quad (23)$$

The sum of the energies of proton and neutron in the laboratory system is thus

$$W_p+W_n=(2Mc^2-\epsilon_0)/\gamma \quad (24)$$

which guarantees the covariance of energy in the collision.

Now we can show that the last term in (23) is considerably smaller than any other, and we will neglect it in the following. If we add to the resulting equation the second of (21), the resulting pair is

$$p_1\cos\theta_1=[Mc^2-\gamma W-\tfrac{1}{2}\epsilon_0]/v,$$
$$p_1\sin\theta_1=p\sin\theta. \quad (25)$$

In the second of Eq. (25), the transverse component of **k** has been neglected. A numerical study shows that this is valid for the cases under consideration. These may be used to calculate the Jacobian of the transformation from (ϵ_1, $\cos\theta_1$) to (T, $\cos\theta$), T being the kinetic energy of one of the ejected particles: $T=W-Mc^2$. We get

$$d\epsilon_1 d(\cos\theta_1)=dTd(\cos\theta) \times\{-4T\cos\theta/v(M\epsilon_1)^{\frac{1}{2}}\}(1+\xi) \quad (26)$$

where $\xi=T_0/2Mc^2$, T_0 being the initial kinetic energy per particle. In obtaining (26) we have for brevity neglected terms of the order $(T-T_0)/Mc^2$ compared to unity, introducing errors of the order of one to two percent.

Since in the present setup the deuteron is incident in the $-x$ direction the angle θ_l that the emergent particle makes with the direction of incidence is the supplement of θ above. If we write $\mu=\cos\theta_l$, then the differential cross section is

$$\sigma dTd\mu=4(Ze^2/\hbar v)^2(\hbar^2/M) \times[(\epsilon_0\epsilon_1^3)^{\frac{1}{2}}/(\epsilon_0+\epsilon_1)^4]dTd\mu(4T\mu/v(M\epsilon_1)^{\frac{1}{2}}) \times(1+\xi)[\tfrac{1}{2}(3\cos^2\theta_1-1)(1-\Gamma^{-2}) +\sin^2\theta_1\cdot\ln\Gamma] \quad (27)$$

where

$$\epsilon_1=2T(1-\mu^2)(1+\xi) +((1+3\xi)/2T_0)(T_0-T-\tfrac{1}{2}\epsilon_0)^2,$$
$$\cos\theta_1=[(T_0-T-\epsilon_0/2)/(2\epsilon_1T_0)^{\frac{1}{2}}](1+\tfrac{3}{2}\xi).$$

Equation (27) has been integrated numerically. Figure 1 shows the angular distribution of all neutrons (or protons) regardless of energy, and Fig. 2 shows the energy distribution integrated over all angle. Both figures are for 185-Mev deuterons on U^{238}.

Graphical integration to get the total cross section from Figs. 1 or 2 yields a value of 1.31×10^{-29} cm^2 which is in reasonable agreement with the value of 1.35×10^{-29} cm^2 found by staying in the deuteron system. The remaining discrepancy is due to the neglect of the center of gravity recoil at several points.

Angle-energy distributions were also derived for the case of 200-Mev deuterons, $A=100$. There are no significantly different features from the case presented in the figures. Again the total cross section checks the value previously obtained.

5. DISCUSSION OF CUT-OFF PROCEDURE

The numerical results obtained here are to some extent sensitive to the details of the cut-off applied to the coulomb potential at the nuclear radius (or to the momentum transferred to the center of gravity of the deuteron. See Eq. (14) and preceding.). The procedure employed above

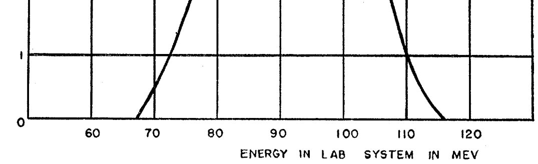

FIG. 2. $N(E)$: Energy distribution of emergent neutrons [Number in dE is $Z^2N(E)d(E/T_0)$, where $T_0=92.5$ Mev].

of cutting off the momentum transfer is equivalent to replacing the singular coulomb potential by one identical with it except for having deleted from it all spatial fourier components with wave number $>(1/R_0)$. In this way it was hoped to calculate only the effect of the acceleration experienced outside R_0 and to omit all effects concerned with the direct nuclear collision. An alternative procedure is to make the cut-off in configuration space, permitting the potential to keep its coulomb shape down to R_0, then flattening it out for smaller separation. The resulting spectrum of fourier components does not end abruptly at $(1/R_0)$ but contains also higher wave numbers whose intensity depends on the precise shape of the fictitious potential. For example, if the potential is taken to be constant inside R_0, the spectrum of components above $(1/R_0)$ is of some importance, due to the kink in the potential at the point R_0. An estimate of the results that would be obtained with this model indicates that the total cross section for 185-Mev deuterons on uranium would be increased from $1.35 \times 10^{-29} Z^2$ cm^2 to $2.5 \times 10^{-29} Z^2$ cm^2 the estimate of the latter number being good to about 15 percent. At the same time the angular distribution would be modified since the large angle scattering processes are somewhat more dependent on the high fourier components than are the small angle processes. Thus the half angle would be increased from 4.05°, as on Fig. 1, to 4.4°.

On the other hand, if the fictitious potential is cut off in a smooth way inside R_0, without the introduction of a kink, the results are very little different from those previously calculated on the basis of the momentum cut-off. On physical grounds, such a smooth model seems more reasonable for the purpose of this calculation.

The point is mentioned because a comparison with experiment seems to indicate that the angular distribution of Fig. 1 (when compounded with the curve of Serber for the stripping process) is too narrow to agree with observation. The use of the "flat" cut-off would leave the discrepancy almost as large. Although from a theoretical point of view the question still contains the uncertainty mentioned above, the extent of this uncertainty seems much too small to permit a hope of reconciling the calculations with observation.

6. CONCLUSIONS

Electric disintegration of 200-Mev deuterons occurs with fairly high probability. For example, in a target with $Z=40$, the disintegration cross section is 3.2×10^{-26} cm^2. The products are widely spread in energy, values of 80 Mev up to 120 Mev occurring with prominence. The resultant neutron or proton beam is well collimated, having a half width at half maximum of slightly over 4°.

The electric disintegration is to be compared in importance with the disintegration involving a direct nuclear collision in which one of the particles collides with the nucleus while the other escapes (shearing). Assuming that the deuteron is so energetic as to be undeflected by the coulomb field, and assuming also that the average proton-neutron separation, ρ, is small compared to the nuclear radius, a, the cross section for the shearing process is $(\pi a \rho/4)$. This is considerably larger than that for the electric process, except for very high Z. For uranium, the processes should be of comparable importance (although the formulae used above to estimate the cross sections for the two processes can hardly be quantitatively trustworthy for so large a nuclear charge). A detailed discussion of the shearing process will be given by Serber, to whom thanks are due for critical discussions of the problems treated above.

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