Note on a Variation with Frequency of the Torsional Modulus of German Silver Wire and its Relation to Gyromagnetic Measurements

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A quite unmistakeable deficit of the mean gyromagnetic ratio for certain ferromagnetic substances as reported some years ago by W. Sucksmith and L. F. Bates from the mean ratio obtained for the same substances in much more elaborate investigations in the laboratories of one of us, might be explained by such a variation alone if the torsional modulus of German silver wire in rapid oscillation (dynamic modulus, M_d) were smaller by four or five percent than the modulus for steady twist (static modulus, M_s)—a result, however, which we should not expect to find. The calibrating operations of the authors referred to were carried out partly with rapid oscillations, partly with steady deflections; and they assumed (implicitly) that the two moduli were equal. We have not worked with the true static modulus, but have determined the ratio of M_d for frequencies ranging approximately from 17 to 50 per second, comparable with those used in the work mentioned, to M_d for periods ranging approximately from 4 sec. to 28 sec.—times doubtless quite comparable with or greater than the times required to produce steady twists in this work. We have detected no certain change of modulus in the high frequency range, or in the range of periods between 4 sec. and 28 sec. However, the modulus has been found to diminish by 1 percent in passing from oscillations with the period 5 sec. to those in the high frequency range. This unexpected diminution reduces somewhat the discrepancy that initiated the experiments described in this note, and it seems important to us for other reasons also. But much the greater part of the discrepancy must be accounted for in other ways.

I. INTRODUCTION AND THEORY

In three extensive investigations on the gyromagnetic ratios of many ferromagnetic substances it has been found that this ratio, ρ, ranges from about $1.00\times m/e$ for Heusler alloy to about 1.08 or $1.09\times m/e$ for cobalt.[1] The value of ρ for iron is about $1.03\times m/e$; that for nickel, Hipernick, and Permalloy about $1.05\times m/e$. There is no disagreement, within the limits of the experimental errors, between these results and the few obtained by others in recent years and sufficiently precise to be at all comparable with them; but the work done some years ago by Sucksmith and Bates[2] gave about $1.00\times m/e$ for both iron and nickel, with an experimental error which they claimed to be not over 1 or 2 percent. Indeed, they considered $1.00\times m/e$ correct for all the substances they tried.

In earlier papers[1] attention has been called to several sources of error which the authors mentioned had not considered. The effects of some of these errors, but not all, tend to disappear in the means from a sufficiently large number of independent observations, which may not have been made. In searching a number of years ago for possible constant sources of systematic error making all values too small, it appeared that an error of this kind might be involved in the work referred to because the authors in one of the calibrating operations used German silver wire in rapid torsional vibration, whereas in another they used a steady twist of the same wire. If the static modulus of torsion were greater than the dynamic modulus, the low values might be at least in part accounted for by this fact.

The matter will become clear from the following brief discussion: In the paper of Sucksmith and Bates the quantity ρ should be calculated from the equation

$$\rho = CA_s,$$

where C is a quantity which does not enter into the present discussion and A_s is the static torsional constant of the wire. Instead of using A_s in calculating ρ, however, they used A_d, the dynamic constant, which they determined from high frequency oscillations. Thus they obtained an *apparent* value, which may be designated by ρ_{app}, from the equation

$$\rho_{\text{app}} = CA_d,$$

whereas the true value is

$$\rho = \rho_{\text{app}}(A_s/A_d).$$

[1] See especially S. J. Barnett, Proc. Am. Acad. **73**, 401–455 (1940); **75**, 109–129 (1944); and *Le Magnétisme* (Proceedings of the Strasbourg *Réunion sur le Magnétisme*) **2**, 203–244 (1940).

[2] W. Sucksmith and L. F. Bates, Proc. Roy. Soc. **A104**, 499 (1923); **A108**, 638 (1925).

With the object of investigating this question, we have recently made some experiments on wires of German silver. It has not been convenient for us to make any experiments of a truly static character, but we have made low frequency observations, in some of which the period was greater than 28 seconds, an interval doubtless much greater than the time required for reaching the steady deflections in the work of Sucksmith and Bates, as well as observations with frequencies up to more than 1000 times as great.

We have used the classical method of torsional oscillations in such ways as to give the ratio of the torsional constant at one frequency to that at another frequency. The higher frequency oscillations were produced magnetically at the natural frequencies determined by resonance. The lower frequency oscillations were produced and their periods determined in the usual way.

