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On the Production of Mesotrons by Nuclear Bombardment

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Mesotron production by nuclear bombardment with fast, heavy particles has been investigated theoretically in a semi-quantitative way to determine the expected threshold energies, the cross sections, and their energy dependence. Whereas a treatment in which the target nucleons are assumed to be at rest predicts a requisite incident energy of ~210 Mev, the present treatment, based on the Fermi degenerate gas model, finds the threshold incident energy as ~95 Mev. The threshold is somewhat higher for positive than for negative mesotrons. The cross section for single mesotron production, evaluated from the accessible volume in momentum space, is found to vary with the fractional excess energy, ϵ, as $\epsilon^{3.5}$ in the scalar or axial-vector theories; at low values of ϵ, a small difference in the energy dependence for negative and positive mesotrons arises from the necessity of giving the former a non-zero initial kinetic energy. For the pseudo-scalar and the polar-vector theories, the matrix element for mesotron emission is proportional to the momentum of the mesotron. This changes the power law to $\epsilon^{4.5}$.

INTRODUCTION

IN treatments of collisions between nuclei and high energy incident particles, it is often assumed as a first approximation that the constituent nucleons of the target nucleus may be considered as essentially free. For the case in which the desired result of the collision is the production of a mesotron (having rest-mass μ), the energy of the incident particle relative to the particular nucleon with which it collides must be at least μc^2. If the nucleon has only the small velocity of the nucleus as a whole, nearly all the relative energy must be supplied by the incident particle; in the limit of zero-nucleon velocity this predicts for an incident proton or neutron a requisite energy just twice the rest-mass of the mesotron,[1] or about 210 Mev.

A more refined calculation should take account of a possible contribution of the target-nucleon velocity to the relative energy, of any change in potential energy for the over-all process, and possibly also the effect of the inter-nucleonic forces. At least the first two of these refinements may be made rather easily. We shall use as our nuclear model the usual[2] degenerate Fermi gas mixture of protons and neutrons at zero temperature. Such a model is admittedly very crude, but will serve to determine orders of magnitude.

The limiting energy—the so-called "Fermi energy," E_F—of the degenerate gas sets an approximate upper limit, p_F, to the permissible momenta of the target nucleons. Using this maximum momentum, directed anti-parallel to the path of the incident particle, it is readily seen that a lowering of the required incident energy is obtained. Furthermore, if the incident particle is captured without ejection of other nucleons its binding energy is released. However, since all the lowest states within the degenerate gas are filled

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[1] We shall employ the value $\mu = 202$ electron masses ($\equiv 103$ Mev), as recently determined by W. B. Fretter, Phys. Rev. **70**, 625 (1946); see also D. J. Hughes, Phys. Rev. **71**, 387 (1947).

[2] H. A. Bethe, Rev. Mod. Phys. **9**, 82 (1937), ¶53A.

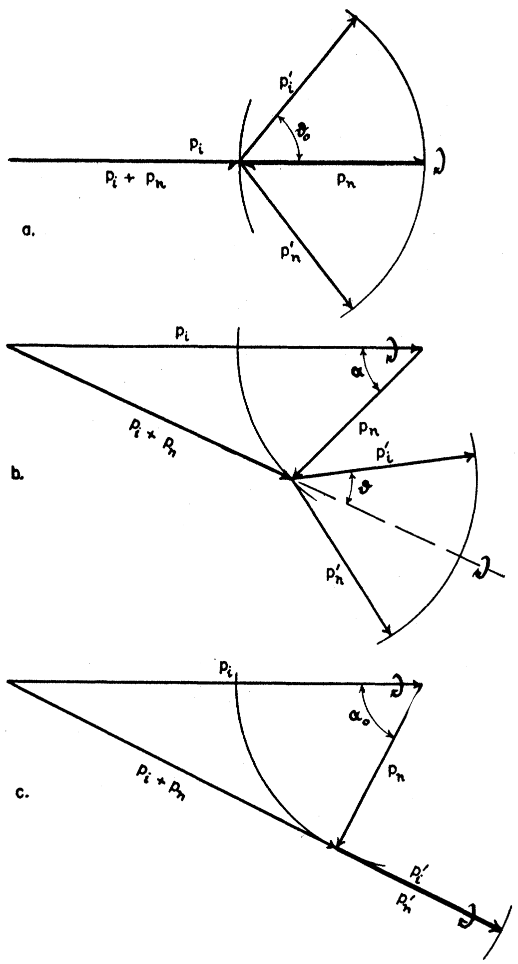

FIG. 1. Possible arrangements of the momentum vectors at the threshold energy with zero-mesotron velocity: (a) and (c) are the two extrema, (b) an intermediate case. Subscripts i and n refer to the incident particle and the nucleon, respectively; primes refer to post-collision conditions. The short curved arrows indicate rotations which leave the total energy unchanged. Here $p_n = p_n' = p_i' = p_F$, the radius of the Fermi sphere.

