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The Theory of Alpha-Radioactivity*

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A number of authors have developed the theory of α-radioactivity in the case of zero angular momentum of the escaping particle. For $l\neq 0$, formulae have also been given but it is shown that these are inaccurate. In order to discuss the case of non-zero values of l, the one-body theory with a rectangular potential well is treated rigorously. A suitable integral representation of the confluent hypergeometric function is developed by the method of steepest descent into an asymptotic series which represents the solution of the wave equation outside the nucleus for $l=0$. Solutions for $l\neq 0$ are obtained from this by recursion operators. Boundary conditions at the nuclear radius give two equations linking the radius of the nucleus and the depth of its potential hole with the decay constant and the energy of the emitted α-particles. The usefulness and reliability for this problem of the one-body model are examined (Section 6). A re-evaluation of nuclear radii has been performed and the results are tabulated. The possibility that the decay constant, for fixed energy, is not a monotonic function of l but has an initial rise is discussed. A quantitative study is made of the $\lambda - E_\alpha$ relationships in a few α-ray spectra with "fine structure."

1. INTRODUCTION

THE quantum-mechanical theory of spontaneous α-particle radioactivity has been treated by a number of authors,[1–8, 11] since the first independent papers of Condon and Gurney[9] and Gamow[10] in 1928. In these papers attention has been centered on the case $l=0$. Gamow[5] has given a formula for $l\neq 0$. In addition to the approximations used in developing the formula for $l=0$, there are two objections which may be raised to Gamow's formula. Firstly, there is no examination of the possibility that the factors which involve the wave function inside the nucleus may depend on l. This point is discussed more fully below. Secondly, ignoring the first objection, Gamow's approximations give incorrect numerical results. For example, his formula (58)[12] gives a correction to the exponent amounting to 0.86σ in the case of Ra decay; accurate numerical integration of the WKB integral gives 0.17σ. (σ is the ratio of "centrifugal" to Coulomb potential at the nuclear radius.) This error in Gamow's theory arises from two sources: (1) Over a significant portion of the range of integration $\sigma r_0/r$ is comparable with $1-Er/2Z\epsilon^2$, although it is treated as small. (2) The ratio of nuclear radius to the classical radius of closest approach is about 0.25 which actually is not sufficiently small to permit some of the approximations necessary to obtain Gamow's formula. Also the WKB method is not too reliable itself in that an expression which should be negligible with respect to unity:[12]

$$\frac{\hbar}{(2m)^{\frac{1}{2}}}\left|\frac{d}{dr}\left[\frac{2Z\epsilon^2}{r}+\frac{\hbar^2 l(l+1)}{2mr^2}-E\right]^{\frac{1}{2}}\right|,$$

is not particularly small, i.e., it is greater than 0.25 for about one quarter of the range of integration.

A valid formula for $l\neq 0$ could be obtained by more refined considerations along the lines employed by Gamow, but the integrals do not lend themselves to simple treatment and it was therefore decided to make the calculations as rigorous as possible from the beginning. Thus the purpose of this paper is to find formulae for $l\neq 0$; to do so it has been found necessary to re-derive the treatment for $l=0$, although here no significant new results are obtained.

The theory should lead to a relationship be-

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[1] T. Sexl, Zeits. f. Physik **81**, 163 (1933).
[2] M. Frenkel, Zeits. f. Physik **95**, 599 (1935).
[3] J. Kudar, Zeits. f. Physik **53**, 61, 95, 134; **54**, 297; **57**, 710 (1929).
[4] C. F. von Weizsacker, *Die Atomkerne* (Akad-Verlagsgesellschaft, Leipzig, 1937), p. 95.
[5] G. A. Gamow, *Constitution of Atomic Nuclei and Radioactivity* (Cambridge, 1937).
[6] O. Rice, Phys. Rev. **35**, 1538 (1930).
[7] H. A. Bethe, Rev. Mod. Phys. **9**, 161 (1937).
[8] F. Rasetti, *Elements of Nuclear Physics* (Prentice-Hall, Inc., New York, 1936).
[9] Condon and Gurney, Nature **122**, 439 (1928) and Phys. Rev. **33**, 127 (1929).
[10] G. A. Gamow, Zeits. f. Physik **51**, 204 (1928).
[11] H. A. Bethe, Phys. Rev. **50**, 977 (1936); Kahan, Comptes rendus **206**, 1289 (1938).
[12] See reference 5, p. 103 and p. 91.

tween the decay constant for the process and the energy of emission of the α-particle, reproducing the experimental results expressed in the Geiger-Nuttall law.[13] We shall see that the relationship is contained in an equation involving two undetermined constants, the nuclear *radius* and *potential*; there is a second equation connecting these quantities and the energy of emission. As has been pointed out,[14] in most of the previous work the depth of the potential has been removed by approximate assumptions from the final results, thus obtaining an equation involving the radius of the nucleus as the only undetermined quantity. Sexl[1] and M. Frenkel[2] have improved on these theories, but their work permits of simplification and, in common with all previous theories, is not adapted to the non-zero values of l.

2. FORMULATION OF THE PROBLEM

We represent the action of the nucleus on the α-particle by a force field with a rectangular hole potential V. The accuracy of this model is discussed in Sec. 6. $V=U$, a constant, for $r<r_0$, $V=2Z\epsilon^2/r$, for $r>r_0$, where Z is the charge number of the product nucleus, ϵ is the elementary charge, r is the distance from the center of the product nucleus, r_0 is the "radius of the product nucleus."

All dynamical quantities are modified to refer to the system in which the product nucleus is at rest. Corrections for recoil are made when numerical calculations are performed. α-particles in the sphere $r<r_0$ have a finite probability λ of piercing the potential barrier and emerging with energy E_α. We assume a solution u of the time-dependent Schrödinger equation in the form

$$u=\psi(x,y,z)\exp(-iE_\alpha t/\hbar)\exp(-\tfrac{1}{2}\lambda t)=\psi(x,y,z)\exp(-iEt/\hbar),$$

introducing the "complex eigenvalue"

$$E=E_\alpha-\tfrac{1}{2}i\hbar\lambda. \qquad (2.1)$$

Then uu^* is proportional to $e^{-\lambda t}$, indicating exponential decay. This computational device of complex eigenvalues was first introduced by Gamow[10] and since used and justified by Kudar,[3] Sexl[1] and von Weiszacker.[4]

We require the following of ψ: (a) at $r=0$, ψ is finite, (b) at $r=r_0$, ψ and $d\psi/dr$ are continuous, (c) for $r>r_0$, ψ represents an outgoing wave. Since V is spherically symmetrical,

$$r\psi=\sum_l X_l(r)P_l(\cos\theta),$$

and

$$d^2X_l/dr^2+(2m/\hbar^2)\times[E-V-\hbar^2l(l+1)/2mr^2]X_l=0, \qquad (2.2)$$

where m is the reduced mass of the α-particle, and P_l are spherical harmonics.

Inside the nucleus, $V=U$ and therefore

$$X_l^{(i)}=\{2m(E-U)/\hbar^2\}^{\frac{1}{4}}r^{\frac{1}{2}}J_{l+\frac{1}{2}}(\{2m(E-U)/\hbar^2\}^{\frac{1}{2}}r). \qquad (2.3)$$

Here J denotes the Bessel function and the superscript (i) refers to the interior of the nucleus. This solution would represent a standing wave, except for the imaginary part of E which permits the "leak" through the potential barrier.

For $r>r_0$, we must write $V=2Z\epsilon^2/r$ in (2.2). In the resulting equation we substitute

$$\begin{aligned} x&=(2mE)^{\frac{1}{2}}r/\hbar=x_R+ix_I,\\ \kappa&=(2mE)^{\frac{1}{2}}(2\epsilon^2Z/\hbar)=\kappa_R+i\kappa_I,\\ y&=X_l^{(o)},\text{ the "outside" solution,}\end{aligned} \qquad (2.4)$$

and we obtain

$$y''+(1-l(l+1)/x^2-\kappa/x)y=0. \qquad (2.5)$$

Because of the experimentally known values of the constants of radioactive elements, we see that it is legitimate to write

$$x_R+ix_I\doteq(2mE_\alpha)^{\frac{1}{2}}(r/\hbar)(1-\tfrac{1}{4}i\hbar\lambda/E_\alpha)\doteq 10-10^{-18}i,$$

and

$$\kappa_R+i\kappa_I\doteq 4\epsilon^2Z/\hbar v+i\epsilon^2Z\lambda/vE_\alpha\doteq 50+10^{-18}i,$$

where we have taken $r_0\doteq10^{-12}$ cm. The small magnitude of the imaginary parts of x and κ plays an essential role in what follows.

3. THE CASE $l=0$.

For $l=0$, Eq. (2.5) is simply

$$y''+(1-\kappa/x)y=0. \qquad (3.1)$$

It is known[15] that $W''+(-\tfrac{1}{4}+p/z)W=0$, has in-

[13] H. Geiger and J. M. Nuttall, Phil. Mag. 22, 613 (1911).
[14] M. A. Preston, Phys. Rev. 69, 535 (1946).
[15] Whittaker and Watson, *Modern Analysis* (Cambridge, 1927), pp. 339, 343.

dependent solutions

$$e^{-\frac{1}{2}z}\int_0^\infty (1+z/t)^p e^{-t}dt,$$

and

$$e^{\frac{1}{2}z}\int_0^\infty (1-z/t)^{-p} e^{-t}dt,$$

if $R(p)\leqslant\frac{1}{2}$. Therefore, since $-\kappa_I<\frac{1}{2}$, we may write two solutions of (3.1) in the form

$$y^{(1)}(x)=e^{-\frac{1}{2}\kappa\pi}e^{ix}\int_0^\infty (1-2ix/t)^{-\frac{1}{2}i\kappa}e^{-t}dt, \quad (3.2)$$

$$y^{(2)}(x)=e^{-\frac{1}{2}\kappa\pi}e^{-ix}\int_0^\infty (1+2ix/t)^{\frac{1}{2}i\kappa}e^{-t}dt. \quad (3.3)$$

Under the substitution $t=2ix(\tau+\frac{1}{2})$, $\tau=\xi+i\eta$,

$$y^{(1)}=e^{-\frac{1}{2}\kappa\pi}2ix\int_C\left(\frac{\tau-\frac{1}{2}}{\tau+\frac{1}{2}}\right)^{-\frac{1}{2}i\kappa}e^{-2ix\tau}d\tau, \quad (3.4)$$

C is the line $\xi=-\frac{1}{2}+(x_I/x_R)\eta$ in the τ-plane from $\eta=0$ to $\eta=-\infty$. Similarly $y^{(2)}$ is the same integral over the path $\xi=\frac{1}{2}+(x_I/x_R)\eta$.

