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The Production of High Energy Neutrons by Stripping*

R. SERBER
Radiation Laboratory, Department of Physics, University of California, Berkeley, California

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When a target is bombarded with high energy deuterons, a narrow beam of high energy neutrons is produced by a process in which the proton in the deuteron strikes the edge of the nucleus and is stripped off, while the neutron misses and continues on its way. The cross section for this stripping process is $\sigma=\frac{1}{2}\pi RR_d$, where R is the nuclear radius and R_d is the deuteron radius, or $\sigma=5A^{\frac{1}{3}}\times10^{-26}$ cm². The yield of neutrons from a $\frac{1}{2}$-in. Be target (in which the energy loss for 190-Mev deuterons is 20 Mev) is nearly 2 percent. The neutrons come out with an energy spread around $\frac{1}{2}E_d$ having a half-width $\Delta E_{\frac{1}{2}}=1.5(E_d\epsilon_d)^{\frac{1}{2}}$. Here E_d is the kinetic energy of the deuteron, ϵ_d its binding energy. For light nuclei the half-width of the neutron angular distribution is $\Delta\theta_{\frac{1}{2}}=1.6(\epsilon_d/E_d)^{\frac{1}{2}}$. The half-width increases somewhat with atomic number, primarily because of the deflection of the deuteron by the Coulomb field as it approaches the nucleus, and, to a lesser extent, because of multiple scattering in the target. The increase in half-width from Be to U is about 25 percent. The calculated half-widths and angular distributions agree well with the measurements of Helmholz, McMillan, and Sewell.

An equal number of high energy protons are produced by stripping processes in which it is the neutron that hits the nucleus.

I. INTRODUCTION

THERE are several processes by which high energy neutrons may be produced when a target is bombarded by high energy deuterons. A deuteron passing at some distance from an atomic nucleus, say two or three times the nuclear radius, may be disintegrated by the Coulomb field of the nucleus.[1] Or, when the deuteron grazes the edge of the nucleus, the proton may strike it and be stripped off, while the neutron misses and continues with almost the velocity of the incident deuteron. And, finally, a high energy neutron can be produced by a direct collision between one of the particles of the deuteron and a nuclear particle.

It is the second process, the stripping process, which will be discussed in this paper. Its characteristics depend primarily on the fact that the deuteron is a very loosely-bound system, the proton and neutron actually spending most of their time outside the range of their mutual forces. For deuterons of kinetic energy considerably larger than the deuteron binding energy, the collision time of the proton with a nuclear particle will be small compared to the period of the relative motion of neutron and proton within the deuteron, and the momentum transferred to the proton will be large compared to the momentum of the relative motion. The proton is thus effectively stripped off instantaneously; there is no reaction on the neutron, which continues its flight with the momentum it had at the instant of collision. This momentum is the sum of the momentum attributable to the motion of the center of mass of the deuteron, plus that attributable to the motion of the neutron within the deuteron. The former is $p_0=(ME_d)^{\frac{1}{2}}$, where M is the proton mass and E_d is the kinetic energy of the deuteron, while the latter is of the order $p_1=(M\epsilon_d)^{\frac{1}{2}}$, with $\epsilon_d=2.18$ Mev the binding energy of the deuteron. The neutron will therefore emerge within an angle to the direction of the deuteron beam of about $\theta\sim(p_1/p_0)=(\epsilon_d/E_d)^{\frac{1}{2}}\sim6°$ for $E_d=190$ Mev. The energy of the neutrons will mostly be in a band given by

$$E=(p_0\pm p_1)^2/2M\sim\tfrac{1}{2}E_d[1\pm2(\epsilon_d/E_d)^{\frac{1}{2}}]\sim\tfrac{1}{2}E_d\pm20\text{ Mev}.$$

The most striking feature of the stripping effect is thus the production of a very narrow cone of neutrons with energy about half that of the deuterons. This prediction has been confirmed on the 184-in. cyclotron in a series of experiments carried out by Helmholz, McMillan, and Sewell; these are reported in the preceding paper.

It is just the narrowness of the cone which distinguishes the stripped neutrons from those produced by direct nuclear encounters, although

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[1] J. R. Oppenheimer, Phys. Rev. **47**, 845 (1935); S. M. Dancoff, Phys. Rev. **72**, 163 (1947).

the total number of neutrons from the two processes are expected to be about the same. For the latter, the formula for the characteristic angle would have, in place of ϵ_d, an energy ϵ_n of the order of the kinetic energy of particles in the nucleus, $\epsilon_n \sim 20$ Mev. The cone would be over three times as wide, and, since the intensity in the forward direction is proportional to the inverse square of the cone width, only a 10 percent contribution might be expected from these neutrons. Furthermore, it can be shown that in consequence of the fact that the struck particle is bound in the nucleus, collisions with small momentum transfers are discouraged, and the forward intensity is smaller even than the above estimate would indicate. The other process which produces high energy neutrons, the disintegration by the electric field, has a cross section which is proportional to Z^2, and small compared to the stripping cross section except for the heaviest nuclei. Even for U, according to estimates by Dancoff,[2] the cross section for the electric field disintegration is only one-quarter the stripping cross section.