initially, both colliding particles must end in states of energy equal to, or greater than the Fermi energy. The program, then, will be: (1) to determine the minimum energy, E_{i0}, of the incident particle necessary to produce a mesotron; (2) to calculate from the accessible volumes in phase space the dependence of the cross section, σ, for mesotron production on the excess incident energy, ϵE_{i0}, above the threshold energy, E_{i0}; and (3), to fix the coefficient of the expression so obtained by considering collisions at energies much greater than E_{i0}.

DETERMINATION OF THE THRESHOLD ENERGY

In evaluating the minimum incident energy required to produce a single mesotron, it is necessary to consider the effect of the charge in some detail. When one compares the energy of the nucleus-mesotron couple at infinite separation (i.e., at the zero of potential energy) with that corresponding to a point in the nucleus, it is evident that for a *negative* mesotron to be observable it must be provided initially with at least the kinetic energy equivalent to the electrostatic potential energy, $-eV_e$, at the position of creation. The minimum observable end state is one in which the mesotron is at rest at infinity, and its corresponding relativistic energy is thus μc^2. On the other hand, a *neutral* mesotron retains any initial kinetic energy it may have had as well as its rest energy μc^2, while a *positive* mesotron at infinite distance has, in addition, the kinetic energy gained in being expelled from the nucleus.

Aside from these effects of the electrostatic potential, a natural permissible lower limit on the initial kinetic energy of the mesotron would seem to be set by the requirement that it have sufficient velocity to traverse a distance of the order of the nuclear radius within a "nuclear period," i.e., in $\sim 10^{-22}$ second. Depending upon the depth within the nucleus at which the creation occurs, the initial kinetic energy might have to be ~ 10 Mev for heavy nuclei. However, since there is no apparent reason why mesotrons should only be produced deep in the nucleus, the minimum initial kinetic energy may well be much less than this figure, especially for the lighter elements, and this factor will be neglected in the subsequent treatment. We will therefore assume[3] that the smallest total energy (including the rest energy) which may be given a negative or neutral

[3] At this point one might inquire as to the "binding energy" of the mesotron to the nucleon and its surrounding mesotron field, i.e., over and above its rest mass and the electrostatic effects. The essential question concerns the existence of a "mesotronic potential" analogous to the electrostatic potential. If such exists, either a potential barrier must be overcome, or the required initial kinetic energy of the mesotron will be raised, depending upon whether the force is a repulsion or an attraction. For lack of sufficient knowledge of this force we only mention the possibility, and hereafter neglect it. However, because of its necessarily short-range character and the comparatively long wave-length of the created mesotron, such a "mesotronic potential barrier" might well have a much smaller effect than would at first be supposed.

mesotron is μc^2, whereas the positive mesotron requires μc^2+eV_c. However, because of the tunnel effect[4] and the fact that the term eV_c is large ($\approx$15 Mev) only for the heavy elements, the effective difference in the threshold energies will generally be small.

Having decided on the minimum mesotron energies, a knowledge of the maximum energies of the target nucleons would permit the calculation of the necessary incident energy. The Fermi energy of the degenerate gas is given by the equation

$$E_F=[(\pi\hbar)^2/2M](3N/\pi v)^{\frac{2}{3}}, \quad (1)$$

where N is the number of neutrons or of protons, each of mass M, in the nuclear volume v. As an example, we consider the uranium nucleus and use for its radius the value $1.0\cdot 10^{-12}$ cm. We find for the neutron and proton gases, respectively,

$$E_{FN}=22.3 \text{ Mev}; \quad E_{FP}=16.4 \text{ Mev}. \quad (2)$$

For a case in which the mesotron may have zero initial kinetic energy, Figs. 1a, c show in cross section the two extremes of the possible orientations of the four momentum vectors representing the two colliding particles before and after collision. All intermediate values of $\theta(0\leq\theta\leq\theta_0)$ and the associated values of $\alpha(0\leq\alpha\leq\alpha_0)$, such as the case shown in Fig. 1b, correspond to the same total energy, the difference being in the relative energy only. Besides these limited variations of θ (and α), the curved arrows indicate other (unrestricted) rotations which do not alter the total energy. These will be of importance in the next section in evaluating the accessible volume in phase space.