We may proceed conveniently by either of two processes: (1) the length of the wire may be kept constant and the moment of inertia changed, or (2) the moment of inertia may be kept constant and the length of the wire changed. It was evidently advantageous to use the first method in the part of the work in which most of the frequencies were relatively high; both methods have been used in the part in which all the frequencies were low. Possible lack of uniformity of the wire is a disadvantage in the second case.

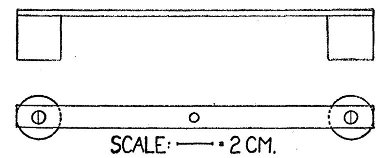

FIG. 2. The system S_3. Approximate masses in grams: bar 69.0; each cylinder 108.4; each screw, 0.5.

Because the result of the experiments seems to us quite different from what would be expected, we give the procedure in some detail, as only in this way can it be shown that the possible errors have been eliminated.

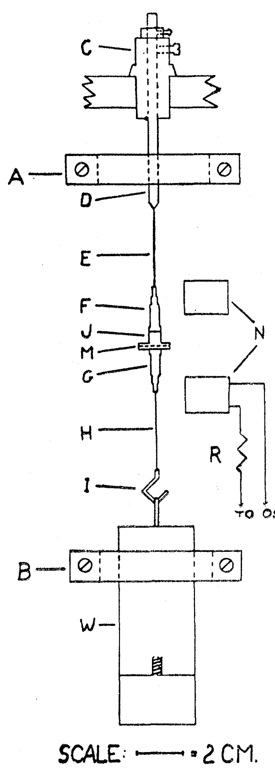

FIG. 1. Apparatus arranged for vibration of the system S alone. Approximate mass of weight $W=567$ grams. In use the axis of the magnet M is normal to that of the coil N, instead of parallel as shown in the figure.

II. WORK AT HIGHER FREQUENCIES

The method used in the part of the work at higher frequencies will be clear from what follows. The essential parts of the inertia apparatus were made as nearly symmetrical about a central vertical line as practicable. The dimensions of most of the apparatus are indicated in Figs. 1–4. The wire under test is divided into two nearly equal parts E and H, about 35 mm long. The upper part E is soldered at the upper end into a brass rod D passing through a torsion head C, and at the lower end into a lug F threaded at the lower end. The lower wire H is soldered at the upper end into a lug G similar to F, except as indicated below, and at the lower end into a brass hook I, on which a stretching weight W can be hung. The hooks have rectangular cross sections and thus form a rigid joint. Between the two lugs and screwed to them is a brass piece J with two cylindrical surfaces. The diameter of the upper cylinder is about 0.476 cm and the height is about the same. The lower cylinder has a diameter of 10 mm and a height of 2 mm. A hard magnetized steel rod, 10 mm long and 0.5 mm in diameter, passes symmetrically through a horizontal hole in the lower cylinder. Details of the system F, J, M, G are given in Fig. 5.

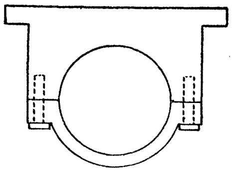

FIG. 3. Ring A or Ring B.

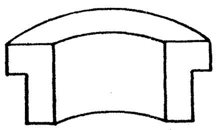

FIG. 4. One of the wooden half-rings.

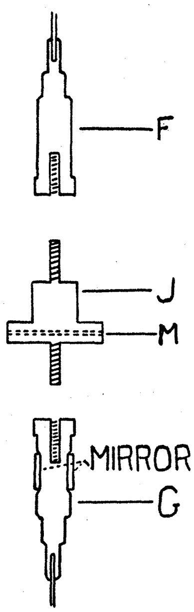

FIG. 5. Details of the system S.

The weight W is divided into two parts, screwed together so that the lower part can readily be removed or attached. Parts of two sides of the lug G are milled flat, parallel, and normal to the axis of the magnet. They carry two similar small plane mirrors (Fig. 5) which make it possible to study the angular motion of the system with lamp, lens, and scale.

The torsion head is so adjusted that the axis of the magnet points approximately east and west. The earth's magnetic field did not alter the frequency appreciably; but if it had done so the effect would have been eliminated in the calculation of the result sought.