II. THE STRIPPING CROSS SECTION

If the incident deuterons have high energy, the neutron passes the nucleus so quickly that its displacement perpendicular to the line of motion of the deuteron during the time of passage is negligible. In a typical impact, the proton will fail to clear the edge of the nucleus by a distance of the order of the "deuteron radius", $R_d = \frac{1}{2}\hbar/(M\epsilon_d)^{\frac{1}{2}} = 2.1\times 10^{-13}$ cm, while the neutron will miss by a like distance. The proton will strike the nucleus a distance $l = (2RR_d)^{\frac{1}{2}}$ in front of a plane through the center of the nucleus. Here R is the nuclear radius, which we suppose appreciably larger than R_d. The neutron traverses this distance in a time l/v, where v is the deuteron velocity. The neutron (or proton) will have a velocity normal to the direction of the deuteron motion of the order $(\epsilon_d/M)^{\frac{1}{2}}$, so its displacement in this direction is $(\epsilon_d/M)^{\frac{1}{2}} l/v$. This displacement is unimportant provided it is small compared to R_d, i.e., $(\epsilon_d/M)^{\frac{1}{2}} l/v < R_d$, a relation which may be rewritten, remembering that $E_d = Mv^2$,

$$E_d > 2(R/R_d)\epsilon_d. \qquad (1)$$

[2] S. M. Dancoff, private communication.

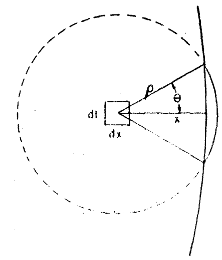

FIG. 1.

The limiting energy given by (1) is, even for the heaviest nuclei, $E_d > 20$ Mev.

The above argument shows that only the (projected) positions of neutron and proton in a plane perpendicular to the deuteron motion need be considered in calculating the cross section: we have only to ask for the probability that at the instant of collision the proton will be within a circle in this plane of radius equal to the nuclear radius, while the neutron will be outside it. Consider a collision in which the separation between proton and neutron (projected in the plane) is ρ. The cross section for the proton hitting a distance x inside the nucleus, within an interval dx, and within an interval dl along the circumference of the nucleus, is just $dxdl$. In the interest of simplicity, we suppose the nuclear radius, R, large compared to the deuteron radius, R_d, so that the curvature of the edge of the nucleus within a distance R_d can be neglected, and the edge considered straight. The probability that the neutron will miss the nucleus is just the fraction of the circumference of a circle of radius ρ which lies outside the nucleus (see Fig. 1); its value is θ/π. The total cross section for proton hitting and neutron missing is thus

$$\sigma(\rho) = \int\int (\theta/\pi) dx dl.$$

The integration over dl gives just the circumference of the nucleus, $2\pi R$. Since $x = \rho\cos\theta$, we have $dx = -\rho\sin\theta d\theta$, and the integral over dx becomes

$$(\rho/\pi)\int_0^{\pi/2} \theta \sin\theta d\theta = \rho/\pi,$$

so

$$\sigma(\rho) = 2R\rho. \qquad (2)$$

Equation (2) gives the cross section when proton and neutron are separated a distance ρ;

to get the total cross section we must multiply (2) by the probability of finding such a separation, and integrate over all values of ρ. If $\psi_d(r)$ is the wave function of the deuteron in its ground state, $|\psi_d(r)|^2 d\mathbf{r}$ is the probability of finding an $n-p$ separation r in the three-dimensional volume element $d\mathbf{r}$. Thus, introducing cylindrical coordinates, the probability of a separation ρ is

$$2\pi\rho d\rho \int_{-\infty}^{\infty} |\psi_d(r)|^2 dz,$$

and the total stripping cross section is

$$\sigma = 4\pi R \int_{-\infty}^{\infty} dz \int_0^{\infty} |\psi_d(r)|^2 \rho^2 d\rho. \quad (3)$$

We can change the variable of integration from z to r by using the relation $r^2=\rho^2+z^2$, which gives $dz = rdr/(r^2-\rho^2)^{\frac{1}{2}}$, and transforms (3) into

$$\sigma = 8\pi R \int_0^{\infty} |\psi_d(r)|^2 r dr \int_0^{r} (\rho^2 d\rho)/(r^2-\rho^2)^{\frac{1}{2}}.$$

The integration over ρ gives $(\pi/4)r^2$, so finally

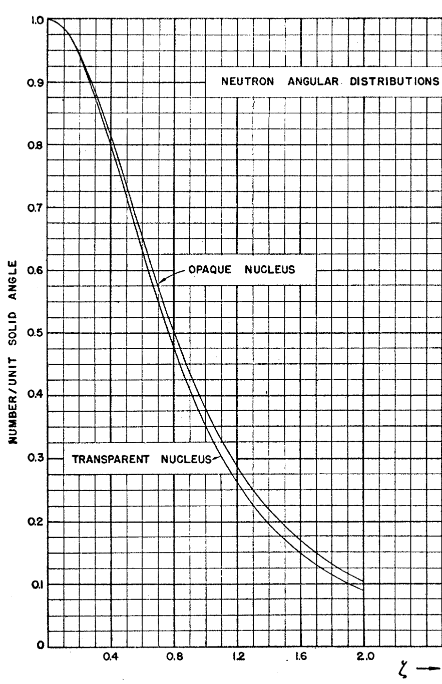

FIG. 2. Neutron angular distributions for transparent and opaque nuclei. Relative number of neutrons per unit solid angle plotted against $\zeta=\theta/\theta_0$.