For the general collision process (in which the mesotron is given an initial momentum) the energy-conservation law reads

$$\frac{p_i^2+p_n^2}{2M}=\mu c^2+V+\frac{p_i'^2+p_n'^2}{2M}+\frac{p_\mu'^2}{2\mu}+zeV_c, \quad (3)$$

where $-(E_F+V)$ is the binding energy of the incident particle; subscripts i and n refer to the incident and nucleon particles, respectively, μ refers to the mesotron, and primes denote the post-collision state. In the last term V_c is the electrostatic potential at the point where the mesotron is born, and ze is the charge of the mesotron. If we neglect for the moment the last two terms on the right-hand side of Eq. (3), and recall that on the basis of the assumed nuclear model

$$p_n^2\leq 2ME_F\begin{cases}\leq p_i'^2\\ \leq p_n'^2,\end{cases} \quad (4)$$

evidently

$$p_i^2/2M\geq\mu c^2+V+E_F. \quad (5)$$

This expression means that the incident energy $p_i^2/2M$ must be greater than, or equal to the rest energy of the mesotron minus the binding energy of the incident particle. However, we are permitted to use the equality only if momentum can be conserved, which in turn implies the possibility of a construction as in Fig. 1. From part b of this figure we obtain the relation[5]

$$(2p_F\cos\theta)^2=(\mathbf{p}_i+\mathbf{p}_n)^2 =p_i^2+p_F^2-2p_ip_F\cos\alpha. \quad (6)$$

The desired construction will therefore be possible if there exist values of α for which $\cos\theta$, calculated from Eq. (6), is not greater than unity. Since $\cos\theta$ has its minimum value for $\alpha=0$, we require that

$$\cos\theta=(p_i-p_F)/2p_F\leq 1,$$

or

$$p_i=3p_F. \quad (7)$$

Using this result in Eq. (5), we see that momentum conservation will be possible provided that

$$E_F\geq(\mu c^2+V)/8. \quad (8)$$

Since $V\approx-25$ Mev, values of $E_F\approx 10$ Mev will permit the momentum conservation corresponding to the minimal energy of Eq. (5). With a binding energy of 8 Mev we get from Eq. (5) 95 Mev for the threshold energy. For positive mesotrons the effective threshold is a few Mev higher. This result is radically different from the estimate of 210 Mev mentioned in the introduction.

THE ENERGY DEPENDENCE OF σ

When the minimal incident energy, E_{i0}, is exceeded, we have some latitude of choice in the magnitudes of the initial and final nucleon momenta and the mesotron momentum, as well as in their directions (cf. the discussion of Fig. 1

[4] There will, of course, be a finite probability for the positive mesotron to penetrate the Coulomb potential barrier when its energy lies between μc^2 and μc^2+eV_c.

[5] Vector quantities are symbolized in **bold-face** type; their absolute magnitudes by the same letters in *italics*.

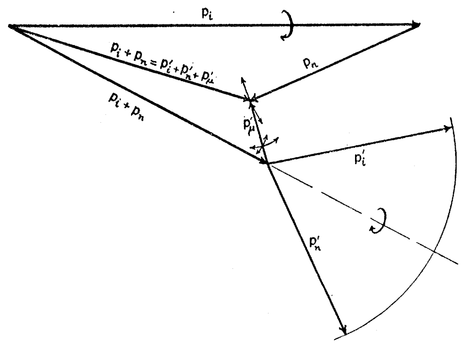

FIG. 2. A possible arrangement of the momentum vectors at the threshold energy for the case in which the minimal mesotron momentum p_μ' is not zero. Other symbols are the same as in Fig. 1. (*Note added in proof:* the vector colinear with the dashed line should be labeled $p_i'+p_n'$.)