The factor e^{ix} in (3.2) shows that $y^{(1)}(x)$ represents the outgoing wave for large x (or r). Therefore we now wish to evaluate $y^{(1)}$ as given by (3.4), remembering that $|\kappa|\doteq 50$ and $|x|\doteq 10$.

The "method of steepest descents"[16] is particularly suited to this problem. It serves to evaluate integrals in the complex plane of the form $\int_C e^{-\kappa f(\tau)}d\tau$, where $R\{f(\tau)\}$ approaches infinity (or a "large" value) at both ends of C. This property of $f(\tau)$ implies that appreciable contributions to the value of the integral come from only a certain finite strip of C. To permit this strip to be as small as possible for purposes of approximation, we replace C by a path on which $R\{f(\tau)\}$ increases most rapidly from its minimum value. The path used is called the *curve of steepest descent*, and the point at which $R\{f(\tau)\}$ is a minimum called the *pass point* or *Sattelpunkt*. It is found that, if τ_s represents a pass point, these points are the roots of

$$(df/d\tau)_{\tau=\tau_s}=f'(\tau_s)=0, \quad (3.5)$$

and the curves of steepest descent have the equation

$$I\{f(\tau)\}=I\{f(\tau_s)\}. \quad (3.6)$$

[16] P. Debye, Math. Ann. 67, 535 (1909).

We write

$$y^{(1)}=2ix\int_C e^{-\kappa f(\tau)}d\tau,$$

where

$$f(\tau)=\tfrac{1}{2}i\log(\tau-\tfrac{1}{2})-\tfrac{1}{2}i\log(\tau+\tfrac{1}{2})+2i(x/\kappa)\tau+\tfrac{1}{2}\pi. \quad (3.7)$$

Putting $\tau=\xi+i\eta$, this reduces to

$$f(\tau)=-[\tfrac{1}{2}\arctan\{\eta/(\xi^2-\tfrac{1}{4}+\eta^2)\}+2(x/\kappa)\eta]+i[\tfrac{1}{4}\log\{(\xi-\tfrac{1}{2})^2+\eta^2\}-\tfrac{1}{4}\log\{(\xi+\tfrac{1}{2})^2+\eta^2\}+2(x/\kappa)\xi].$$

Now neglecting 10^{-16} with respect to unity, $x/\kappa=x_R/\kappa_R+i\gamma$, say, where $\gamma>0$. Therefore

$$R\{f(\tau)\}=-[\tfrac{1}{2}\arctan\{\eta/(\xi^2-\tfrac{1}{4}+\eta^2)\}+2(x_R/\kappa_R)\eta+2\gamma\xi], \quad (3.8)$$

and

$$I\{f(\tau)\}=\tfrac{1}{4}\log[\{(\xi-\tfrac{1}{2})^2+\eta^2\}/\{(\xi+\tfrac{1}{2})^2+\eta^2\}]+2(x_R/\kappa_R)\xi-2\gamma\eta. \quad (3.9)$$

Also setting $f'(\tau_s)=0$ and solving for the pass points, we find

$$\tau_s=\pm\tfrac{1}{2}(1-\kappa/x)^{\frac{1}{2}}=\pm\tfrac{1}{2}i\tan\alpha, \quad (3.10)$$

where α is defined by

$$\cos^2\alpha=x/\kappa,\quad \alpha=\alpha_R+i\alpha_I,\quad \pi/2>\alpha_R>0. \quad (3.11)$$

Then $x/\kappa=\cos^2(\alpha_R+i\alpha_I)=\cos^2\alpha_R+i\alpha_I\sin2\alpha_R$, to the order 10^{-16}. Therefore

$$\cos^2\alpha_R=x_R/\kappa_R,\quad\text{and}\quad \alpha_I=\gamma/\sin2\alpha_R. \quad (3.12)$$

We now study the pass point $\tau_-=-\frac{1}{2}i\tan\alpha$. Substituting in (3.7), we find easily

$$f(\tau_-)=-(\alpha-\cos\alpha\sin\alpha)=-(\alpha_R-\cos\alpha_R\sin\alpha_R)-2i\alpha_I\cos^2\alpha_R. \quad (3.13)$$

Thus from (3.12) the imaginary part of $f(\tau_-)$ is $-\gamma\cot\alpha_R$. From (3.9), the curves of steepest descent through τ_- are

$$\tfrac{1}{4}\log[\{(\xi-\tfrac{1}{2})^2+\eta^2\}/\{(\xi+\tfrac{1}{2})^2+\eta^2\}]+2(x_R/\kappa_R)\xi=\gamma(2\eta-\cot\alpha_R). \quad (3.14)$$

The right side of (3.14) is important only for $\eta>10^{12}$, say, because of the size of γ. In this region the logarithmic term is negligible and the curve is the line

$$\eta-\tfrac{1}{2}\cot\alpha_R=(x_R/\kappa_R)\xi, \quad (3.15)$$

which has a slope of about 10^{15}. For smaller η, (3.14) is approximately

$$\tfrac{1}{4}\log[\{(\xi-\tfrac{1}{2})^2+\eta^2\}/\{(\xi+\tfrac{1}{2})^2+\eta^2\}]+2(x_R/\kappa_R)\xi=0. \tag{3.16}$$

This curve is more conveniently studied in the form

$$\eta^2=\frac{(\xi+\frac{1}{2})^2\exp(-4\xi\cos^2\alpha_R)-(\xi-\frac{1}{2})^2\exp(4\xi\cos^2\alpha_R)}{\exp(4\xi\cos^2\alpha_R)-\exp(-4\xi\cos^2\alpha_R)}. \tag{3.17}$$

It is easily verified that

1. the curve is centrally symmetric about the origin,
2. the curve meets $\xi=0$ in $\eta=\pm\frac{1}{2}\tan\alpha_R$,
3. the curve intersects $\xi=\pm\frac{1}{2}$ in real points,
4. $\eta^2=0$ only for two specific values $\xi=\pm\xi_0$, and consequently η is real only for $-\xi_0\leqslant\xi\leqslant\xi_0$,
5. $\xi_0>\frac{1}{2}$ by fact 3 above. Actually its value is about 0.8 and it is a root of $(\xi+\frac{1}{2})/(\xi-\frac{1}{2})=\exp(8\xi\cos^2\alpha_R)$.
6. $\xi=0$ is a part of (3.16).

Figure 1 shows the curve (3.14) and the corresponding one through $\tau_+(=\frac{1}{2}i\tan\alpha)$. The imaginary part of α is exaggerated in the figure.

The path of integration for $y^{(1)}(x)$ is the straight line L_1 in Fig. 1. The integrand is without singularities in the region concerned and the integral over L_1+L_4 equals the integral over L_2+L_3. L_3 is the path of steepest descent. We must show that the integrals over L_4 and L_2 are negligible. We have

$$\left|\int_{L_4}\exp\{-\kappa f(\tau)\}d\tau\right|$$
$$=\left|\lim_{\eta\to-\infty}\int_{(x_I/x_R)\eta}^{(\kappa_R/x_R)\gamma\eta}\exp(2x\eta-2xi\xi)d\xi\right|$$
$$=\lim_{\eta\to-\infty}\frac{1}{2x_R}(e^{2(x_R-x_I)\eta}-e^{2(x_R-\gamma\kappa_R)\eta})$$
$$=0.$$

On L_2, $\eta=0$ and the integral is

$$\int_{-\frac{1}{2}}^{-\xi_1}\exp\{-i[\tfrac{1}{2}\kappa\log(\xi+\tfrac{1}{2})/(\tfrac{1}{2}-\xi)-2x\xi]\}d\xi,$$

if L_3 cuts $\eta=0$ at $\xi=-\xi_1$. The absolute values of both real and imaginary parts of the integral are less than

$$\int_0^{\frac{1}{2}}\exp[-\tfrac{1}{2}\kappa_I\log(\xi+\tfrac{1}{2})/(\tfrac{1}{2}-\xi)+2x_I\xi]d\xi,$$

which, since $x_I<0$, is less than

$$(\tfrac{1}{2})^{\frac{1}{2}\kappa_I}\int_0^{\frac{1}{2}}(\xi+\tfrac{1}{2})^{-\frac{1}{2}\kappa_I}d\xi\doteq 0.5,$$

disregarding terms of order 10^{-16}. This is negligible in comparison with the integral along L_3, which is of the order 10^{13} (see Eq. (3.14)). Therefore L_1 may be replaced by L_3 for the integration.