$$\sigma = 2\pi^2 R \int_0^{\infty} |\psi_d(r)|^2 r^3 dr$$

$$= (\pi/2) R \int r |\psi_d(r)|^2 d\mathbf{r}, \quad (4)$$

remembering that $d\mathbf{r}=4\pi r^2 dr$. The integral in (4) has a simple interpretation: it gives just $\bar{r}$ the average separation of neutron and proton in the deuteron. Calling this separation R_d, we have for the stripping cross section

$$\sigma = (\pi/2) R R_d. \quad (5)$$

If we take

$$\psi_d = (\alpha/2\pi)^{\frac{1}{2}} e^{-\alpha r}/r, \quad \alpha = (M\epsilon_d)^{\frac{1}{2}}/\hbar, \quad (6)$$

we find $R_d = 1/(2\alpha) = 2.1\times 10^{-13}$ cm, the result previously quoted.

In actuality, the deuteron wave function has the form (6) only outside the range of $n-p$ forces. If the finite range of the forces were taken into account, a somewhat larger value of R_d would be obtained. However, in considering the stripping effect we are only interested in the narrow neutron beam which comes off nearly in the forward direction, and these neutrons are produced in collisions in which the neutron and proton are outside the range of their forces at the instant of collision. Collisions which occur with neutron and proton within the range of the forces will give rise to a wide angular distribution, similar to that resulting from direct nuclear encounters, and may be lumped with the latter effect. Thus, within the limits of unambiguity inherent in the separation of the effects, it is proper to ignore the finite range of the forces.

The derivation of (5) has been carried out as if the nucleus were completely opaque to the neutron and proton. In fact, there will be a finite mean-free path, of the order of 4×10^{-13} cm, for a particle to make a collision in traversing nuclear matter. So in some cases, even though the neutron does not miss the nucleus, it may pass through the edge without being disturbed. However, this effect is balanced out by the approximately equal number of cases in which the proton passes through the edge without a collision.

If we take $R=1.5A^{\frac{1}{3}}\times 10^{-13}$ cm, (5) becomes

$$\sigma = 5A^{\frac{1}{3}}\times 10^{-26}\ \text{cm}^2. \quad (7)$$

The stripping cross section ranges from 0.1 barn

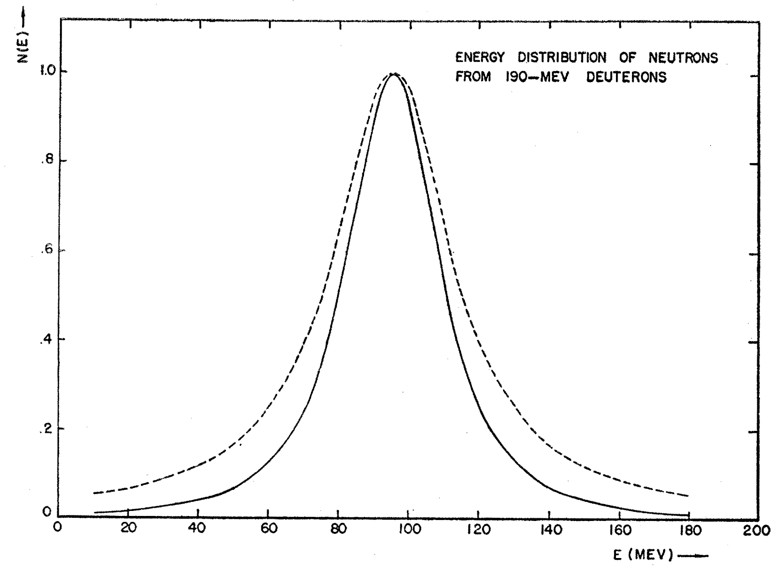

FIG. 3. Energy distribution of neutrons from 190-Mev deuterons. Solid curve, opaque nucleus; dotted curve, transparent nucleus.

for Be to 0.3 barn for U. The yield of neutrons in one passage of the deuteron beam through a $\frac{1}{16}$-in. target, such as is often used in the 184-in. cyclotron, is 1/500 for Be, 1/400 for U.

There is, of course, an equal yield of high energy protons from collisions in which it is the neutron that strikes the nucleus, and the proton which misses.

III. ANGLE AND ENERGY DISTRIBUTIONS

We shall first calculate the angular distribution with the assumption that the nucleus is completely transparent to neutrons. The reason, in addition to the simplicity of this case, is that the model of a transparent nucleus is one limiting case, of which the model of a completely opaque nucleus is the other. Since, as we shall see, the angular distributions to be expected of these two limits turn out to be very little different, we gain by the comparison a considerable confidence in the reliability of the results. Alternatively, we can describe the transparent case as that to be expected in the limit of a very small nucleus, $R \ll R_d$. In our treatment of the opaque case we consider $R \gg R_d$. Treatment of the opposite limit therefore provides some insight into the error likely to be caused by applying the opaque model to light nuclei, where R is not very large compared to R_d, and effects of curvature of the edge of the nucleus and transparency might be expected to show.

The simplicity of the transparent nucleus case lies in the fact that the distribution of neutron momenta due to its motion in the deuteron is just that characteristic of the ground state of the deuteron, without any modification resulting from adding a condition that the neutron has to miss the nucleus.