To change the moment of inertia and frequency of the system small toroidal rings, not illustrated in the figures, or a weighted bar (Figs. 2 and 6) can be slipped over the upper cylinder of J.[3] The axis of the system passes through two fixed horizontal rings A and B, Figs. 1 and 3. The upper ring, together with a small bar laid across it, was used to hold one of the inertia rings when not in use; the other, or one half of it, to clamp the weight W in position, by means of cotton wads, two half-cylinders of wood, or one half-cylinder and soft wax. One of the half-cylinders of wood is illustrated in Fig. 4.

There are three processes involved in each series of measurements: (1) With the complete weight W in place, the system S, as illustrated in the figure, is set into torsional oscillations, and the frequency, ν_1, determined. Let K_1 denote the moment of inertia of this (unloaded) system, A_1 the torsional constant. Then

$$A_1 = \nu_1^2 4\pi^2 K_1. \quad (1)$$

(2) The system is loaded with one or more small rings, with moment of inertia K, the total moment of inertia being now $K_2 = K + K_1$, and the frequency ν_2 is determined. If the torsional constant is now A_2, we have

$$A_2 = \nu_2^2 4\pi^2 K_2. \quad (2)$$

Assume (consistently with experiment) $A_2 = A_1 = A_{12}$. Then

$$K_1 = \frac{(K)\nu_2^2}{\nu_1^2 - \nu_2^2}; \quad K_2 = \frac{(K)\nu_1^2}{\nu_1^2 - \nu_2^2}, \quad (3)$$

and

$$A_{12} = \frac{4\pi^2 (K)\nu_1^2 \cdot \nu_2^2}{\nu_1^2 - \nu_2^2}. \quad (4)$$

(3) The ring is replaced by a system S_3 with a much larger moment of inertia K', such that the total moment of inertia is now $K_3 = K_1 + K'$. At the same time the detachable part of the weight

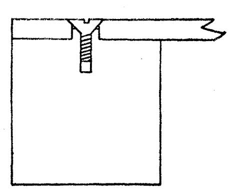

FIG. 6. Details of one end of S_3.

[3] To insure that relative motion between the rings and the ring holder should always be impossible we have used, in some cases, very small quantities of wax, which did not affect the moments of inertia appreciably.

W is removed. The frequency ν_3, very much lower than either ν_1 or ν_2, is determined. The torsional content A_3 is now computed from the relation

$$A_3=\nu_3^2\cdot 4\pi^2\cdot K_3=\nu_3^2 4\pi^2(K'+K_1). \quad (5)$$

Let T_u denote the tension of the upper wire and T_L that of the lower wire in (1); and T_u' and T_L' the tensions in (3). Let W_1 and W_3 be the weights of the systems S_1 and S_3, and let B_1 and B_3 denote the stretching weights in (1) and (3). Then we have

$$T_u=T_L+W_1=B_1+W_1,$$
$$T_u'=T_L'+W_3=B_3+W_3.$$

Then

$$T_u'-T_u=T_L'-T_L+(W_3-W_1).$$

If now $T_L-T_L'\equiv B_1-B_3$ is made equal to $\frac{1}{2}\cdot(W_3-W_1)$, as was done in this work, we get

$$T_u'-T_u=-(T_L'-T_L)=\tfrac{1}{2}\cdot(W_3-W_1).$$

Thus in changing from (1) to (3), the tension of the upper wire is increased and that of the lower wire decreased by the same amount. Hence if there is any dependence of the torsional constant of the material on tension and stretch, there is nevertheless no change on that account in the constant or the double wire systems used in this part of the work.

As a matter of fact experiment shows that if no other change is made the removal or addition of the lower part of W has not more than a minute effect on the frequency.

The ratio $R\equiv A_{12}/A_3$ is the quantity sought.

Temperature changes were so small as to be of no consequence in this work; and even if this were not so it is easy to prove that their effects would be almost completely eliminated in taking the ratio R.

All the higher frequencies in this work were measured with reference to the frequency of the power line from Hoover Dam, which is now ordinarily constant to at least a tenth or twelfth percent at 60 cycles per second. We have continually checked this frequency with a standard Weston meter permanently connected to the line. The low frequencies have been determined in the usual way, with negligible error, by means of a synchronous clock driven by the 60 cycle per second supply, or by means of a reliable watch.