above). In order to determine the dependence of the mesotron-production cross section σ on the fractional excess energy ϵ, we shall investigate the total volume in momentum space which corresponds to a successful collision. In this we should allow for the possibility that the minimum mesotron momentum is not zero; such a case is illustrated in Fig. 2. Starting from such an initial orientation of the momentum vectors appropriate to the minimal energy, and denoting increment vectors away from this orientation by $\boldsymbol{\pi}$'s with corresponding subscripts, we obtain

$$\epsilon E_{i0}=\left(\frac{\mathbf{p}_i'\cdot\boldsymbol{\pi}_i'}{M}\right)+\left(\frac{\mathbf{p}_n'\cdot\boldsymbol{\pi}_n'}{M}\right)+\left(\frac{\mathbf{p}_\mu'\cdot\boldsymbol{\pi}_\mu'}{\mu}\right)+\left(-\frac{\mathbf{p}_n\cdot\boldsymbol{\pi}_n}{M}-\frac{\pi_n^2}{2M}\right)+\left(\frac{\pi_i'^2}{2M}\right)+\left(\frac{\pi_n'^2}{2M}\right)+\left(\frac{\pi_\mu'^2}{2\mu}\right), \quad (9)$$

in which the (minimal) mesotron momentum $\mathbf{p}_\mu'$, and its increment $\boldsymbol{\pi}_\mu'$, are determined by momentum conservation:

$$\mathbf{p}_\mu'=\mathbf{p}_i+\mathbf{p}_n-\mathbf{p}_i'-\mathbf{p}_n', \quad (10a)$$

$$\boldsymbol{\pi}_\mu'=\boldsymbol{\pi}_i+\boldsymbol{\pi}_n-\boldsymbol{\pi}_i'-\boldsymbol{\pi}_n'. \quad (10b)$$

Each term in parentheses in Eq. (9) is intrinsically positive, as may be easily demonstrated with the help of Eq. (4) and the additional restriction that $|\mathbf{p}_\mu'+\boldsymbol{\pi}_\mu'|\geq p_\mu'$.

Since $\boldsymbol{\pi}_i$ is set by the initial conditions, Eq. (10b) gives three relations between the twelve components of $\boldsymbol{\pi}_i'$, $\boldsymbol{\pi}_n'$, $\boldsymbol{\pi}_n$, and $\boldsymbol{\pi}_\mu'$. From the remaining nine independent variables we eliminate from consideration those whose range is not appreciably altered by the excess energy. Such variations in momenta give rise to a practically constant factor in the volume of momentum space, and do not affect the energy dependence of the cross section. One such degree of freedom is the rotation of $\mathbf{p}_i'$ and $\mathbf{p}_n'$ about the dashed line, as indicated in Figs. 1b and 2; other possibilities are indicated by the curved arrows.

For simplicity we take first the case in which $p_\mu'=0$. We should consider only such variations of the $\boldsymbol{\pi}$'s for which

$$\mathbf{p}_n\times\boldsymbol{\pi}_n=0, \quad (11a)$$

and

$$(\mathbf{p}_i'\times\mathbf{p}_n')\cdot(\boldsymbol{\pi}_i'-\boldsymbol{\pi}_n')=0. \quad (11b)$$

Of the nine independent variables these relations eliminate three. One of the six remaining variables is evidently to be chosen parallel to $\mathbf{p}_n$, and by Eq. (9) will be of the order ϵ. Of the six coordinates of $\boldsymbol{\pi}_i'$ and $\boldsymbol{\pi}_n'$, Eq. (11b) eliminates only the one corresponding to a pure rotation. We are, therefore, left with three degrees of freedom in directions perpendicular to $\mathbf{p}_n'$ and $\mathbf{p}_i'$, and two parallel. By Eq. (9) the maximum value of each perpendicular component is of the order $\epsilon^{\frac{1}{2}}$, and of each parallel component, of order ϵ. These conditions do not violate momentum conservation since the largest mesotron momentum is of the order $\epsilon^{\frac{1}{2}}$, the same as the largest nucleon-momentum increment. The volume in momentum space corresponding to a successful collision for all energies below $(1+\epsilon)E_{i0}$ is thus proportional to the $1+3\cdot\frac{1}{2}+2=4.5$ power of ϵ. Differentiating with respect to ϵ, the volume per unit energy range, and therefore the cross section for the process, is proportional to $\epsilon^{3.5}$. This is the expected energy dependence for positive and neutral mesotrons.