The probability, $P(\mathbf{p})$, that the neutron in the deuteron has a momentum $\mathbf{p}$ in the interval $d\mathbf{p}$ is

$$P(\mathbf{p}) = |\psi(\mathbf{p})|^2,$$

$$\psi(\mathbf{p}) = (1/h^{\frac{3}{2}}) \int \psi_d \exp[-(i/\hbar)\mathbf{p}\cdot\mathbf{r}]d\mathbf{r}. \quad (8)$$

Using (6), we find

$$\psi(\mathbf{p}) = (1/\pi)[(M\epsilon_d)^{\frac{1}{4}}/(M\epsilon_d + p^2)], \quad (9)$$
$$P(\mathbf{p}) = (1/\pi^2)[(M\epsilon_d)^{\frac{1}{2}}/(M\epsilon_d + p^2)^2].$$

To get the total momentum of the emergent neutron, we have to add to $\mathbf{p}$ the momentum in the z direction

$$p_0 = (ME_d)^{\frac{1}{2}}[1 + (E_d/8Mc^2)]$$

caused by the motion of the center of mass of the deuteron. The second term in the bracket is a small relativistic correction term.

If we denote by $p_\perp$ the magnitude of the component of $\mathbf{p}$ perpendicular to z, the angle of emergence of the deuteron is

$$\theta = (p_\perp/p_0). \quad (10)$$

More strictly, we should write $\theta=p_\perp/(p_0+p_z)$. However, p_z is small compared to p_0, and since p_z is equally likely to be positive or negative, the correction terms in the angular distribution linear in p_z/p_0 drop out, and we are left with a correction only of the order $(p_z/p_0)^2\sim\epsilon_d/E_d\sim 1$ percent. It can readily be verified explicitly that this correction term is negligible.

The probability of a given value of $p_\perp$ is

$$P(p_\perp)2\pi p_\perp dp_\perp=\int_{-\infty}^{\infty}\frac{(M\epsilon_d)^{\frac{1}{2}}dp_z}{\pi^2(M\epsilon_d+p_\perp^2+p_z^2)^2}\cdot 2\pi p_\perp dp_\perp$$
$$=\frac{(M\epsilon_d)^{\frac{1}{2}}}{2\pi(M\epsilon_d+p_\perp^2)^{\frac{3}{2}}}\cdot 2\pi p_\perp dp_\perp.$$

Expressing this in terms of θ by means of (10), we find

$$P(\theta)d\Omega=(1/2\pi)[\theta_0/(\theta_0^2+\theta^2)^{\frac{3}{2}}]d\Omega, \qquad (11)$$

where

$$\theta_0=(\epsilon_d/E_d)^{\frac{1}{2}}[1-(E_d/8Mc^2)], \qquad (12)$$

and $d\Omega=2\pi\theta d\theta$ is the element of solid angle. Or, if angles are measured in terms of θ_0, $\zeta=\theta/\theta_0$,

$$P(\zeta)d\Omega_\zeta=(1/2\pi)[1/(1+\zeta^2)^{\frac{3}{2}}]d\Omega_\zeta, \qquad (13)$$

with $\Omega_\zeta=2\pi\zeta d\zeta$.

A graph of $2\pi P(\zeta)$ is given in Fig. 2. $P(\zeta)$ falls to half its maximum value at $\zeta=0.7664$. Thus the half-width of the angular distribution (full width at half-maximum) is $\zeta_{\frac{1}{2}}=2\times 0.7664=1.533$, or $\theta_{\frac{1}{2}}=1.533\theta_0$.

The energy of the emergent neutron is

$$E=(1/2M)[(p_0+p_z)^2+p_\perp^2]$$
$$=(1/2M)[p_0^2+2p_0p_z+p^2].$$

Since, for the main part of the distribution, $p_0\gg p$, we may neglect the last term and write

$$E=\tfrac{1}{2}E_d+(E_d/M)^{\frac{1}{2}}p_z. \qquad (14)$$

It will be noted that, while the angular distribution depends on $p_\perp$, the energy distribution depends on p_z.

From (9) we find for the probability of a given p_z

$$P(p_z)dp_z=(2/\pi)(M\epsilon_d)^{\frac{1}{2}}\int_0^\infty\frac{p_\perp dp_\perp}{(M\epsilon_d+p_z^2+p_\perp^2)^2}dp_z$$
$$=\frac{(M\epsilon_d)^{\frac{1}{2}}}{\pi(M\epsilon_d+p_z^2)}dp_z.$$

Changing variables from p_z to E by means of (14), we find for the energy distribution

$$P(E)dE=\frac{(\epsilon_dE_d)^{\frac{1}{2}}}{\pi[(E-\frac{1}{2}E_d)^2+\epsilon_dE_d]}dE. \qquad (15)$$

This gives an energy distribution centered around $E=\frac{1}{2}E_d$, with a half-width $\Delta E_{\frac{1}{2}}=2(\epsilon_dE_d)^{\frac{1}{2}}=41$ Mev for $E_d=190$ Mev. A plot of Eq. (15) is given by the dotted curve in Fig. 3.