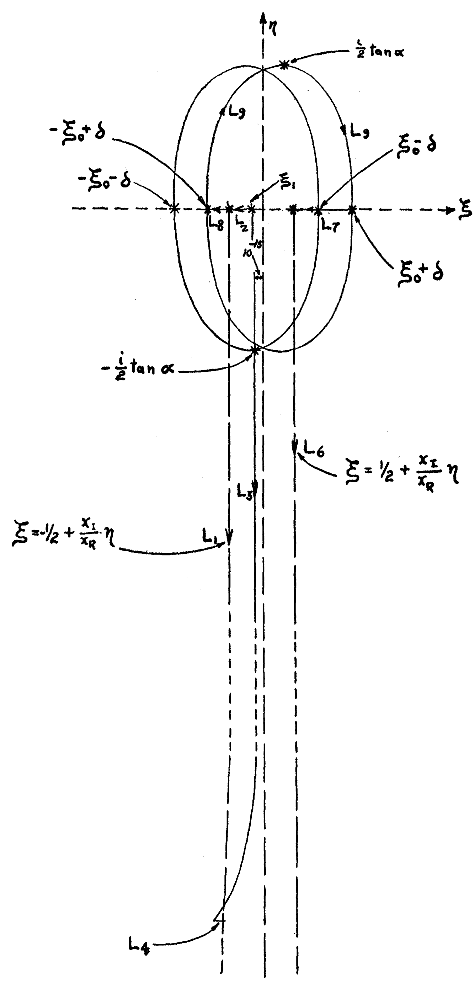

FIG. 1. Lines of steepest descent.

We now proceed to obtain an approximate expression for $y^{(1)}(x)$. The method of the remainder of this section is not mathematically rigorous. Moreover it provides no estimate of the percentage error of its results. To overcome these drawbacks requires a somewhat lengthy argument which is given in Appendix A. In this section we shall obtain the leading terms of the result in a simple manner.

Since L_3 is a path of steepest descent,

$$\int_{L_3} \exp\{-\kappa f(\tau)\}d\tau \doteq \int_{\tau_-+i\beta}^{\tau_--i\beta} \exp\{-\kappa f(\tau)\}d\tau.$$

Remembering that $f'(\tau_-)=0$, we put

$$\begin{aligned} f(\tau) &\doteq f(\tau_-)+\tfrac{1}{2}f''(\tau_-)(\tau-\tau_-)^2 \\ &= f(\tau_-)-\tfrac{1}{2}f''(\tau_-)\eta^2, \quad \text{on } L_3, \end{aligned}$$

where $\tau=\tau_-+i\eta$. It is found that $f''(\tau_-)=-8\cos^4\alpha\tan\alpha$. Then the integral is

$$\begin{aligned} i\int_{\beta}^{-\beta} &\exp\{-\kappa f(\tau_-)\}\exp\{-4\kappa\cos^4\alpha\tan\alpha\eta^2\}d\eta \\ &= \frac{i\exp\{-\kappa f(\tau_-)\}}{2\cos^2\alpha(\kappa\tan\alpha)^{\frac{1}{2}}}\int_{2\cos^2\alpha(\kappa\tan\alpha)^{\frac{1}{2}}\beta}^{-2\cos^2\alpha(\kappa\tan\alpha)^{\frac{1}{2}}\beta} \\ &\quad\times\exp(-\eta^2)d\eta \\ &\doteq \frac{i\exp\{-\kappa f(\tau_-)\}}{2\cos^2\alpha(\kappa\tan\alpha)^{\frac{1}{2}}}\int_{\infty}^{-\infty} \\ &\quad\times\exp(-\eta^2)d\eta, \text{ for sufficiently large } \kappa, \\ &= -\tfrac{1}{2}i\sec^2\alpha(\pi\cot\alpha/\kappa)^{\frac{1}{2}}\exp\{-\kappa f(\tau_-)\}. \end{aligned}$$

Therefore

$$\begin{aligned} y^{(1)}(x) &\doteq x\sec^2\alpha(\pi\cot\alpha/\kappa)^{\frac{1}{2}}\exp\{-\kappa f(\tau_-)\} \\ &= (\pi\kappa\cot\alpha)^{\frac{1}{2}}e^{\omega}, \end{aligned} \tag{3.18}$$

where

$$\omega=\kappa(\alpha-\cos\alpha\sin\alpha). \tag{3.19}$$

Notice that in this approximation we have ignored the fact that x, κ, α have small imaginary parts. Then (3.18) gives the real part of $y^{(1)}$. The imaginary part of $y^{(1)}$ is much smaller in absolute value and hence does not appear in the expansion about τ_-. However (3.2) and (3.3) show that (if x and κ are real) $y^{(2)}$ is the complex conjugate of $y^{(1)}$. Therefore $I\{y^{(1)}\}$ is $-\frac{1}{2}i(y^{(1)}-y^{(2)})$. This quantity is $x\int\exp\{-\kappa f(\tau)\}d\tau$ integrated over L_1-L_6 or equivalently over $L_8+L_9+L_7$. It is easily seen that the integral over L_8+L_7 is

$$-2\int_{\frac{1}{2}}^{\xi_0}\cos[\tfrac{1}{2}\kappa\log(\xi-\tfrac{1}{2})/(\xi+\tfrac{1}{2})+2x\xi]d\xi. \tag{3.20}$$

When x and κ are real, this integral is zero (see Appendix A). Therefore the integral is taken over L_9 only, which is a path of steepest descent from the pass point $\tau_+=\frac{1}{2}i\tan\alpha$. We have

$$f''(\tau_+)=8\cos^4\alpha\tan\alpha,$$

and

$$\begin{aligned} \int_{L_9} &\exp\{-\kappa f(\tau)\}d\tau \\ &\doteq \int_{-\beta'}^{\beta'}\exp\{-\kappa f(\tau_+)\}\exp\{-4\kappa\cos^4\alpha\tan\alpha\xi^2\}d\xi \\ &= \tfrac{1}{2}\sec^2\alpha(\pi\cot\alpha/\kappa)^{\frac{1}{2}}e^{-\omega}. \end{aligned}$$

Therefore

$$y^{(1)} \doteq (\pi\kappa\cot\alpha)^{\frac{1}{2}}(e^{\omega}+\tfrac{1}{2}ie^{-\omega}). \tag{3.21}$$

The accuracy of this formula is examined in Appendix A.

4. EQUATIONS FOR r_0 AND U FOR $l=0$

If superscripts (i) and (o) refer to properly normalized solutions inside and outside the nucleus, the condition of continuity requires that

$$\left.\begin{aligned} \psi^{(i)} &= \psi^{(o)} \\ d\psi^{(i)}/dr &= d\psi^{(o)}/dr \end{aligned}\right\} \quad \text{at} \quad r=r_0,$$

which is equivalent to the condition

$$\frac{1}{X_l^{(i)}}\frac{dX_l^{(i)}}{dr}=\frac{1}{X_l^{(o)}}\frac{dX_l^{(o)}}{dr} \quad \text{at} \quad r=r_0. \tag{4.1}$$

When the explicit expressions for X are substituted in (4.1), it becomes a complex equation for E, from which we can deduce two real equations, one for the energy E_α of the α-particle and one for the decay constant λ.

For $l=0$, $X_0^{(o)}=y^{(1)}$ and $X_0^{(i)}$ is given by Eq. (2.3). Noting that $d\alpha/dx=-(\kappa\sin2\alpha)^{-1}$, and $dx/dr=mv/\hbar$, straightforward calculation gives, using (3.21),

$$\begin{aligned} \frac{1}{X_0^{(o)}}\frac{dX_0^{(o)}}{dr} &= -\frac{mv}{\hbar\sin2\alpha} \\ &\quad\times\left\{2\sin^2\alpha\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}-\frac{1}{\kappa\sin2\alpha}\right\}. \end{aligned}$$

$(1/\kappa_R)$ cosec$2\alpha_R$ is 1 to 2 percent of $2\sin^2\alpha_R$, which is the value of the first term. (The accurate value of $y^{(1)}$ given by Eq. (A.12) adds other terms, the greatest of which is proportional to $e^{-2\omega}$ which is of the order 10^{-20} to 10^{-30} with $r \doteq 10^{-12}$ cm, and with experimentally known values of E_α.) Therefore we have 2 percent numerical accuracy in writing

$$\frac{1}{X_0^{(o)}}\frac{dX_0^{(o)}}{dr} = -(2mE/\hbar^2)^{\frac{1}{2}}\tan\alpha\left(\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}\right).$$

Also, from (2.3),

$$\frac{1}{X_0^{(i)}}\frac{dX_0^{(i)}}{dr} = \{2m(E-U)/\hbar^2\}^{\frac{1}{2}}\cot[r\{2m(E-U)/\hbar^2\}^{\frac{1}{2}}].$$

These two expressions are to be equated at $r=r_0$ and the substitution $E=E_\alpha-\frac{1}{2}i\hbar\lambda$ is to be made. In the initial factors $E^{\frac{1}{2}}$ and $(E-U)^{\frac{1}{2}}$ we may neglect the imaginary part of E, because it introduces corrections at most 10^{-27} times the real and imaginary parts on each side. Similar arguments apply to the imaginary parts of $\tan\alpha$, and of κ, and α in the exponents.

Introducing

$$\mu=(1-U/E_\alpha)^{\frac{1}{2}},\quad k=mv/\hbar,\quad \alpha_0=\alpha_R \text{ for } r=r_0,$$

and

$$\omega_0=\kappa_R(\alpha_0-\sin\alpha_0\cos\alpha_0),$$

the condition becomes

$$\mu\cot\{\mu kr_0(1-i\hbar\lambda/2E_\alpha\mu^2)\} = -\tan\alpha_0\frac{1-\frac{1}{2}ie^{-2\omega_0}}{1+\frac{1}{2}ie^{-2\omega_0}}. \tag{4.2}$$

Also the cotangent term is

$$\frac{1+(i\hbar\lambda kr_0/4E_\alpha\mu)\tan(\mu kr_0)}{\tan(\mu kr_0)-i\hbar\lambda kr_0/4E_\alpha\mu}.$$

Separating real and imaginary parts of (4.2) we thus obtain

$$\mu=-\tan\alpha_0\tan(\mu kr_0), \tag{4.3}$$

$$\lambda=\frac{2v}{r_0}\frac{\mu^2\tan\alpha_0}{\mu^2+\tan^2\alpha_0}e^{-2\omega_0}. \tag{4.4}$$

TABLE I. Values of r_0 and B for $l=0$.