For the case in which $p_\mu'\neq 0$, the additional condition

$$\mathbf{p}_\mu'\times\boldsymbol{\pi}_\mu'=0 \quad (11c)$$

restricts the vector $\mathbf{p}_\mu'+\boldsymbol{\pi}_\mu'$ to a spherical shell of thickness $\sim\epsilon$. An argument similar to that above now finds the total volume in momentum space proportional to ϵ^4, and the volume per unit-energy-interval proportional to ϵ^3. This is the case for negative mesotrons.[6]

[6] It is to be noted, however, that this result holds only as long as $\pi_\mu'\ll p_\mu'$. Since this condition will be violated

MULTIPLE MESOTRON PRODUCTION

For multiple production, the minimal requisite incident energy is approximately proportional to the number n of mesotrons produced. If the condition analogous to Eq. (8), namely,

$$E_F \geq (n\mu c^2 + V)/8, \qquad (8')$$

is satisfied, a generalization of Eq. (5) gives for the threshold energy

$$E_{i0} = (s+t)\mu c^2 + (V + E_F) + teV_c, \qquad (12a)$$

where s is the number of negative plus neutral mesotrons, and t is the number of positive mesotrons produced. If Eq. (8′) does not hold, E_{i0} is to be determined from the condition for conservation of the *relative* energy:

$$\tfrac{1}{2}(E_{i0}^{\frac{1}{2}} + E_F^{\frac{1}{2}})^2 = (s+t)\mu c^2 + V + teV_c. \qquad (12b)$$

Comparison of the values of E_F obtained from Eq. (8′) with the actual values of Eq. (2), shows that Eq. (12b) must be used for $n = s+t > 2$. The case $n=2$ is borderline, so neither (12a) nor (12b) applies exactly; the threshold by either equation, however, is not far from 200 Mev. The energy dependence of the production cross section for n mesotrons (in case $n>2$) is found to be $\epsilon^{(3n+5)/2}$. For $n=2$ the ambiguity as to whether (12a) or (12b) applies leads to a power law between ϵ^5 and $\epsilon^{5.5}$, with the threshold $\sim$200 Mev. This is mentioned since recent considerations[7–9] suggest that the creation of a *single* charged mesotron may be forbidden.

So far we have dealt only with the energy dependence of the density of energy states in phase space, and have calculated the cross sections with the tacit assumption that the square of the matrix element between initial and final states (i.e., before and after collision) is independent of the momentum of the mesotron. Such an assumption is justified only in the case of the scalar and axial-vector theories of mesotron forces.[10] However, in order to explain the existence of a nuclear quadrupole moment, it has been necessary in the mesotron theory to introduce the first derivative of the mesotron field into the interaction terms (pseudo-scalar and polar-vector theories). When the initial mesotron kinetic energy is zero, the matrix element is proportional to $\epsilon^{\frac{1}{2}}$, and thus for each mesotron produced an additional factor of ϵ is introduced into the expression for the cross section.

even at relatively low values of $\epsilon(\sim 0.1)$, the cross section for negative mesotrons will have essentially the same energy dependence as that given for positive mesotrons. Other approximations vitiate these results very near the threshold, and it is thus questionable if the indicated behavior difference caused by the charge would be experimentally significant.

[7] J. A. Wheeler, Phys. Rev. **71**, 462 A (1947).

[8] E. Fermi, E. Teller, and V. Weisskopf, Phys. Rev. **71**, 314 (1947).

[9] M. Conversi, E. Pancini, and O. Piccioni, Phys. Rev. **71**, 209 (1947).

[10] W. Pauli, *Meson Theory* (Interscience Publishers, Inc., New York, 1946), Chap. 1.

THE ABSOLUTE MAGNITUDE OF σ

We may carry the discussion one step further and determine the order of magnitude of the absolute cross sections for mesotron production. For high energies the above restrictions on the accessible volume of momentum space are less important, and at $\epsilon=1$ can be considered to be absent.[11] At this energy ($E_i \sim 200$ Mev for single mesotron production), the energy E relative to an *average* target nucleon is about 100 Mev; the corresponding wave-length, $\lambda\!\!\!^- = \hbar/(2mE)^{\frac{1}{2}} \approx 0.45 \cdot 10^{-13}$ cm, is evidently smaller than the range of nuclear forces, $\hbar/\mu c = 2\cdot 10^{-13}$. This fact justifies a consideration of the collision in terms of a particle picture.