The extreme tails of this distribution are, of course, not to be believed, in particular the part for which $E>E_d$, which violates energy conservation. Here it is no longer true that $p_0\gg p$, and, concomitantly, our assumption that the collision of the proton with the nucleus can be regarded as sudden, with no reaction on the neutron, is evidently no longer valid.

We now turn to the calculation of the angular distribution for an opaque nucleus. Referring to Fig. 1, we introduce the coordinates z in the direction of the deuteron, x perpendicular to the edge of the nucleus, and y parallel to the edge of the nucleus. In doing the calculation analogous to (8) we now have to impose the condition that the neutron misses and the proton hits the nucleus, i.e., we are to take $\psi_d=0$ unless $x_n>0$ and $x_p<0$. The y and z integrations in (8) are unaltered; after performing them we are left with a wave function which can be written

$$\psi(p_y,p_z,x_n,x_p)=h^{-\frac{1}{2}}\int_{-\infty}^{\infty}\psi(p_x',p_y,p_z)$$
$$\times\exp[(i/\hbar)p_x'(x_n-x_p)]dp_x', \qquad (16)$$

with $\psi(p_x,p_y,p_z)$ given by (9). We next express the wave function in terms of the momentum variables p_x for the neutron, p_{xp} for the proton:

$$\psi(p_y,p_z,p_x,p_{xp})$$
$$=h^{-1}\int_0^\infty dx_n\int_{-\infty}^0 dx_p\psi(p_y,p_z,x_n,x_p)$$
$$\times\exp[-(i/\hbar)(p_xx_n+p_{xp}x_p)]$$
$$=-\hbar^{\frac{1}{2}}(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\frac{\psi(p_x'p_yp_z)}{(p_x'-p_x)(p_x'+p_{xp})}dp_x'.$$

The poles in the denominator are to be avoided by deforming the contour of integration into the upper half-plane. Since $\psi(p_x', p_y, p_z)$ has one pole in the upper half-plane, this integral can be evaluated in terms of the residue at the pole. We find

$$\psi(p_y, p_z, p_x, p_{xp}) = \frac{\hbar^{\frac{1}{2}}(M\epsilon_d)^{\frac{1}{4}}}{(2\pi)^{\frac{3}{2}}P(p_x - iP)(p_{xp}+iP)},$$

where

$$P = (M\epsilon_d + p_y^2 + p_z^2)^{\frac{1}{2}}.$$

The probability of a given neutron momentum is

$$P_1(\mathbf{p}) = \int_{-\infty}^{\infty} |\psi(p_y, p_z, p_x, p_{xp})|^2 dp_{xp}$$

$$= \frac{\hbar(M\epsilon_d)^{\frac{1}{2}}}{8\pi^3 P^2(p_x^2+P^2)} \int_{-\infty}^{\infty} \frac{dp_{xp}}{(p_{xp}^2+P^2)}$$

$$= \frac{\hbar}{8\pi^2} \frac{(M\epsilon_d)^{\frac{1}{2}}}{P^3(p_x^2+P^2)}. \quad (17)$$

Equation (17) gives the differential cross section per unit length of the circumference of the nucleus. We have now to integrate around the circumference, in analogy with the integral over dl in the derivation of the total cross section. We must remember that the x and y axes rotate as we go around the nucleus. If ϕ is the azimuthal angle around the circumference, we can write $p_x = p_\perp \cos\phi$, $p_y = p_\perp \sin\phi$, $dl = Rd\phi$, and the differential cross section is

$$d\sigma = \frac{\hbar(M\epsilon_d)^{\frac{1}{2}}R}{8\pi^2(M\epsilon_d+p^2)} \times \int_0^{2\pi} \frac{d\phi}{(M\epsilon_d + p_z^2 + p_\perp^2 \sin^2\phi)^{\frac{3}{2}}} d\mathbf{p}. \quad (18)$$

To get the angular distribution, we integrate (18) over p_z,

$$d\sigma = [\hbar(M\epsilon_d)^{\frac{1}{2}}R/8\pi^2] \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{dp_z}{(M\epsilon_d + p_\perp^2 + p_z^2)(M\epsilon_d + p_z^2 + p_\perp^2 \sin^2\phi)^{\frac{3}{2}}} \cdot 2\pi p_\perp dp_\perp. \quad (19)$$

To carry out the integrations,[3] we change variable from p_z to a new variable ψ, defined by

$$\tan\psi = p_z/(M\epsilon_d + p_\perp^2 \sin^2\phi)^{\frac{1}{2}}.$$

The double integral in (19) becomes

$$2\int_0^{\pi/2} \cos^3\psi d\psi \int_0^{2\pi} \frac{d\phi}{(M\epsilon_d + p_\perp^2 \sin^2\phi)(M\epsilon_d + p_\perp^2 \cos^2\psi + p_\perp^2 \sin^2\psi \sin^2\phi)}$$

$$= \frac{2}{(M\epsilon_d + p_\perp^2)} \int_0^{\pi/2} \cos\psi d\psi \int_0^{2\pi} \left[\frac{1}{M\epsilon_d + p_\perp^2 \sin^2\phi} - \frac{\sin^2\psi}{M\epsilon_d + p_\perp^2 \cos^2\psi + p_\perp^2 \sin^2\psi \sin^2\phi}\right] d\phi$$

