Some Stationary Distributions of Neutrons in an Infinite Medium

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(Received March 13, 1947)

Some stationary solutions of the transport equation describing the diffusion and slowing down of neutrons in an infinite homogeneous medium (Sections 1 and 2) are presented for the following cases: constant mean free path, power spectrum in energy and exponential behavior in space (Section 3); constant m.f.p., point source of fast neutrons (Section 4); variable m.f.p. proportional to a power of velocity, point source of fast neutrons (Section 5). For this last case the possibility of actual numerical computation of the neutron distribution is displayed on one example, approximately describing conditions in water or paraffin.

I. INTRODUCTION

THE present paper is concerned with the development of methods for the solution of the transport equation, describing the diffusion and slowing down of neutrons. Attention will be focused especially on the case of a point-source of fast neutrons embedded in an infinite hydrogenous medium.

Usually problems of this kind are attacked by means of a more elementary diffusion equation ("age theory"), which represents a reasonable approximation for most practical purposes. Nevertheless, the greater effort required in solving the more exact equation is not always wholly unrewarding.

The present work was undertaken in 1943 with a view to fill what seemed an unnatural gap in the literature. Since then the writers have become aware that problems of this kind have received a great deal of attention in connection with the development of self-sustaining chain-reactions. Accordingly, this paper has been rewritten, so as to lay the main stress on the methodological aspects of the subject, rather than on the detailed computation of special examples. The numerical results which have been obtained in one case are presented mainly as an illustration of the method. The authors hope in this way that their work will not appear as a wholly useless duplicate of other more concentrated and purposeful efforts, and that some details at least in the method will be found sufficiently new to justify publication.

The main assumptions made throughout are: (a) the medium is homogeneous and infinite, so that a Fourier transform in space can be usefully applied, and: (b) the energy of the neutrons is neither too low (say >1 ev), so that effects due to binding and thermal motion are negligible, nor too high (say <10 Mev), so that the angular distribution after an elastic collision of a neutron with a nucleus is isotropic in the center-of-gravity system.

From the start, the principal aim of this investigation was to develop a method capable of dealing with velocity-dependent mean free paths. The method finally evolved is described in Sec. 5 for a rather special example; some comments on the adaptability of the method to other cases will be found there. The study of the case of a constant mean free path, presented in Secs. 3 and 4, was almost unavoidable as a preliminary to the more complicated case. The authors first believed that if a workable complete solution could not be found for the simpler case, one could not even attempt to study the more difficult case of a variable mean free path. This proved not to be the case; for some types at least of velocity dependence it is perfectly feasible to work out the solution numerically very accurately, although the results for the simpler case are disappointingly inconclusive. One of the writers, however, hopes to show in a forthcoming paper that also the results of Secs. 3 and 4 can be refined in such a way as to yield some information about the asymptotic behavior of the neutron density at large distances from the source.

2. THE TRANSPORT EQUATION

We shall use, among others, the following notations:

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$\mathbf{v}$ and $\mathbf{r}$: velocity and position vectors for a neutron;
v_0: velocity of the fast neutrons emitted by the source;
$u=2\ln(v_0/v)$: logarithmic energy variable;
$\boldsymbol{\omega}=\mathbf{v}/v$: unit vector specifying direction of motion;
$d\omega$: element of solid angle for $\boldsymbol{\omega}$;
$l(u)$: total mean free path of a neutron (including all possible processes, as e.g., capture) considered as a function of the logarithmic energy variable;
l_s: scattering mean free path; if several species of nuclei are present in the medium an additional index may be appended to l_s, specifying the nature of the nucleus.

The Boltzmann equation for stationary distributions is too well known as to require any comment. It is written customarily in terms of the distribution function $f(\mathbf{r}, \mathbf{v})$, or the particle-density in phase-space. It is often more convenient, however, to work with the "collision-density" defined by:

$$\psi(\mathbf{r}, u, \boldsymbol{\omega}) = \tfrac{1}{2}v^3(v/l(u))f(\mathbf{r}, \mathbf{v}), \tag{1}$$

the factor $(\frac{1}{2})v^3$ being introduced because we use $d\mathbf{r}dud\omega$ as the phase-space element, and v/l being the collision probability per second.