Nuclei	Radius (10^{-13} cm)	Total decay energy (Mev)	λ (sec.$^{-1}$)	B (Mev)	$R=r_0A^{-\frac{1}{3}}$
UI-UXI	9.37	4.31[a]	4.82×10^{-18}[c]	23.9	1.52
	9.34	4.32[b]		23.9	1.52
UII-Io	9.26	4.89[a]	8.14×10^{-14}[c]	25.6	1.51
	9.21	4.91[b]		25.7	1.51
Io-Ra	9.26	4.80[d]	2.6×10^{-13}[e]	23.1	1.52
*Ra-Rn[f]	9.29	4.879	1.35×10^{-11}	22.3	1.53
Rn-RaA	9.28	5.589	2.097×10^{-6}	21.0	1.54
RaA-RaB	9.14	6.112	3.78×10^{-3}	20.3	1.53
RaF-RaG	8.27	5.410[g]	5.886×10^{-8}[g]	23.8	1.40
Th-MthI	9.92	3.99[h]	1.2×10^{-8}	22.0	1.62
	10.01		1.7×10^{-8}[i]	21.8	1.64
*RdTh-ThX[f]	9.33	5.517	9.7×10^{-9}	22.1	1.54
ThX-Thn	9.29	5.786	2.20×10^{-6}	21.4	1.54
Thn-ThA	9.28	6.400	1.27×10^{-2}	20.2	1.55
ThA-ThB	9.12	6.904	4.95	19.5	1.53
*Pa-Ac[f]	8.68	5.09[k]	5.5×10^{-13}[m]	27.2	1.34
*An-AcA	8.70	6.953	1.22×10^{-1}	21.5	1.45
AcA-AcB	8.99	7.508	3.47×10^{2}	19.3	1.51

The letters in the body of the table refer to the remarks below.
[a] N. Feather, see reference 17.
[b] From the values of range given by Sizoo, Physica 4, 791 (1937); and graphs of range against energy given by M. S. Livingston and H. A. Bethe, Rev. Mod. Phys. 9, 266 (1937). Value then corrected to include recoil energy, since the tabulated figure is total energy.
[c] A. O. Nier, Phys. Rev. 55, 150 (1939).
[d] Obtained as in (b), using range given by Mme. Curie, reference 17.
[e] Hernegger, Akad. Wiss. Wien 143, 367 (1934).
[f] For fuller information see section 5.
[g] W. Y. Chang, Phys. Rev. 69, 60 (1946).
[h] Obtained as in (b), using range given by G. H. Henderson and G. C. Laurence, Phys. Rev. 52, 46 (1937).
[i] The value 1.7×10^{-8} is given by Feather and by Stranathan, reference 17.
[k] Tsien San-Tsiang, M. Bachelet and G. Bouissieres, Phys. Rev. 69, 39, 1946.
[m] Total decay constant 6.7×10^{-13} from Stranathan, reference 17; $l=0$ line is 80 to 85 percent of this according to Tsien San-Tsiang et al, reference (k).

In this we have, summarizing the symbolism for convenience,

$$\left.\begin{aligned}\mu&=(1-U/E_\alpha)^{\frac{1}{2}},\\ \alpha_0&=\arccos(mv^2r_0/4\epsilon^2Z)^{\frac{1}{2}},\\ k&=mv/\hbar,\\ \kappa_R&=4\epsilon^2Z/\hbar v,\\ \omega_0&=\kappa_R(\alpha_0-\sin\alpha_0\cos\alpha_0).\end{aligned}\right\} \tag{4.5}$$

Equations (4.3) and (4.4) are to be regarded as two relations between the four variables E_α, λ, r_0, U. This is the form in which the theory expresses the Geiger-Nuttall law. What is more important however is that when E_α and λ are known experimentally, the radius r_0 and the potential U of the nucleus may be calculated.

Table I gives the values of r_0 and the depth B of the potential, calculated from (4.3) and (4.4).

$$B=(2Z\epsilon^2/r_0)-U. \tag{4.6}$$

Only rays for which $l=0$ are listed. Except where indicated by a note in the Table, the values of λ and E_α which we have used are those found generally in reference works.[17] In the case of an

[17] N. Feather, *Nuclear Physics* (Cambridge University Press, 1936); I. Curie, *Radioactivité* (Hermann, Paris,

α-ray spectrum of more than one line, the λ used should refer to the probability of decay with the one energy E_α only. Such cases are indicated by an asterisk in Table I where partial decay constants are used. Birge's[18] values of ϵ, m, h, etc., are used.

The last column in Table I allows a comparison of the results with the empirical law that the nuclear radius is proportional to the cube root of the atomic number.

5. THE CASE $l\neq 0$

When $l\neq 0$, we have, instead of (3.1),

$$\frac{d^2y}{dx^2}+[1-\kappa/x-l(l+1)/x^2]y=0. \qquad (5.1)$$

This equation can be studied by means of the factorization method as described by Infeld.[19] The recursion formula is:

$$X_{l+1}^{(o)}=G_l\left(\frac{l+1}{x}+\frac{\kappa}{2(l+1)}-\frac{d}{dx}\right)X_l^{(o)}, \qquad (5.2)$$

where G_l is independent of x. Since our theory requires only the ratio of $X_l^{(o)}$ to its derivative (see (4.1)), the form of G_l is immaterial.

It is then apparent that the results for a general l can be expressed in terms of the first $(l+1)$ derivatives of $X_0^{(o)}$. The calculations are contained in Appendix B. Here we shall simply state the results. Equations (4.3) and (4.4) for μ and λ are replaced by:

$$\mu=-\tan\alpha_0\frac{S_l(M)\tan D+C_l(M)}{K_l(M)-H_l(M)\tan D}, \qquad (5.3)$$

$$\lambda=(2v/r_0)\frac{\mu^2\tan\alpha_0(H_lC_l+K_lS_l)Q_le^{-2\omega_0}}{\mu^2(H_l^2+K_l^2)+\tan^2\alpha_0(C_l^2+S_l^2)+2\mu\tan\alpha_0(C_lK_l-S_lH_l)}. \qquad (5.4)$$

The notation of these equations is:

$$D=1/M=\mu kr_0, \qquad (5.5)$$

Q_l is a rational function of $\tan\alpha_0$ and r_0 (or M), C_l, S_l, H_l, K_l are polynomials in M.

These functions do not have a useful general form for arbitrary l, but are calculated separately for each l. The first five results are:

$l=0$:

$$Q_0=1,\quad S_0=1,\quad C_0=0,\quad H_0=0,\quad K_0=1.$$

$l=1$:

$$Q_1=(\kappa-2\tan\alpha_0)/(\kappa+2\tan\alpha_0);$$

$$S_1=M;\quad C_1=-1;\quad H_1=M^2-1;\quad K_1=M.$$

$l=2$:

$$Q_2=(\kappa+10\mu M-6\tan\alpha_0)/(\kappa+10\mu M+6\tan\alpha_0);$$

$$S_2=3M^2-1;\quad C_2=-3M;$$

$$H_2=3M(2M^2-1);\quad K_2=6M^2-1.$$

$l=3$:

$$Q_3=\frac{\kappa+28\mu M-12\tan\alpha_0+(44/\kappa)\tan^2\alpha_0}{\kappa+28\mu M+12\tan\alpha_0+(44/\kappa)\tan^2\alpha_0};$$

$$S_3=-6M+15M^3;\quad C_3=1-15M^2;$$

$$H_3=1-21M^2+45M^4;\quad K_3=-6M+45M^3.$$

$l=4$:

$$Q_4=\frac{\kappa(\kappa+60\mu M)-20\tan\alpha_0(40\mu M+\kappa)+140}{\kappa(\kappa+60\mu M)+20\tan\alpha_0(40\mu M+\kappa)+140};$$

$$S_4=1-45M^2+105M^4;\quad C_4=10M-105M^3;$$

$$H_4=10M-195M^3+420M^5;$$

$$K_4=1-55M^2+420M^4.$$

In making calculations for r_0 and U in the case of α-ray spectra with fine structure, it is necessary to determine the angular momentum quantum number l. If the spin number of the disintegrating nucleus is s', and that of the relevant level of the product nucleus is s, l is an integer between $|s+s'|$ and $|s-s'|$. Thus if s and s' are known, the possible values of l are easily found. In particular if s' is zero, $l=s$.

1935), Vol. II; J. Hoag, *Electron and Nuclear Physics* (D. Van Nostrand Company, Inc., New York, 1938); J. Stranathan, *The "Particles" of Modern Physics* (The Blakiston Company, Philadelphia, 1942); W. B. Lewis and B. V. Bowden, Proc. Roy. Soc. **A145**, 235 (1934); International Commission, J. de phys. et rad. **2**, 273 (1931).

[18] R. T. Birge, Rev. Mod. Phys. **13**, 233 (1941).

[19] L. Infeld, Phys. Rev. **59**, 743 (1941).