$$= \frac{4\pi}{(M\epsilon_d + p_\perp^2)^{\frac{3}{2}}} \int_0^{\pi/2} \left[\frac{1}{(M\epsilon_d)^{\frac{1}{2}}} - \frac{\sin^2\psi}{(M\epsilon_d + p_\perp^2 \cos^2\psi)^{\frac{1}{2}}}\right] \cos\psi d\psi$$

$$= \frac{4\pi}{M^2\epsilon_d^2} \frac{1}{(1+\zeta^2)^{\frac{3}{2}}} \{1 - (1/2\zeta^3)[(1+\zeta^2)\tan^{-1}\zeta - \zeta]\},$$

where ζ is the same variable used in (13), $\zeta = p_\perp/(M\epsilon_d)^{\frac{1}{2}} = \theta/\theta_0$. Putting this in (19) gives

$$d\sigma = [RR_d/\pi(1+\zeta^2)^{\frac{3}{2}}] \times \{1 - (1/2\zeta^3)[(1+\zeta^2)\tan^{-1}\zeta - \zeta]\} d\Omega_\zeta. \quad (20)$$

It can readily be verified that integration over $d\Omega_\zeta$ again gives (5) for the total cross section.

[3] I am indebted to Dr. Joseph Weinberg for this integration.

Comparing (20) and (13), we see that the angular distribution for the opaque nucleus differs from that for the transparent nucleus by the factor in the curly bracket. The angular distribution given by (20) is also plotted in Fig. 2, and we see the two distributions are not very different. The half-width given by (20) is $\theta_{\frac{1}{2}} = 1.601\theta_0$, only 4 percent wider than that given by (13). The distribution (20) has a higher tail at large angles than (13), an effect that can be interpreted as caused by the additional diffraction of the neutrons around the edge of the nucleus.

The calculation of the energy distribution of the neutrons parallels the derivation of (15). We first integrate (18) over $p_\perp$; this integral is elementary, and gives

$$\int_0^\infty \frac{p_\perp dp_\perp}{(M\epsilon_d+p_z^2+p_\perp^2)(M\epsilon_d+p_z^2+p_\perp^2\sin^2\phi)^{\frac{3}{2}}} = \frac{1}{2(M\epsilon_d+p_z^2)^{\frac{3}{2}}\cos^2\phi}\left\{\frac{1}{\cos\phi}\ln\left(\frac{1+\cos\phi}{1-\cos\phi}\right)-2\right\}.$$

The integration over ϕ gives just a numerical factor, which is equal to π^2. Finally, changing variables from p_z to E by means of (14), (18) becomes

$$d\sigma=\tfrac{1}{4}\pi RR_d\frac{E_d\epsilon_d}{[(E-\frac{1}{2}E_d)^2+E_d\epsilon_d]^{\frac{3}{2}}}dE. \quad (21)$$

The energy distribution (21) has a half-width $\Delta E_{\frac{1}{2}}=1.533(E_d\epsilon_d)^{\frac{1}{2}}$, i.e., $\Delta E_{\frac{1}{2}}=31$ Mev for $E_d=190$ Mev. A plot of the energy distribution for this deuteron energy is given by the solid curve in Fig. 3. It will be remarked that some neutrons are to be expected with an energy considerably larger than $\frac{1}{2}E_d$: 3 percent of the area of the curve lies between 150 Mev and 190 Mev.

Comparison of (15) and (21) shows that the energy distribution, unlike the angular distribution, is appreciably different in the transparent and opaque cases, the latter giving a narrower distribution.

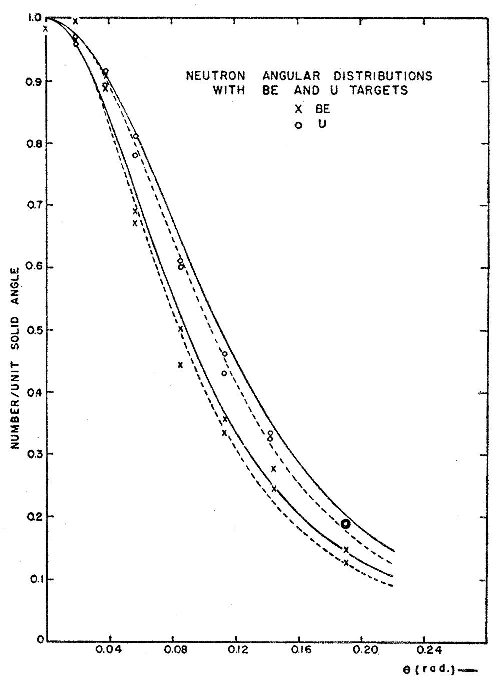

FIG. 4. Measured and calculated angular distributions for Be and U targets. Experimental points from measurements by Helmholz, McMillan, and Sewell. Curves for opaque nucleus (solid line) and transparent nucleus (dotted line) are included for comparison.

IV. EFFECT OF THE COULOMB FIELD ON THE ANGULAR DISTRIBUTION

The observed half-widths of the neutron distribution are found to increase slowly with atomic number. This effect can be understood as being caused by the nuclear Coulomb field, which deflects the deuteron slightly before the stripping process takes place. There are two sources of deflection. The first is an intrinsic one: the bending of the deuteron's orbit in the field of the nucleus at whose surface the deuteron is stripped. The second is due to the finite thickness of the target; multiple scattering produces a fanning out of the deuteron beam as it traverses the target. The angle of deflection attributable to the first cause is[4]

$$\theta_c=E_b/2E_d,$$

where $E_b=Ze^2/R$ is the barrier height. For 190-Mev deuterons bombarding U, $\theta_c=0.037$. The angle of deflection caused by multiple scattering in passing through the target can be calculated from the usual formula[5]; the results for a number of elements are tabulated in the preceding paper by Helmholz, McMillan, and Sewell. Although both angles are small, they are by no means negligible compared to the width of the neutron distribution.