The vibrations at higher frequencies, with one exception, have been obtained in the following way. A coil N is mounted with its axis A normal the length of the small magnet M, Fig. 1 (instead of parallel, as indicated in the figure). A sine wave oscillator OSC is connected to the coil N through a high resistance R and has its frequency varied while the amplitude of the torsional oscillation is measured, and the frequency of resonance thus determined. The e.m.f. supplied by the oscillator and the current in the coil N were independent of the frequency for ranges far beyond those of any resonance curve. The oscillator was provided with a special circle graduated in degrees, which were read to 0.°1, and was very steady after being operated for two hours. It was calibrated in the usual way with the help of an oscilloscope; and the calibration curve was checked, and corrected when necessary, for every frequency determination. Ordinarily, for each such determination, three closely concordant resonance curves were obtained.

The lowest of the higher frequencies was below the range of the oscillator and was therefore obtained from an electrically driven tuning fork. The wave form in this case was by no means sinusoidal, but this was of no consequence since the resonance was exceedingly sharp.

The inertia system S_3, used to obtain the low frequency, was made of brass. Its various parts had the approximate dimensions indicated in Fig. 2. One end is illustrated in more detail in Fig. 6. Its moment of inertia is 19.65×10^3 g cm^2 with negligible error. The brass bar alone has the moment of inertia 23.41×10^2 g cm^2, also with negligible error.

Two double suspensions were used, almost exactly alike. Only the wires and mirrors differed in the two. Both were made with No. 28 German silver wire, probably from the same stock. The wire of the first system was cut from wire actually used in experiments made here on the Einstein-deHaas effect; those of the second system were cut from the ends of the 60-cm wire of the long suspension referred to below.

The period of oscillation of the first double suspension vibrator with S_3 in place was 4.81_5

TABLE I. Data with regard to inertia systems and frequencies with first double suspension.

1 System	2 Approx. thickness	3 Approx. outer diameter	4 Approx. mass	5 Approx. moment of inertia	6 Observed frequency and mean error	7 Approx. calc. max. error in frequency	8 $100\times\frac{\delta A_{12}}{A_{12}}$ $\left(\frac{\delta\nu_1}{\delta\nu_2}>0\right)$	9 $100\times\frac{\delta A_{12}}{A_{12}}$ $\left(\frac{\delta\nu_1}{\delta\nu_2}<0\right)$
Ring holder				0.33 g cm²	50.02±0.03	0.08		
Ring 1	0.16 cm	2.0 cm	0.75 g	0.25	37.92±0.08	0.06	0.27	1.14
Ring 2	0.10	2.0	2.42	1.28	22.78±0.03	0.02	0.17	0.34
Rings 1+2				1.52	21.2	0.02	0.14	0.28
Ring 3	1.6	2.0	1.29	0.68+	28.79±0.04	0.03	0.2	0.5
Ring 4	1.6	2.0	1.29	0.68+				
Rings 3+4				1.37−	22.16±0.01	0.02	0.16	0.31
Ring 5	3.2	2.0	2.59	1.37	22.12			
Ring 6	3.2	2.0	2.59+	1.37				
Rings 5+6				2.74	16.58	0.02	0.3	0.4

seconds; that of the second, and very similar, suspension, 4.82_2 seconds.

For use with the higher frequencies six small inertia rings were constructed and used, each in the form of a precise toroid with rectangular cross section. Their *approximate* thicknesses, outer diameters, masses, and moments of inertia are given in Table I. All were cut from rolled aluminum sheet except No. 2, which was turned from a brass cylinder. All had the same internal diameter, *viz.*, 0.476 cm. Table I includes also data with regard to the ring holder, and other (frequency) data referred to below.

All the moments of inertia except that of the ring holder were calculated from the standard formulae, which it is unnecessary to quote here. In making the calculations the precise dimensions and masses were of course used, not the approximate values given above. All the linear measurements and masses were measured with such precision that possible errors due to them were negligible. For the precise determination of all the masses we are much indebted to Professor Yost and Mr. Whittaker of the Gates and Crellin Laboratories.

The aluminum rings 3 and 4 were cut from adjoining parts of the same aluminum sheet, one just after the other in the direction of rolling. Each was marked to indicate this direction appropriately. When mounted together (3+4) the marks were ordinarily placed in opposition to compensate for possible variation of density and thickness. But tests in which the marks were placed in the same direction gave the same frequency, showing that no error from this source occurred when a single ring was used.

Numbers 5 and 6 were cut in the same manner from another aluminum sheet and were marked like Nos. 3 and 4. When both rings were used the marks were opposed.