The Boltzmann equation is then:

$$l(u)\boldsymbol{\omega}\cdot\mathrm{grad}\psi+\psi-a(\psi)=S(\mathbf{r}, u), \tag{2}$$

where S is the function representing the source distribution, and a is an operator representing the effect of collisions. Specifically, $a(\psi)d\mathbf{r}dud\omega$ represents the number of collisions per second taking place within $d\mathbf{r}$ and such that a neutron of arbitrary velocity and direction is thrown into the element $dud\omega$ by the collision. Assuming that a neutron of velocity specified by u' and $\boldsymbol{\omega}'$ suffers a collision (of any kind) let $dud\omega P(u, \boldsymbol{\omega}; u', \boldsymbol{\omega}')$ be the probability for the collision to be such that the neutron is thrown into the element $dud\omega$. Then one finds:

$$a(\psi)=\int_0^u du'\int P(u, \boldsymbol{\omega}; u', \boldsymbol{\omega}')d\omega'\psi(\mathbf{r}, u', \boldsymbol{\omega}'). \tag{3}$$

It may be remarked that if only one nuclear species is present, the above-mentioned probability is the product of $l(u')/l_s(u')$ (representing the probability that the collision be of the scattering type) by a factor depending only on the cosine $\boldsymbol{\omega}\cdot\boldsymbol{\omega}'$ and on the ratio of the velocities (or on the difference $u-u'$). For instance in hydrogen, where $l/l_s=1$, one has

$$P(u, \boldsymbol{\omega}; u', \boldsymbol{\omega}')=(1/2\pi)e^{u'-u}\delta(\boldsymbol{\omega}\cdot\boldsymbol{\omega}'-e^{\frac{1}{2}(u'-u)}). \tag{4}$$

If many nuclear species are present, the probability is a composite of expressions of the above described type. One especially simple case arises when all mean free paths are velocity independent, or more generally if their ratios are velocity independent; then the function P is of the type:

$$P(u, \boldsymbol{\omega}; u', \boldsymbol{\omega}')=p(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', u-u') \tag{5}$$

as is clearly the case for Eq. (4).

3. THE CASE OF CONSTANT MEAN FREE PATH

When all relevant cross sections—or the corresponding m.f. paths—are constant, one is naturally led to apply a Laplace-Mellin transformation with respect to the velocity, i.e., a Laplace transformation with respect to the variable u. The customary inversion formula then yields the distribution function as a superposition of terms of the type:

$$\psi\sim e^{\eta u}=(v_0/v)^{2\eta}, \tag{6}$$

η being the variable of the Laplace transform. Any such term, if taken by itself, represents a power spectrum in velocity. One finds, moreover, as in the cascade theory of showers, that such power spectra are self-sustaining, that is, they are capable of existence even if the source term S on the right-hand side of Eq. (2) is equated to zero, provided the variable u is also allowed to take negative values.[1] More accurately, this means that the source terms are relegated to infinite energies. The properties of the non-homogeneous Eq. (2) are closely related, as usual, to these solutions of the homogeneous equation, which we are now going to investigate.

3.1 Homogeneous Equation. ($S=0$).

The total mean free path being a constant, it can be chosen as a unit of length. It has to be remembered that, in the absence of a source, v_0 means an arbitrary constant velocity. With a power spectrum of type (6), we can write:

$$\psi(\mathbf{r}, u', \boldsymbol{\omega}')=e^{\eta(u'-u)}\psi(\mathbf{r}, u, \boldsymbol{\omega}'); \tag{7}$$

inserting this into (3) and remembering Eq. (5)

[1] In that case the integral in Eq. (3) with respect to u' must also extend from $-\infty$ to u.

and footnote 1, we easily find

$$a(\psi)=\int g(\omega\cdot\omega',\eta)\psi(\mathfrak{r},u,\omega')d\omega', \tag{8}$$

where

$$g(\omega\cdot\omega',\eta)=\int_0^\infty e^{-\eta u}p(\omega\cdot\omega',u)du. \tag{9}$$

We may also separate the spatial coordinates, setting:

$$\psi=e^{\eta u-kz}\phi(\mu) \tag{10}$$

where μ is the z-component of ω. The choice of the z axis as the direction of exponential decrease entails no loss of generality.