TABLE II. Calculations for line zero, ThC-ThC″. r_0 and l in the first two columns are used to obtain the λ's in the third column. Experimental value is $\lambda = 1.83\times 10^{-5}$ sec^{-1}.

r_0 (10^{-13} cm)	l	λ (10^{-5} sec^{-1})	r_0 (10^{-13} cm)	l	λ (10^{-5} sec^{-1})
7.57	0	6.93	7.20	2	2.44
	1	10.84		3	2.02
	2	11.29		4	0.48
	3	9.49	7.40	4	1.18
7.20	0	1.52	7.48	4	1.81
	1	2.37	7.49	4	1.89

As an example we take the α-decay ThC→ThC″. This is a particularly convenient case, as Oppenheimer[20] has assigned spin numbers to the various levels of the ThC″ nucleus; these values were determined by a study of γ-ray data. Where the subscripts refer to the levels of the ThC″ nucleus, Oppenheimer gives

$$s'=1,\quad s_0=3,\quad s_1=1,\quad s_2=3,\quad s_3=3,\quad s_4=0.$$

Thus

$$l_0=2,\ 3 \text{ or } 4;\quad l_1=0,\ 1 \text{ or } 2;\quad l_2=2,\ 3 \text{ or } 4;$$
$$l_3=2,\ 3 \text{ or } 4;\quad l_4=1.$$

Before considering the calculation in detail, two general remarks will be useful. It is found that, when (5.3) and (5.4) are used to calculate λ for a given r_0 and E_α, the values of λ at first increase with l and then decrease continually. This is caused by the presence of two opposite effects, the increase of μ with l and the decrease of Q. The latter finally becomes of greater importance. Also the value of λ decreases if r_0 is decreased. Table II illustrates these facts.

In calculating r_0 for complex spectra, our procedure is to assume a value for r_0 and for l, and to calculate μ and λ from (5.3) and (5.4), employing experimentally known E_α. The process is repeated until the chosen r_0 and l reproduce the experimental value of λ.

In the case of ThC→ThC″, we examine first line four, for which $l_4=1$. We find $r_0=7.5_4\times 10^{-13}$ cm. We then turn to line one and find that, if $l=0$, $r_0=7.57\times 10^{-13}$, if $l=1$, $r_0=7.38\times 10^{-13}$, and if $l=2$, r_0 is still smaller. These figures suggest that to obtain agreement with line four we should take for line one $r_0=7.57\times 10^{-13}$ and $l=0$. For line zero, the results in Table II are obtained. Since for line zero l must be 2, 3, or 4, it is seen that the greatest possible radius for line zero is 7.48×10^{-13} cm. This agrees reasonably well with the values obtained for lines four and one. Line two gives $r_0=7.40\times 10^{-13}$ for $l=3$, and $r_0=7.60\times 10^{-13}$ for $l=4$. Line three gives $r_0=7.5\times 10^{-13}$ with $l=4$.

Therefore, the most plausible results are those in Table III. Any other combination of l's would yield less consistent (and smaller) radii.

Calculations have been made for certain other complex spectra as in Table IV. For elements in the Ra and Th series, occurring before the C-products in the disintegration series, it is assumed that all ground state transitions have $l=0$; for the lines corresponding to excited states $l=1$ or 2 in virtue of the connection between spin and the emission of γ-rays by the product nucleus in falling to its ground state. If s_0 is the spin of the ground state of the product nucleus, the character of the γ-ray from level j is determined by $|s_j-s_0|$. This quantity must be 0, 1, or 2; dipole radiation occurs when it is 1, otherwise quadrupole; $s_j=s_0$ is forbidden. In particular, if $s'=s_0=0$, $l(=s_j)$ must be either 1 or 2.

It may be mentioned that it is presumably possible that any one line contains particles of various possible l's, present with relative intensities related to the respective λ's. This would seem to require smaller r_0's, but the available data does not in any case permit a test of the hypothesis.

The formulae (5.3) and (5.4) differ considerably from those previously given for $l\neq 0$ by Sexl[1] and Gamow.[21] Gamow's formula makes λ a monotonically decreasing function of l for fixed r_0, as opposed to the initial rise possible with our formula. Numerical results with Gamow's formula have been published by Gamow and Rosenblum.[21] Their values of radii are presented by means of a graph of a quantity r_{eff} introduced by these authors. Numerical values of r_{eff} have been

TABLE III. r_0 for ThC-ThC″.

Line	λ (10^{-5} sec^{-1})	Total energy (Mev)	l	r_0 (10^{-13} cm)
0	1.83	6.201	4	7.48
1	4.65	6.161	0	7.57
2	0.12	5.873	4	7.60
3	0.01	5.728	4	7.50±.05
4	0.07	5.709	1	7.54±.02

[20] F. Oppenheimer, Proc. Camb. Phil. Soc. **32**, 328 (1936).

[21] G. Gamow and S. Rosenblum, Comptes rendus **197**, 1620 (1933).

determined from this graph and converted to r_0 by means of a relation given in their paper, *viz.*

$$r_{\text{eff}}=r_0-\frac{\hbar^2}{m\epsilon^2 Z}l(l+1).$$

For this purpose, the l's of Tables III and IV have been used. These r_0's are compared with our results in Table V. The large differences in the radii of corresponding ground states are caused by the employment by Gamow of a more approximate formula for $l=0$. Nevertheless, it will be seen that there is considerable disparity even in the *relative* values of "fine structure radii"; the spread of the r_0's from Gamow's data for ThC→ThC″ is 25 percent of the smallest radius; for our values the corresponding spread is 1.6 percent. In some cases the radius given by Gamow's formula increases when that given by (5.3) and (5.4) decreases. It is clear, therefore, that the earlier approximate formulae, besides giving appreciably lower radii for $l=0$, are unsuited for adaptation to the case $l\neq 0$.

6. EFFECTIVENESS OF ONE-BODY MODEL

It is necessary to examine the objection that possibly too much weight is given to spurious effects resulting solely from the assumption of a rectangular well potential. As Gamow points out, it should not be used to calculate the energy levels of the α-particle in the nucleus. Thus Gamow obtains only one $E_\alpha-\lambda$ relationship not involving U. Now the two equations in this paper do allow a numerical evaluation of U (see Table VI); but this result should be regarded not as a determination of an energy level but rather as a step in the implicit elimination of U to give one equation connecting E_α and λ.

TABLE IV. r_0 for some complex spectra. In Pa-Ac, the separate λ's for lines 1 and 2 are not known: r_0 is chosen to give their sum. The difference between $l=1$ and $l=2$ lies within the limits indicated for r_0.

Nuclei	Line	λ (sec^{-1})	Total energy (Mev)	l	r_0 (10^{-13} cm)
Pa-Ac	0	5.5×10^{-13}	5.09	0	8.68
	1} 2}	1.2×10^{-13}	4.80 / 4.77	1 or 2 / 1 or 2	9.09
RdTh-ThX	0	9.7×10^{-9}	5.517	0	9.33
	1	1.9×10^{-9}	5.431	{1 / 2	9.08 / 9.05
Ra-Rn	0	1.35×10^{-11}	4.879	0	9.29
	1	0.03×10^{-11}	4.695	1 or 2	$8.92\pm.02$

TABLE V. Comparison of r_0's for complex spectra from Gamow's formula and present results.

Nuclei	Line	l	r_0 calculated from r_{eff}	r_0 from Table III or IV
ThC-ThC″	0	4	7.9×10^{-13} cm	7.48×10^{-13} cm
	1	0	6.4	7.57
	2	4	8.1	7.60
	3	4	8.0	7.50
	4	1	6.8	7.54
RdTh-ThX	0	0	7.8_5	9.33
	1	1	7.9_1	9.08
	1	2	8.0_0	9.05
Ra-Rn	0	0	7.9_7	9.29
	1	1	7.9_7	8.92
	1	2	8.0_6	8.92

TABLE VI. Values of U. ThC-ThC″.

l	0	1	4	4	4
Line	1	4	0	2	3
U (Mev)	5.38	4.10	0.74	0.58	0.35
E	6.16	5.71	6.20	5.87	5.73

The decay constant may in a sense be considered as dependent on two factors: the frequency with which the α-particle "strikes the wall" of the potential barrier and the probability of penetration for any one collision. This latter factor is not sensibly dependent on the model of internal forces; it corresponds to the exponential factor $Q_le^{-2\omega_0}$ in (4.4) and (5.4). The remaining factors in these formulae may be considered to express the "frequency of striking"; they take the form g_lv/r_0, where g_l are factors which are dependent on the model of nuclear forces employed. $(v/r_0)Q_le^{-2\omega_0}$ determines the power of ten in the value of λ; g_l changes the value by a factor of the order of 10 or less for different models.

Now Gamow's results are apparently independent of the model of internal potential. But this is somewhat misleading, since his formulae depend on his Eq. (41) (reference 5, p. 100), which is equivalent to using the potential well and ignoring the dependence of g_l on l.

The assumption here is presumably that the manner in which the change in l affects the internal solution is not highly dependent on the model of internal potential. However, in that case an accurate evaluation should be permissible, and, in fact, a new qualitative effect appears, for with the rectangular well λ increases slightly for small $l(\lesssim 3)$, whereas with Gamow's formula it decreases monotonically.

That g_l increases with l may be seen as follows. Given a fixed radius and internal energy, the introduction of an increased "centrifugal force" raises the value of the potential near the center of the nucleus, implying greater velocity near the surface; the maximum of the internal wave function is also shifted to a greater radius. These effects correspond to a greater frequency of "collision with the barrier" and, with the rectangular well, an over-all increase in λ for the first few l's. More accurate models might well produce the same effect; there is at present insufficient knowledge of the spins of the nuclear levels involved to determine from the experimental results whether or not such an initial rise occurs in the λ *vs.* l curve.