Accurate measurements of the thicknesses showed that no appreciable errors could be produced by their departure from uniformity.

Error from imperfect knowledge of the frequencies is more difficult to eliminate. If we assume that for each frequency a maximum error of 0.1 degree on the oscillator circle is possible, we obtain from the calibration curve the values of $\delta\nu_1$ and $\delta\nu_2$ given in column 7 of Table I. The actual mean errors in the frequencies are given in column 6 of the same table for the first double suspension.

To show how A_{12} depends on the errors in frequency we differentiate Eq. (4) and thus obtain

$$\frac{\delta A_{12}}{A_{12}}=2\left(\frac{\delta\nu_1}{\nu_1}+\frac{\delta\nu_2}{\nu_2}-\frac{\nu_1\delta\nu_1-\nu_2\delta\nu_2}{\nu_1^2-\nu_2^2}\right). \quad (6)$$

The maximum calculated value of $\delta A_{12}/A_{12}$ is calculated for each case by inserting in Eq. (6) the tabular maximum magnitudes of $\delta\nu_1$ and $\delta\nu_2$ and giving them opposite signs; the minimum, by giving them the same sign. These values are given in percent in columns 8 and 9 of Table I. The maxima are of course much greater than the mean experimental errors to be expected. All these values assume the constancy of the frequency of the mains.

The observations with the second double suspension were still more precise and the frequency errors less.

The resulting values of the ratio $R\equiv A_{12}/A_3$

obtained with both double suspensions are given in Table II.

From each series of observations the difference between unity and the mean value of the ratio $R \equiv A_{12}/A_3$ is much greater than would be expected, even from column 9 of Table I, on the hypothesis of no real effect of frequency on A. The change in A with frequency is in the right direction to account for the discrepancy which initiated this work, but its magnitude is much too small—about 1 percent instead of some 4 or 5 percent.

The precision is not sufficient to detect any differences in A for the different values of the higher frequency ν_2.

III. WORK AT LOWER FREQUENCIES

To investigate the possible variation of the torsion modulus when the period of vibration ranged from about 4 seconds to about 28 seconds, three additional suspensions, designated as Nos. 1, 2, and 3, were prepared and the period of each determined when vibrated (A) with the bar only on the ring holder and (B) with the cylindrical weights added. They were all cut from a continuous length of No. 28 German silver wire from the same stock from which the wires of the second double suspension were cut, and probably also those of the first double suspension. Suspension No. 3, about 60 cm long, was cut from the central part of the wire; No. 1 and No. 2, each about 10 cm long, from the ends.

The periods for these slow oscillations, except in preliminary work, were determined with precision by means of a reliable watch, the frequency of the mains happening to be less than usually stable while these observations were in progress.

TABLE II. Results obtained with double suspensions I and II. (Periods from about 1/50 second to about 5 seconds.)

Suspension	Group	Rings Used	$R \equiv A_{12}/A_3$	Mean A_{12}/A_3
I	1	1, 2, 1+2, 3+4, 5+6	0.987±0.002	
	2	1, 2, 3, 3+4, 5+6, 5	0.987±0.003	
	3	3	1.004±0.001	0.992±0.007
	4	3	0.998±0.000	
	5	3+4	0.985±0.001	
II	1	3+4 (opposed)	0.990	
	2	3	0.990	
	3	3+4 (together)	0.987	0.989±0.002
	4	3+4 (opposed)	0.987	
	5	3	0.991	

TABLE III. Data for low frequency vibrators.

	A. Inertia bar without cylinders			B. Inertia bar with cylinders		
Suspensions	l_A	T_A	$R_A = \frac{l_A}{T_A^2}$	l_B	T_B	$R_B = \frac{l_B}{T_B^2}$
No. 1	9.98 cm	3.990	0.6270	9.98 cm	11.57_3	0.0745_2
No. 2	9.95	3.976	0.6294	9.95	11.52_6	0.0749_0
No. 3	59.95	9.769	0.6282	59.97	28.28	0.0749_8

For this work the small magnet was removed from the holder.

On the assumption that the material of the wire, the radius and the tension are the same in two sets (1) and (2) of vibrations we have the relation

$$\frac{M_1}{M_2} = \frac{(l_1/T_1^2)}{(l_2/T_2^2)} \frac{K_1}{K_2}$$

for the ratio of the moduli. Table III gives the length l, the period T, and the quantity $R \equiv l/T^2$ for each of the wires for the cases designated above as A and B.