For two nucleons to interact strongly the distance of closest approach must, of course, be less than the range of nuclear forces. A closer collision is actually necessary for the following reason. In order to produce a "quantum" of rest-mass μ, the relative frequency ω of the two particles during collision must be equivalent to at least an energy μc^2; thus for $E = \mu c^2$,

$$\omega \geq E/\hbar. \qquad (13)$$

With the collision parameter b (the distance of closest approach) the predominant angular frequency will be of the order of u/b, where u is the relative velocity. This gives

$$u/b \geq E/\hbar, \quad \text{or} \quad b \leq \hbar u/E = 2\hbar/p = 2\lambda\!\!\!^-, \qquad (14)$$

and there will be a high enough frequency only if the collision parameter is less than twice the relative wave-length $\lambda\!\!\!^-$. The corresponding creation cross section, σ, for a single nucleon is therefore

$$\sigma = \pi b^2 = 4\pi\lambda\!\!\!^-{}^2 = 4\pi\hbar^2/2mE.$$

[11] The fact that the mesotron has only one-tenth the mass of a nucleon has the consequence that at $\epsilon=1$ the restrictions have not yet vanished completely. Thus the cross sections to be given for $\epsilon=1$ are somewhat overestimated.

With $\dot{E}\sim 100$ Mev we find $\sigma=0.025$ barn per nucleon; this sets an approximate upper limit on the cross section.

Since the *scattering* of neutrons by protons presumably corresponds to the emission and reabsorption of a mesotron, one would not expect the cross section for mesotron production to be of an order of magnitude lower than that for scattering. Estimates of the scattering cross section at these energies give a result comparable with the value of σ obtained above. The total cross section for an intermediate element of mass number A will thus be simply 0.025A barn. For the heavier elements this may give a figure larger than the geometrical area of the nucleus, and correspondingly must be corrected downwards.

For the sake of definiteness, the total cross section for the creation of a single mesotron (in the pseudo-scalar or polar-vector theories) will be given approximately by

$$\Sigma\sim 0.025A\epsilon^{4.5} \text{ barn}, \tag{15a}$$

with

$$\epsilon\approx (E_i-100)/100, \tag{15b}$$

while that for mesotron pair creation (again in the pseudo-scalar theory) is

$$\Sigma\sim 0.025A\epsilon^{7.5}, \tag{16a}$$

with

$$\epsilon\approx (E_i-200)/200. \tag{16b}$$

CONCLUDING REMARKS

In determining the probable relation of the estimates given above to the true cross sections, we distinguish between two different types of errors: (1) those which are introduced through the approximate character of the assumed nuclear model and of the calculations employed; and (2) those which enter through failure to take account of all relevant factors in the domain of nuclear forces, as was for example suggested in reference 3. This second category merely reflects the lack of conclusive knowledge concerning nuclear forces, and it is impossible at this stage to estimate the over-all effect of these errors.

In the first group, the nature and magnitudes of the approximations are more evident. The assumption of a continuum of energy levels at all energies above the threshold leads to a dependence on a higher power of ϵ than would have been obtained at low energies by using the correct, discrete spectrum. This means that the cross sections obtained from Eqs. (15) and (16) have been decreased (from their values at $\epsilon=1$) by too high a power of ϵ, and are therefore underestimated. Furthermore, the fluctuations in the target-nucleon momenta, whose terminae nominally lie in the surface of the Fermi sphere, cause the volume in momentum space to be underestimated for small ϵ, again indicating that the cross sections given above are, if anything, too low.

Other complications, possibly having opposite effects, would enter if, for example, there were an unfavorable radial distribution of energy among the nucleons such that fast internal particles were shielded by slower external ones. However, with such high energies the incident particle will be only slightly deflected in scattering collisions and the corresponding energy loss will be small. Even with the generous estimate of several Mev per scattering process, the effect on the mesotron-production cross section would not be very noticeable when the incident energy is, say, 20 Mev above the threshold. The assumption that at $\epsilon=1$ no restrictions apply to the momentum volume also gives rise to an overestimate of σ. As mentioned in reference 11, the small mass of the mesotron is likely to cause some restrictions in the region of $\epsilon=1$.

When these approximations are considered, it seems a reasonable estimate that at $\epsilon=0.1$ the probable error of the cross sections given above is of the order of magnitude of a factor 10, although an unfavorable accumulation of errors might raise this figure.[12] It should be borne in mind, however, that this does not include those inaccuracies which may be introduced through ignorance of the correct form of the interaction to be used in the mesotron field, and of the selection rules which specify the number and kind of mesotrons which can be produced.

ACKNOWLEDGMENT

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[12] *Note added in proof:* Messrs. W. Horning and M. Weinstein have kindly communicated to us their manuscript on "Meson materialization," to appear in the *Physical Review*, in which they confirm our result for the energy dependence of the cross section for single mesotron production (for the scalar theory) but obtain a significantly lower value for the coefficient.