Clearly we may assume ϕ to be symmetric about the z axis as is done in Eq. (10). There are also other solutions in which we are not interested just now. Developing the $\phi(\mu)$ function into a series of Legendre polynomials, we write:

$$\phi=(1/4\pi)\sum_{n=0}^{\infty}(2n+1)\phi_nP_n(\mu). \tag{11}$$

Developing also g in a series of Legendre polynomials:

$$g(\omega\cdot\omega',\eta)=(1/4\pi)\sum_{n=0}^{\infty}(2n+1)\times g_n(\eta)P_n(\omega\cdot\omega'), \tag{11a}$$

and making use of the sum-rule for spherical harmonics:

$$P_n(\omega\cdot\omega')=P_n(\mu)P_n(\mu')+\sum_{m\neq0}Y_n^{(m)}(\omega)\,Y_n^{(m)}(\omega'), \tag{12}$$

we easily find that all terms with $m\neq0$ cancel in Eq. (8), so that:

$$a(\psi)=(1/4\pi)e^{\eta u-kz}\sum_{n=0}^{\infty}(2n+1)g_n(\eta)\phi_nP_n(\mu). \tag{13}$$

Moreover, with ψ given by (10), the first term on the left-hand side of (2) becomes:

$$\begin{aligned}\omega\cdot\mathrm{grad}\psi&=-ke^{+\eta u-kz}(1/4\pi)\textstyle\sum_n\phi_n(2n+1)\mu P_n(\mu)\\&=-ke^{+\eta u-kz}(1/4\pi)\textstyle\sum_n\phi_n\\&\quad\times[nP_{n-1}(\mu)+(n+1)P_{n+1}(\mu)],\end{aligned} \tag{14}$$

owing to a well-known property of Legendre polynomials. Substituting (10) and (14) into Eq. (2) and rearranging terms, this becomes $\sum x_nP_n(\mu)=0$, where the coefficients x_n, given by the left-hand side of Eq. (15) below, do not depend on μ. This gives the equations $x_n=0$, or:

$$(2n+1)[1-g_n(\eta)]\phi_n-k[n\phi_{n-1}+(n+1)\phi_{n+1}]=0 \tag{15}$$

for $n=0,\ 1,\ 2,\ \cdots$. A further condition is obtained on remarking that for (11) to be meaningful one must have:

$$\lim_{n=\infty}\phi_n=0. \tag{16}$$

Since the solution of (15) contains an arbitrary factor, we may set for instance $\phi_0=1$. Then the first of the equations (15) or:

$$[1-g_0(\eta)]\phi_0-k\phi_1=0 \tag{17}$$

determines ϕ_1, while the remaining equations determine successively $\phi_2,\ \phi_3\ \cdots$. Thus, Eq. (15) is sufficient in itself to determine the solution completely, and Eq. (16) is an additional condition *implying a relation between η and k.*

In order to see this more clearly, we notice that since $\lim_{n=\infty}g_n=0$ the asymptotic form of Eq. (15), for $n\to\infty$, is:

$$2\phi_n-k(\phi_{n-1}+\phi_{n+1})=0, \tag{18}$$

of which two solutions are readily obtained in the form $\phi_n\sim\epsilon^n$ where ϵ is either of the two roots of the characteristic equation:

$$2\epsilon-k(1+\epsilon^2)=0. \tag{19}$$

If k is complex, or if it is real and satisfies the condition $-1<k<1$, the two roots are one larger and one smaller than unity, so that one of the solutions of (18) tends to ∞, the other to 0 when $n\to\infty$. According to a theorem by Poincaré,[2] the same applies to the system (15), Eq. (17) being omitted. The solution selected by Eq. (17) will in general be a linear combination of the two, and will thus tend to infinity. Thus, for every value of η, there will be special values

[2] N. Nörlund, *Differenzenrechnung* (Verlagsbuchhandlung Julius Springer, Berlin, 1924), Chap. 10.

of k, to be called eigenvalues henceforward, for which condition (16) may be satisfied.[3]

3.2 Series Development for Small Values of k.

We now wish to investigate the connection between η and k when k is small ($k \ll 1$), neglecting capture for simplicity. In the limiting case $k=0$, system (15) reduces to:

$$[1-g_n]\phi_n=0 \quad \text{for} \quad n=0, 1, 2, \cdots$$

showing that either all ϕ's are $=0$, which is trivial, or one at least of the g's must be $=1$.

Equations (9) and (11a) show that $g_0=1$ when $\eta=0$; this merely expresses the normalization condition for probability. One then has:

$$k=0, \quad \eta=0, \quad \phi_1=\phi_2=\cdots=0, \quad \text{or:} \quad \psi=\text{const.} \tag{20}$$

This solution is constant in space ($k=0$) and isotropic and corresponds to the well-known Fermi solution for the energy spectrum of slowed-down neutrons. One may get also other solutions by putting $g_1=1$, or $g_2=1$, $\cdots$ but one immediately suspects that they are less significant from a physical point of view. This is also borne out by the study of the inhomogeneous problem.