Since the angular distributions given by (20) and (13) are very little different, we shall simplify our treatment of the Coulomb effects by treating the transparent nucleus case. The results can then be translated to the opaque case by multiplying by the factor by which (20) and (13) differ.

[4] Since $Ze^2/\hbar v=1.5$ for 190-Mev deuterons and a U target, it seems adequate to use a classical description to obtain an estimate of the spreading due to the Coulomb field.

[5] E. J. Williams, Proc. Roy. Soc. 169, 531 (1939).

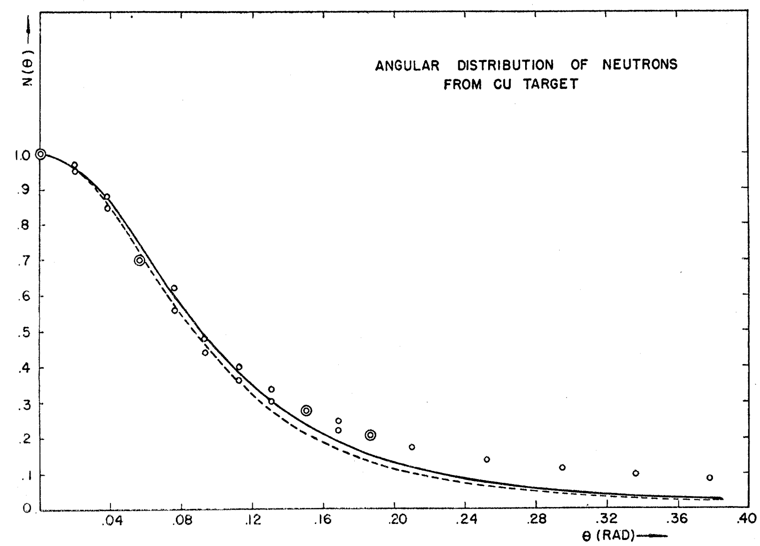

FIG. 5. Measured and calculated angular distributions for a Cu target. Solid curve, opaque nucleus; dotted curve, transparent nucleus. The fact that at large angles the experimental points lie above the curves is presumably due to the production of neutrons, with a wider angular distribution, by processes other than stripping.

We first treat the effect of the deflection by the field of the nucleus which does the stripping. Using the same coordinate system employed in the calculation of the angular distribution from an opaque nucleus, the effect of the bending of the deuteron's orbit is to give the neutron an added momentum $p_0\theta_c$ in the x direction. The distribution function (9) is altered by having p_x replaced by $p_x - p_0\theta_c$. Equation (13) is changed to

$$P(\zeta)d\Omega_\zeta = \int_0^{2\pi} \frac{d\phi}{2\pi[1+\zeta^2\sin^2\theta+(\zeta\cos\theta-\zeta_c)^2]^{3/2}}\zeta d\zeta \tag{22}$$

$$= \int_0^{2\pi} \frac{d\phi}{2\pi[1+\zeta^2+\zeta_c^2-2\zeta\zeta_c\cos\phi]^{3/2}}\zeta d\zeta,$$

where

$$\zeta_c = p_0\theta_c/(M\epsilon_d)^{\frac{1}{2}} = \tfrac{1}{2}E_b/(E_d\epsilon_d)^{\frac{1}{2}}.$$

The integral appearing in (22) is a Legendre function[6] of argument

$$u = \frac{1+\zeta^2+\zeta_c^2}{[(1+\zeta^2+\zeta_c^2)^2-4\zeta^2\zeta_c^2]^{\frac{1}{2}}},$$

$$P(\zeta)d\Omega_\zeta = \frac{u^{\frac{3}{2}}P_{\frac{1}{2}}(u)}{2\pi(1+\zeta^2+\zeta_c^2)^{\frac{3}{2}}}d\Omega_\zeta. \tag{23}$$

The function $P_{\frac{1}{2}}(u)$ is given by the rapidly convergent series

$$P_{\frac{1}{2}}(u) = 1+\frac{3}{4}\left(\frac{u-1}{2}\right)-\frac{15}{64}\left(\frac{u-1}{2}\right)^2 +\frac{35}{256}\left(\frac{u-1}{2}\right)^3-\cdots.$$

For the purpose of making the further correction for multiple scattering, it is more convenient, and sufficiently accurate, to expand the integrand of (22) in powers of ζ_c. This gives

$$P(\zeta)d\Omega_\zeta = [1/2\pi(1+\zeta^2)^{\frac{3}{2}}] \times\{1+\zeta_c^2 f_1(\zeta)+\zeta_c^4 f_2(\zeta)\}d\Omega_\zeta, \tag{24}$$

where

$$f_1(\zeta) = 3(\tfrac{3}{2}\zeta^2-1)/2(1+\zeta^2)^2,$$

$$f_2(\zeta) = 15[1-5\zeta^2+(15/8)\zeta^4]/8(1+\zeta^2)^4.$$

Since (22) is symmetrical in ζ and ζ_c, a better approximation for $\zeta<\zeta_c$ can be obtained simply by interchanging ζ and ζ_c in (24).