A remark similar to that made in regard to the numerical values of U applies also to r_0, although less strongly. Bethe[11] estimates that a many-body theory would increase the values of nuclear radii by amounts up to about 30 percent of our values in Table I. Thus, although our values are, on the one-body model, usually accurate to the three figures given, it is clear that such a degree of significance cannot be attributed to them if they are considered as actual nuclear dimensions. The quantity r_0 should be regarded as a parameter introduced in the one-body mathematical model, and adjusted to fit experimental data.

It is perhaps surprising that such a complicated problem as the formation and escape of an α-particle from a heavy nucleus can be handled as satisfactorily as it is with a one-body model. The crudest approximation depends only on the penetrability of the barrier, and complete confidence may be felt in this portion of the theory. As we have seen, the behavior of the internal solution is also important, and calculations with the one-body model throw considerable light on its effects, producing new results (for g_l) in agreement with those indicated by general considerations. To obtain further significant progress would require a theory describing the internal forces in detail and, of course, no such technique is at present available.

I wish to express my appreciation of the generous guidance and counsel given by Professor L. Infeld throughout the course of this investigation. I am also indebted to Professor R. Peierls of Birmingham University for helpful criticism, and to Professors A. F. C. Stevenson and W. J. Webber of Toronto for suggestions in connection with particular parts of the work.

APPENDIX A

Accurate Evaluation of $y^{(1)}(x)$

We require the following lemma given by Watson.[22] If $|\kappa|$ is large and $|\arg\kappa| \leqslant \frac{1}{2}\pi - \Delta$, where $\Delta > 0$ but otherwise arbitrary, and $n > 0$,

$$\int_0^\infty e^{-\kappa\tau} F(\tau) d\tau \sim \sum_{m=1}^{\infty} a_m \Gamma(m/n) \kappa^{-m/n}, \tag{A.1}$$

subject to the following conditions:

1. $F(\tau)$ is analytic for $|\tau| \leqslant a+\delta$, $a>0$, $\delta>0$.
2. $F(\tau) = \sum_{m=1}^{\infty} a_m \tau^{m/n-1}$ for $|\tau| \leqslant a$. (A.2)
3. $|F(\tau)| < Ne^{b\tau}$, $|\tau| \geqslant a$, $N>0$, $b>0$ and independent of τ.
4. $b < \kappa \sin\Delta$.

The symbol $\sim$ is used to denote asymptotic equivalence for large κ. The series is semi-convergent in the sense of Poincaré, i.e., the sum of m terms differs from the function the series represents by less than $K_m \kappa^{-m}$, where K_m is independent of κ.

To use this lemma for the evaluation of $y^{(1)}$ by the method of steepest descents, we let

$$\zeta = f(\tau) - f(\tau_-), \tag{A.3}$$

where τ is on L_3. Then ζ is real and positive. For any real $\zeta < \zeta_0$, where

$$\zeta_0 = f(\xi_1) - f(\tau_-) = \alpha - \cos\alpha\sin\alpha + 2\gamma\xi_1,$$

there are two values of τ which satisfy (A.3), one above, and one below τ_-. Call these points τ_1 and τ_2, and let τ_1 be below τ_-, τ_2 above τ_-. Then

$$\begin{aligned} y^{(1)} &= 2ixe^{-\kappa f(\tau_-)} \left\{ \int_{\zeta_0}^{0} e^{-\kappa\zeta} \frac{d\tau_2}{d\zeta} d\zeta + \int_0^\infty e^{-\kappa\zeta} \frac{d\tau_1}{d\zeta} d\zeta \right\} \\ &= 2ixe^{-\kappa f(\tau_-)} \int_0^{\zeta_0} e^{-\kappa\zeta} \frac{d(\tau_1 - \tau_2)}{d\zeta} d\zeta. \end{aligned} \tag{A.4}$$

It is permissible to neglect the integral from ζ_0 to ∞ because of the smallness of the integrand for $\zeta > \zeta_0$. This is the essence of the Sattelpunkt method. Numerical indications of error appear later.

We wish to see now that $d(\tau_1 - \tau_2)/d\zeta$ satisfies the conditions for $F(\tau)$ in the lemma (A.1), and to find the a_m's explicitly.

Since $\zeta = d\zeta/d\tau = 0$ for $\tau = \tau_-$, the expansion of ζ in terms of $(\tau - \tau_-)$ begins with the term in $(\tau - \tau_-)^2$. Hence the inverted series are

$$\tau_1 - \tau_- = \sum_{m=0}^{\infty} \frac{a_m \zeta^{\frac{1}{2}(m+1)}}{m+1},$$

$$\tau_2 - \tau_- = \sum_{m=0}^{\infty} (-1)^{m+1} \frac{a_m \zeta^{\frac{1}{2}(m+1)}}{m+1};$$

[22] Watson, *Bessel Functions* (The Macmillan Company, New York, 1944), p. 235ff.

and we find

$$\left.\begin{aligned} d\tau_1/d\zeta &= \sum_0^\infty \tfrac{1}{2}a_m\zeta^{\frac{1}{2}(m-1)} \\ d\tau_2/d\zeta &= \sum_0^\infty (-1)^{m-1}\tfrac{1}{2}a_m\zeta^{\frac{1}{2}(m-1)}. \end{aligned}\right\} \quad \text{(A.5)}$$

Therefore,

$$a_m=(2\pi i)^{-1}\int (d\tau_1/d\zeta)\zeta^{-\frac{1}{2}(m+1)}d\zeta,$$

where the path of integration circles the origin of the ζ-plane twice. However if we think of $\zeta^{\frac{1}{2}(m+1)}$ expressed as a function of τ, we can write

$$a_m=(2\pi i)^{-1}\int_{(0+)}\zeta^{-\frac{1}{2}(m+1)}d\tau,$$

where the path circles the origin of the τ-plane once in the positive sense.

Let $T=\tau-\tau_-$. Then

$$\zeta=T^2\sum_{j=0}^\infty B_jT^j,$$

where B_j are certain coefficients, $B_0\neq 0$. We now define coefficients $a_j(m)$ by the following:

$$\zeta^{-\frac{1}{2}(m+1)}=T^{-(m+1)}\left(\sum_0^\infty B_jT^j\right)^{-\frac{1}{2}(m+1)}=T^{-(m+1)}\sum_0^\infty a_j(m)T^j. \quad \text{(A.6)}$$

Then

$$\begin{aligned} a_m&=(2\pi i)^{-1}\int_{(0+)}\zeta^{-\frac{1}{2}(m+1)}d\tau, \\ &=\text{coefficient of } T^{-1} \text{ in } \zeta^{-\frac{1}{2}(m+1)}, \\ &=a_m(m). \end{aligned} \quad \text{(A.7)}$$

Thus (A.6) and (A.7) indicate how the quantities a_m may be found. These are the coefficients desired for application of (A.1). For, by (A.5),

$$d(\tau_1-\tau_2)/d\zeta=\sum_{m=0}^\infty a_{2m}\zeta^{m-\frac{1}{2}}=\sum_1^\infty b_m\zeta^{\frac{1}{2}m-1},$$

where $b_{2m+1}=a_{2m}$, and $b_{2m}=0$, and thus the last series is in the form quoted in (A.2).

Also

$$|d(\tau_1-\tau_2)/d\zeta|<Ke^{b\zeta},\quad K>0,\quad b>0$$

for $\zeta>a$. This follows by noting that

$$|(d\zeta/d\tau_i)^{-1}|=|1/(4\tau_i^2-1)+x/\kappa|^{-1},\quad (i=1,2)$$

is bounded for $\zeta>0$. Since this expression is bounded, b can be taken as small as desired by increasing a.

Also $\arg\kappa=\arctan(\kappa_I/\kappa_R)\doteq 10^{-16}$. In the notation of Watson's lemma, then, Δ can be as great as $\frac{1}{2}\pi-10^{-15}$, say. That is, $\sin\Delta\doteq 1$. Since b can be very small, the condition $b<\kappa\sin\Delta$ is satisfied.

Therefore, all the conditions of the lemma are satisfied by κ and $d(\tau_1-\tau_2)/d\zeta$, and we may write

$$\int_0^\infty e^{-\kappa\zeta}\frac{d(\tau_1-\tau_2)}{d\zeta}d\zeta\sim\sum_1^\infty b_m\Gamma(m/2)\kappa^{-\frac{1}{2}m}. \quad \text{(A.8)}$$

The limits of the integral for $y^{(1)}$ in (A.4) are 0 to ζ_0, rather than 0 to ∞. This has the effect of replacing $\Gamma(m/2)\kappa^{-\frac{1}{2}m}$ in (A.8) by

$$\int_0^{\zeta_0}e^{-\kappa\zeta}\zeta^{\frac{1}{2}m-1}d\zeta.$$

However,

$$\zeta_0\doteq .5,\quad |\kappa|\doteq 50$$

and it can be shown that, if $m<10$,

$$\int_{.5}^\infty e^{-50\zeta}\zeta^{\frac{1}{2}m-1}d\zeta<10^{-13}.$$

Therefore, we may employ Eq. (A.8) as it stands, or rather in the form

$$\int_0^\infty e^{-\kappa\zeta}\frac{d(\tau_1-\tau_2)}{d\zeta}d\zeta\sim\sum_0^\infty a_{2m}\Gamma(m+\tfrac{1}{2})\kappa^{-(m+\frac{1}{2})}. \quad \text{(A.9)}$$

It is now necessary to calculate a_{2m} explicitly. The first step is to find the B_j's. We have

$$\begin{aligned} B_0T^2+B_1T^3+\cdots=\zeta&=f(\tau)-f(\tau_-) \\ &=\tfrac{1}{2}i\log(\tau-\tfrac{1}{2})-\tfrac{1}{2}i\log(\tau+\tfrac{1}{2}) \\ &\quad +2ix\tau/\kappa-\tfrac{1}{2}\pi+\alpha-\cos\alpha\sin\alpha. \end{aligned}$$

The algebra of writing $\tau=T-\frac{1}{2}i\tan\alpha$ and expanding need hardly be reproduced. In this way B_j is found as a function of α.