We shall proceed first by method (2) of p. 897. Here $K_1 = K_2$. For case A the ratio R_1 for No. 1 to R_3 for No. 3 is 0.998. The ratio R_2 for No. 2 to R_3 for No. 3 is 1.002. Neither gives correctly the ratio of the modulus for the period 4 sec. to that for the period 10 sec., as the radii of Nos. 1 and 2 and their materials cannot be exactly the same. But the mean, *viz.* 1.000, must give this ratio with close approximation.

For case B, the corresponding ratios are 0.994 and 0.999, with the mean of 0.9965, which, if there were no errors, would be the value of the ratio of the modulus at period about 11.5 sec. to that at about 28.3 sec. In view of the ratio unity obtained in case A, we should expect to obtain the same ratio for the slower vibrations of case B. It is important to notice that, as would be expected if the observations are correct, $(R_3 - R_1)_A = 0.004$ is practically equal to $(R_3 - R_1)_B = 0.005$.

If now we proceed by method (1), and assume that the frequency is not appreciably affected by the change in the tension from case A to case B, we apply the complete formula, with $K_2/K_1 (\equiv K_B/K_A) = 8.395$. This process gives for suspensions Nos. 1 and 2 the ratios 1.002 and 1.001 for the change of period from 4 sec. to 11.5 sec. approximately; and for No. 3, the ratio 0.998 for the change from 10 sec. to 28 sec.

In view of the approximations made in the assumptions and the (slight) observational errors both methods agree in showing no certain variation of the modulus for periods between 4 sec. and 28 sec.

IV. CHECK ON CALCULATIONS OF MOMENTS OF INERTIA

If we assume that this result is established by method (2) we may of course reverse method (1) to give a valuable check on the correctness of the calculations of the moments of inertia of the inertia system S_3 and the bar which forms a part of it. Thus, for each suspension, if in the experiments there is no appreciable change of modulus with frequency or tension, we should have

$$K_B/K_A = (l_A/l_B)(T_B/T_A)^2,$$

where K_B and K_A are the moments of inertia in cases B and A, *viz.*, 19.65×10^3 g cm^2 and 2.341×10^3 g cm^2. (The [negligible] moment of inertia, *ca.* 0.33, g cm^2, of the holder is included.) The ratio $K_B/K_A = 8.395$; while the ratios $l_A/l_B(T_B/T_A)^2$ for the three suspensions are 8.413, 8.395, and 8.377, with the mean 8.395, exactly equal to K_B/K_A.

This work has been done in the Norman Bridge Laboratory of the California Institute with facilities provided by the University of California, the Institute, the Carnegie Institution of Washington, and the National Research Council.

We have desired to make further observations on German silver, with wires of different diameters and in the frequency range between our higher and lower values, as well as precise observations on other substances; but the pressure of other work has hitherto prevented this.

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A Note on Weinstein's Variational Method

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Weinstein's modification of the Ritz principle is used to derive (1) a lower bound for the n-th energy-level of a quantum mechanical system if a lower bound for the $(n+1)$-st level is known; and (2) an upper bound for the n-th level if an upper bound for the $(n-1)$-st level is known.

1. INTRODUCTION

By applying the Ritz variational principle to the equation

$$(H-\lambda)^2\psi = W\psi \tag{1}$$

Weinstein[1] was able to obtain both lower and upper bounds for the energy levels of the Schroedinger equation

$$H\psi = E\psi. \tag{2}$$

The chief theoretical weakness of this method is that it gives no hint as to which one of the energy levels the bounds obtained refer to.

Stevenson and Crawford[2,3] have made use of Weinstein's method in an improved form to establish a theoretical lower bound for the ground-level of the helium atom. In their calculations both the lower and upper bounds obtained lie well below the experimental value of the second energy level and therefore, of necessity, refer to the ground state.

In section 2 of the present note the method of Stevenson and Crawford is generalized to give a lower bound for E_n if a lower bound of E_{n+1} is known. Since no general theoretical method is available for determining the latter, its value may

[1] D. H. Weinstein, Proc. Nat. Acad. of Sci. **20**, 529 (1934).

[2] A. F. Stevenson, Phys. Rev. **53**, 199 (1938).

[3] A. F. Stevenson and M. F. Crawford, Phys. Rev. **54**, 374 (1938).