The effect of multiple scattering is to give the deuteron beam, after traversing a thickness t of the target, a Gaussian spread of directions with a mean-square angle of scattering proportional to t. The angles θ_s quoted by Helmholz, McMillan, and Sewell are the root mean-square angles after traversing the full target thickness, T. At thickness t, the mean square angle of the Gaussian distribution is thus $\theta_s^2 t/T$, or expressed

[6] See, for example, E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Teddington, England, 1920), p. 314.

in terms of the spread in ζ, $\zeta_s^2 t/T$, with $\zeta_s=\theta_s(E_d/\epsilon_d)^{\frac{1}{2}}$. The effect of spreading the distribution (24) by this additional Gaussian distribution is given by applying to (24) the integral operator $G=\exp\{(\zeta_s^2 t/4T)\Delta\}$, where

$$\Delta=(\partial/\zeta\partial\zeta)(\zeta\partial/\partial\zeta)$$

is the Laplacian operator. Since the width of the Gaussian distribution is small compared to the width of the distribution (24), the exponential can be expanded, and the operator written

$$G=\left\{1+\frac{\zeta_s^2 t}{4T}\Delta+\frac{\zeta_s^4 t^2}{32T^2}\Delta^2\right\}.$$

Averaging over the thickness of the target, we get

$$G=\left\{1+\frac{1}{8}\zeta_s^2\Delta+\frac{1}{96}\zeta_s^4\Delta^2\right\}.$$

Applying this operator to (24), and to the corresponding formula with ζ and ζ_c interchanged, we find

$$\begin{aligned}P(\zeta)d\Omega_\zeta&=\frac{1}{2\pi(1+\zeta^2)^{\frac{3}{2}}}\{1+(\zeta_c^2+\tfrac{1}{2}\zeta_s^2)f_1(\zeta)\\&\quad+(\zeta_c^4+2\zeta_c^2\zeta_s^2+\tfrac{2}{3}\zeta_s^4)f_2(\zeta)\}d\Omega_\zeta,\quad \zeta\geqslant\zeta_c,\\&=\frac{1}{2\pi(1+\zeta_c^2)^{\frac{3}{2}}}\{1+\tfrac{1}{2}\zeta_s^2 f_1(\zeta_c)+\tfrac{2}{3}\zeta_s^4 f_2(\zeta_c)\\&\quad+[f_1(\zeta_c)+2\zeta_s^2 f_2(\zeta_c)]\zeta^2\}d\Omega_\zeta,\quad \zeta<\zeta_c.\end{aligned}\tag{25}$$

Equation (25) gives the neutron angular distribution for the transparent model; the formula for the opaque model is obtained by multiplying (25) by the correction for the opacity of the nucleus, the factor in curly brackets in (20).

The increase in half-width caused by the intrinsic scattering is, in first approximation, proportional to ζ_c^2, i.e., to $E_b^2\sim Z^2/R^2\sim Z^2/A^{\frac{2}{3}}$. The increase caused by multiple scattering is proportional to ζ_s^2, or, for given target thickness, to $Z^2\rho/A$. Because of the factor ρ this is not a smooth function of atomic number. With the $\frac{1}{16}$-in. thick targets used in the experiments, the intrinsic Coulomb effect contributes between 90 percent (in Be) and 60 percent (in U) of the total increase in half-width.

The Coulomb field will widen the proton distribution even more than the neutron distribution, since the proton in leaving the nucleus is bent through twice the angle the deuteron is in approaching it. The intrinsic Coulomb effect here is much larger than the multiple scattering, and the angular distribution can be obtained from (23), with ζ_c replaced by $3\zeta_c$.

An additional, smaller effect of the Coulomb field has been pointed out to me by Professor McMillan. The kinetic energy of the deuteron when it reaches the nucleus has been reduced by the amount E_b. Thus E_d in the foregoing formulae is to be taken not as the bombarding energy (corrected for energy loss in the target), but as this minus E_b. An interesting consequence is that the center of the neutron energy distribution will be shifted to lower energy by an amount $\frac{1}{2}E_b$ (7 Mev in U), while the proton, since it regains the energy E_b in escaping, will have its energy distribution shifted upwards by this amount.

Figures 4 and 5 show the calculated neutron angular distributions for targets of Be, U and Cu, and the distributions measured by Helmholz, McMillan, and Sewell. Figure 3 of their paper shows the measured half-widths for a number of elements, and the calculated half-widths The agreement is seen to be quite satisfactory.

A word remains to be said about the neutrons produced by the Coulomb-field disintegration of the deuteron. As previously mentioned, estimates by Dancoff indicate that about one-quarter as many neutrons would be produced in this way as by stripping. Whether, and to what extent, the experimental data might be taken to show the smallness of such an effect depends on the expected angular distribution of the neutrons. This will be the subject of a forthcoming paper by Professor Dancoff, whom I wish to thank for a number of discussions of the electric-field breakup, as well as of the stripping effect.

I am indebted also to Mr. T. B. Taylor, who carried out most of the computations.

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