Next, from (A.6) we see that a formal expansion of $(\sum_0^\infty B_jT^j)^{-\frac{1}{2}(m+1)}$ gives $a_j(m)$ in terms of B_j and m. Finally. $a_{2m}=a_{2m}(2m)$ gives the coefficients of (A.9) in terms of α, The end results of this tedious computation are:

$$\begin{aligned} a_0&=-\tfrac{1}{2}i\cos^{-2}\alpha\tan^{-\frac{1}{2}}\alpha, \\ a_2&=-\tfrac{3}{4}i\tan^{-\frac{3}{2}}\alpha\cos^{-2}\alpha(\tfrac{1}{6}+\tfrac{1}{4}\tan^2\alpha+5/36\cot^2\alpha), \\ a_4&=-\tfrac{1}{32}i\tan^{-5/2}\alpha\cos^{-2}\alpha \\ &\quad\times(-\tfrac{5}{8}\tan^4\alpha-\tfrac{5}{6}\tan^2\alpha+31/12+77/18\cot^2\alpha \\ &\quad+(5.7.11)/(8.27)\cot^4\alpha). \end{aligned}$$

Returning now to (A.4) and remembering that $x=\kappa\cos^2\alpha$, and $f(\tau_-)=\cos\alpha\sin\alpha-\alpha$, we have

$$\begin{aligned} y^{(1)}&\sim 2i\kappa\cos^2\alpha\exp\{\kappa(\alpha-\cos\alpha\sin\alpha)\} \\ &\quad\cdot(a_0\Gamma(\tfrac{1}{2})\kappa^{-\frac{1}{2}}+a_2\Gamma(\tfrac{3}{2})\kappa^{-\frac{3}{2}}+\cdots); \\ y^{(1)}&\doteq(\pi\kappa\cot\alpha)^{\frac{1}{2}}e^\omega(1+A\kappa^{-1}+B\kappa^{-2}), \end{aligned} \quad \text{(A.10)}$$

where

$$A=\tfrac{3}{4}\cot\alpha(\tfrac{1}{6}+\tfrac{1}{4}\tan^2\alpha+5/36\cot^2\alpha);$$

$$B=-(\tfrac{3}{64})\cot^2\alpha\left(\tfrac{5}{8}\tan^4\alpha+\tfrac{5}{6}\tan^2\alpha-31/12-77/18\cot^2\alpha-\frac{5.7.11}{8.27}\cot^4\alpha\right).$$

This expression (A.10), in which κ and α have small imaginary parts, is the part of $y^{(1)}$ which is predominantly real. The part of $y^{(1)}$ which is predominantly imaginary is much smaller in absolute value, and hence does not appear in the expansion about τ_-. However, the predominantly imaginary part of $y^{(1)}$ is $x\int e^{-\kappa f(\tau)}d\tau$, integrated over $L_8+L_9+L_7$ (cf. Section 3). The integral over L_8+L_7 is approximately

$$-2\int_{\frac{1}{2}}^{\xi_0-\delta}\cos[\tfrac{1}{2}\kappa\log\{(\xi-\tfrac{1}{2})/(\xi+\tfrac{1}{2})\}+2x\xi]d\xi, \quad \text{(A.11)}$$

where δ is the correction to ξ_0 caused by the imaginary part of α. It can be shown that the magnitude of the integral (A.11) is less than 10^{-17}. For (A.11) is

$$-2\int_{\frac{1}{2}}^{\xi_0-\delta}\cos z_1\cosh z_2d\xi+2i\int_{\frac{1}{2}}^{\xi_0-\delta}\sin z_1\sinh z_2d\xi,$$

where

$$z_1=\tfrac{1}{2}\kappa_R \log\{(\xi-\tfrac{1}{2})/(\xi+\tfrac{1}{2})\}+2x_R\xi,$$
$$z_2=\tfrac{1}{2}\kappa_I \log\{(\xi-\tfrac{1}{2})/(\xi+\tfrac{1}{2})\}+2x_I\xi.$$

Take a number θ of the order 10^{-17}. Then, for $\frac{1}{2}\leqslant\xi\leqslant\frac{1}{2}+\theta$

$$-\infty<\tfrac{1}{2}\kappa_I\log(\xi-\tfrac{1}{2})\leqslant\tfrac{1}{2}\kappa_I\log\theta<-8\times10^{-17},$$
$$0<\tfrac{1}{2}\kappa_I\log(\xi+\tfrac{1}{2})<\tfrac{1}{2}\kappa_I\log(1+\theta)<2\times10^{-35},$$
$$0>2x_I\xi>-10^{-18},$$

since over the whole range of natural α-emitters,

$$10^{-18}>-x_I>5\times10^{-40},$$
$$4\times10^{-18}>\kappa_I>2\times10^{-39}.$$

Therefore, for $\frac{1}{2}<\xi\leqslant\frac{1}{2}+\theta$,

$$|\sinh z_2|<\tfrac{1}{2}(\xi-\tfrac{1}{2})^{-\frac{1}{2}\kappa_I},$$
$$|\cosh z_2|<\tfrac{1}{2}(\xi-\tfrac{1}{2})^{-\frac{1}{2}\kappa_I}+\tfrac{1}{2}(\xi-\tfrac{1}{2})^{\frac{1}{2}\kappa_I},$$

and the integrals over the range $\frac{1}{2}$ to $\frac{1}{2}+\theta$ are less than

$$\left[\frac{(\xi-\frac{1}{2})^{1-\frac{1}{2}\kappa_I}}{2-\kappa_I}+\frac{(\xi-\frac{1}{2})^{1+\frac{1}{2}\kappa_I}}{2+\kappa_I}\right]_{\frac{1}{2}}^{\frac{1}{2}+\theta}\doteq\theta.$$

Also for $\frac{1}{2}+\theta\leqslant\xi\leqslant\xi_0-\delta$, the sinh term in the integral may be replaced by z_2, and the cosh term by 1. Then the real part is

$$-2\int_{\frac{1}{2}+\theta}^{\xi_0-\delta}\cos z_1 d\xi.$$

Now choose θ more exactly so that $z_1=n\pi$ for $\xi=\frac{1}{2}+\theta$. n is approximately 200. Note also that $z_1=0$ for $\xi=\xi_0$ and, therefore, for $\xi=\xi_0-\delta$, $z_1=\epsilon=-\delta\{x_R+\kappa_R\xi_0/(\xi_0^2-\frac{1}{4})\}$, which is of the order of 10^{-17}. Then the real part is

$$-2\int_{n\pi}^{\epsilon}\frac{\cos z_1 dz_1}{2x_R+\frac{1}{2}\kappa_R/(\xi^2-\frac{1}{4})}>-2\int_{n\pi}^{\epsilon}\cos z dz\doteq2\epsilon<0.$$

Also the imaginary part is less than

$$2\int_{\frac{1}{2}+\theta}^{\xi_0-\delta}z_2 d\xi.$$

This can be explicitly integrated and, on putting $\xi_0=.8$, the integral is seen to be less than 2×10^{-18}. Therefore, the integral over L_8+L_7 is less than 10^{-17}. We shall see later that the integral over L_9 is of the order of 10^{-14} or more. Hence the path L_8+L_7 may be neglected. (Note that if x_I and κ_I approach zero, (A.11) approaches zero, as stated in section 3.)

L_9 is a path of steepest descent. The expansion of the integral along L_9 about the pass-point $\tau_+(=\frac{1}{2}i\tan\alpha)$ is the same as that about τ_- along L_3, except that 1. α is replaced by $-\alpha$, and 2. the limits of ζ on both branches of the path are finite, *viz.* $R\{f(\xi_0+\delta)\}-R\{f(\tau_+)\}$, and $R\{f(-\xi_0+\delta)\}-R\{f(\tau_+)\}$, but as before these may both be replaced by inifinity. Therefore the imaginary part of $y^{(1)}$ is

$$I\{y^{(1)}\}\sim\tfrac{1}{2}(\pi\kappa\cot\alpha)^{\frac{1}{2}}e^{-\omega}(1-\tfrac{1}{2}a_2/a_0\kappa+\tfrac{3}{4}a_{\frac{1}{2}}/a_0\kappa^2+\cdots).$$

Finally, we have

$$y^{(1)}\doteq(\pi\kappa\cot\alpha)^{\frac{1}{2}}\{e^{\omega}(1+A\kappa^{-1}+B\kappa^{-2})+\tfrac{1}{2}ie^{-\omega}(1-A\kappa^{-1}+B\kappa^{-2})\}.\quad\text{(A.12)}$$

APPENDIX B

Derivation of Formulae for $l\neq0$

We saw in Section 5, that $X_l^{(o)}$ can be expressed in terms of the first $(l+1)$ derivatives of $X_0^{(o)}$ by means of the recursion formula

$$X_{l+1}^{(o)}=G_l\left(\frac{l+1}{x}+\frac{\kappa}{2(l+1)}-\frac{d}{dx}\right)X_l^{(o)}.\quad\text{(B.1)}$$

Let us then study these derivatives. We have seen that

$$X_0^{(o)}=2ixe^{-\frac{1}{2}\kappa\pi}\int_C(\tau-\tfrac{1}{2})^{-\frac{1}{2}i\kappa}(\tau+\tfrac{1}{2})^{\frac{1}{2}i\kappa}e^{-2ix\tau}d\tau.\quad\text{(B.2)}$$

Hence differentiation with respect to x introduces integrals of the form

$$I_n=e^{-\frac{1}{2}\kappa\pi}\int_C\tau^n(\tau-\tfrac{1}{2})^{-\frac{1}{2}i\kappa}(\tau+\tfrac{1}{2})^{\frac{1}{2}i\kappa}e^{-2ix\tau}d\tau,\quad n=1,2,3.\cdots$$

The lth derivative involves integrals from I_0 to I_l linearly. This differentiation assumes that C, the path of integration, is independent of x. Actually C is the line $\xi=-\frac{1}{2}+(x_I/x_R)\eta$, but, as we have seen, the small term dependent on x does not affect the value of any result obtained by the method of steepest descents, which method we shall now apply to I_n.

$$I_n\doteq\int_C\exp\{-\kappa f_n(\tau)\}d\tau,$$

where

$$f_n(\tau)=\tfrac{1}{2}i\log(\tau-\tfrac{1}{2})-\tfrac{1}{2}i\log(\tau+\tfrac{1}{2})+2i(x/\kappa)\tau-(n/\kappa)\log\tau+\tfrac{1}{2}\pi.$$

Possible pass points are the roots of $f_n'(\tau)=0$, that is of

$$1/(2\tau-1)-1/(2\tau+1)+ni/\kappa\tau+2x/\kappa=0.$$

This is a cubic equation for τ. Provided n/κ is not large (and normally it ranges from 0.02 to 0.1), two of the roots will be near the former pass points τ_+ and τ_-. The exact values can be found numerically if x and κ are given. The third root will be found to be near $\tau=0$ for small n. To the first order it is $\frac{1}{2}ni/(\kappa-x)\doteq ni/80$.

Disregarding the imaginary parts of x and κ which we have seen do not modify the form of the paths of steepest descent appreciably, the imaginary part of $f_n(\tau)$ is

$$G(\xi,\eta)=\tfrac{1}{4}\log\frac{(\xi-\frac{1}{2})^2+\eta^2}{(\xi+\frac{1}{2})^2+\eta^2}+2(x/\kappa)\xi-(n/\kappa)\arctan(\eta/\xi).$$

The paths of steepest descent for I_n would be $G(\xi,\eta)=\text{constant}$.

The term $(n/\kappa)\arctan(\eta/\xi)$ is a "perturbation" which slightly deforms the curves found for $l=0$. The equation still includes $\xi=0$, the constant term being increased by $n\pi/2\kappa$. Also the real part of $f_n(\tau)$ is

$$-[\tfrac{1}{4}\arctan\{\eta/(\xi^2+\eta^2-\tfrac{1}{4})\}+2(x/\kappa)\eta+(n/\kappa)\log|\tau|].$$

Now $\log|\tau|$ goes to infinity at both ends of the integration path. Hence the method of steepest descents seems applicable, and it can be checked that the conditions for the lemma (A.1) still hold. Therefore, we know that the method of steepest descents can be applied to I_n, and that an asymptotic series in $1/\kappa$ will be obtained.

There is a theorem[23] that, if a function and its derivative both possess expansions in asymptotic series, the term-by-term derivative of an asymptotic series for the function is an asymptotic series for the derivative. Therefore, since I_n possesses an asymptotic series, we may obtain the solutions

[23] E.g., Bromwich, *Theory of Infinite Series* (The Macmillan Company, London, 1926), p. 345.

$X_l^{(o)}$ by continued application of Eq. (B.1) in which for $X_0^{(o)}$ we substitute its asymptotic expansion.

Explicit general formulae for any l are not obtained, but the method is clear. In this paper, the case $l=2$ will be considered in some detail, and the procedure in the general case will be indicated.

By (B.1), $X_2^{(o)}$ is proportional to

$$\begin{aligned}(2/x+\kappa/4-d/dx)(1/x+\kappa/2-d/dx)X_0^{(o)}\\ =(3/x^2+5\kappa/4x+\kappa^2/8)X_0^{(o)}\\ -(3/x+3\kappa/4)\frac{d}{dx}X_0^{(o)}+\frac{d^2}{dx^2}X_0^{(o)}.\end{aligned}$$

Therefore $\frac{d}{dx}X_2^{(o)}$ is proportional to

$$\begin{aligned}-(6/x^3+5\kappa/4x^2)X_0^{(o)}+(6/x^2+5\kappa/4x+\kappa^2/8)\frac{d}{dx}X_0^{(o)}\\ -(3/x+3\kappa/4)\frac{d^2}{dx^2}X_0^{(o)}+\frac{d^3}{dx^3}X_0^{(o)}.\end{aligned}$$

By performing a calculation involving the first three terms of the series for $X_0^{(o)}$ and then neglecting terms of the order $1/\kappa$ against unity, committing about a three percent error, we find

$$\frac{d}{dx}X_0^{(o)}\Big/X_0^{(o)}=-\tan\alpha\cdot(1-\tfrac{1}{2}ie^{-2\omega})/(1+\tfrac{1}{2}ie^{-2\omega}),$$

$$\frac{d^2}{dx^2}X_0^{(o)}\Big/X_0^{(o)}=\tan^2\alpha,$$

$$\frac{d^3}{dx^3}X_0^{(o)}\Big/X_0^{(o)}=-\tan^3\alpha\cdot(1-\tfrac{1}{2}ie^{-2\omega})/(1+\tfrac{1}{2}ie^{-2\omega}).$$

Therefore,

$$\frac{d}{dx}X_2^{(o)}\Big/X_2^{(o)}=-\tan\alpha\cdot\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}\cdot\frac{1+\frac{8\tan^2\alpha}{\kappa(\kappa+10/x)}+\{6\tan\alpha/(\kappa+10/x)\}\frac{1+\frac{1}{2}ie^{-2\omega}}{1-\frac{1}{2}ie^{-2\omega}}}{1+\frac{8\tan^2\alpha}{\kappa(\kappa+10/x)}+\{6\tan\alpha/(\kappa+10/x)\}\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}}.$$

In the last term $8\tan^2\alpha$ may be neglected in comparison with κ^2, i.e., 32 in comparison with 2500, in both numerator and denominator.

In general, it is found that

$$\begin{aligned}\frac{d}{dx}X_l^{(o)}\Big/X_l^{(o)}&=-\tan\alpha\cdot\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}\cdot\frac{1+\sigma_l+\rho_l\frac{1+\frac{1}{2}ie^{-2\omega}}{1-\frac{1}{2}ie^{-2\omega}}}{1+\sigma_l+\rho_l\frac{1-\frac{1}{2}ie^{-2\omega}}{1+\frac{1}{2}ie^{-2\omega}}}\\ &=-\tan\alpha\cdot\frac{1-\frac{1}{2}iQ_le^{-2\omega}}{1+\frac{1}{2}iQ_le^{-2\omega}},\end{aligned}\tag{B.3}$$

where ρ_l, σ_l, Q_l are rational functions of $\tan\alpha$ and x, and

$$Q_l=(1-\rho_l+\sigma_l)/(1+\rho_l+\sigma_l).\tag{B.4}$$

The solution inside the nucleus, given by Eq. (2.3), depends on the properties of

$$\begin{aligned}(\pi z/2)^{\frac{1}{2}}J_{l+\frac{1}{2}}(z)=\sin(z-l\pi/2)\cdot\sum_{m=0}^{\leq\frac{1}{2}l}\frac{(-1)^m(l+2m)!}{(2m)!(l-2m)!}(2z)^{-2m}\\ +\cos(z-l\pi/2)\cdot\sum_{m=0}^{\leq\frac{1}{2}(l-1)}\frac{(-1)^m(l+2m+1)!}{(2m+1)!(l-2m-1)!}(2z)^{-2m-1}\\ =S_l(1/z)\sin z+C_l(1/z)\cos z,\end{aligned}\tag{B.5}$$

thus defining S_l and C_l as polynomials in $1/z$. Then

$$\frac{d/dz[z^{\frac{1}{2}}J_{l+\frac{1}{2}}(z)]}{z^{\frac{1}{2}}J_{l+\frac{1}{2}}(z)}=\frac{K_l\cot z-H_l}{C_l\cot z+S_l},$$

where

$$H_l=C_l+(1/z)^2(dS_l/dz^{-1}),\tag{B.6}$$

$$K_l=S_l-(1/z)^2(dC_l/dz^{-1}).\tag{B.7}$$

Writing

$$z=\{2m(E-U)/\hbar^2\}^{\frac{1}{2}}r_0$$

and expanding the cotangent to the first order in $\lambda ir_0/v$ as in the case of $l=0$, it follows that

$$\left(\frac{d}{dx}X_l^{(i)}\Big/X_l^{(i)}\right)_{r=r_0}=-\frac{H_l(M)\tan D-K_l(M)-(i\lambda r_0/2\mu v)(H_l+K_l\tan D)}{S_l(M)\tan D+C_l(M)-(i\lambda r_0/2\mu v)(S_l-C_l\tan D)}\mu,\tag{B.8}$$

where

$$D=\mu kr_0=1/M.\tag{B.9}$$

Equating now the right-hand members of Eqs. (B.3) and (B.8) at $r=r_0$, and separating real and imaginary parts, we find the Eqs. (5.3) and (5.4), which are our final results.

In the case $l=2$, we have found

$$\rho_2=6\tan\alpha_0/(\kappa+10/kr_0)\doteq12/50,$$
$$\sigma_2=8\tan^2\alpha_0/\kappa(\kappa+10/kr_0)\doteq32/2500\doteq0.$$

Then

$$\begin{aligned}Q_2&\doteq(1-\rho_2)/(1+\rho_2)\\ &=(\kappa+10/kr_0-6\tan\alpha_0)/(\kappa+10/kr_0+6\tan\alpha_0).\end{aligned}$$

Other Q_l's are found in a similar manner. S_l, C_l, H_l, and K_l are found from (B.5), (B.6), and (B.7). The results for l's as high as 4 are given in section 5.