We shall therefore limit ourselves to studying the solution which reduces to Eq. (20) for $k=0$. For k small, one is naturally tempted to develop $\phi_0, \phi_1, \cdots$ and η in powers of k. Closer inspection of Eq. (15) then shows immediately that these series must have the form:

$$\phi_n=k^n(\phi_{n0}+k^2\phi_{n1}+k^4\phi_{n2}+\cdots) \tag{21}$$
$$\eta=\eta_1k^2+\eta_2k^4+\cdots.$$

We shall not carry out this development in detail. It should be mentioned, that there is a certain arbitrariness in the development for ϕ_0, ϕ_1, $\cdots$ (not in that for η!) owing to the fact that the solution can be multiplied by an arbitrary factor, which may be a function of k. One may remove this arbitrariness by setting $\phi_0=1$, for instance.

The coefficients in Eq. (21) are positive and indicate that η is a rapidly increasing function of k. For intermediate values of k it is, however, quite impracticable to try to compute all the coefficients that may be necessary in the development (21). It is much easier to ascertain the connection between η and k by a numerical investigation. One computes $g_0, g_1, \cdots$ for a given value of η, and then solves the system (15) for a tentative value of k, starting for instance with $\phi_0=1$, and determining $\phi_1, \phi_2, \cdots$ by means of the recurrence relation. Unless k is already very near to the eigenvalue, it will very soon become apparent that the solution does not satisfy the boundary condition at infinity. One then starts with a better guess for k, and so on. The work is quite similar to finding an eigenvalue of a Schrödinger equation by means of a step-by-step integration, only it is somewhat simpler.

Until now we have generally assumed that k was real; Eq. (21) shows however that η is real also for purely imaginary values of k, corresponding to a "plane wave" distribution in space. This conclusion holds as long as the series converges. We have carried out a numerical investigation for the case of pure hydrogen, Eq. (4), and the result is given in Fig. 1. When the absolute value of k exceeds a certain limit there is no longer a real value of η corresponding to it; the curve representing η as a function of k joins a lower branch corresponding to an $\eta<0$ for $k=0$, that is one of the branches we have previously neglected (and precisely the one for $g_1=1$ when $k=0$).

4. NON-HOMOGENEOUS EQUATION. CONSTANT MEAN FREE PATHS

We assume now the existence of sources distributed in space and emitting neutrons of velocity v_0. We set therefore in Eq. (2):

$$S(\mathbf{r}, u)=s(\mathbf{r})\delta(u).$$

We choose as before the total mean free path as a unit of length. Moreover we perform a Laplace-transformation with respect to u or a Laplace-Mellin transformation with respect to velocity, setting:

$$\Phi(\mathbf{r}, \eta, \omega)=\int_0^\infty e^{-\eta u}\psi(\mathbf{r}, u, \omega)du, \tag{22}$$

[3] In the forthcoming paper mentioned in Sec. 1, it will be shown that the values of k on the real strips $k<-1$ and $k>+1$, for which the roots of Eq. (19) are complex numbers of modulus "unity," and the solutions of Eq. (18) neither tend to zero nor to infinity, exhibiting an oscillating behavior, may also be considered as eigenvalues. There is thus both a discrete spectrum and a continuum of eigenvalues.

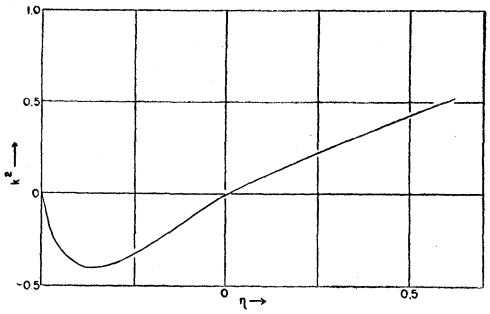

FIG. 1. Dependence of η on eigenvalue k, for real and imaginary values of k (positive and negative k^2). The descending part of the curve on the left corresponds to the lower branch mentioned in the text.

which can be inverted by means of:

$$\psi=(1/2\pi i)\int_{\beta-i\infty}^{\beta+i\infty}\Phi(\mathbf{r},\eta,\omega)e^{\eta u}d\eta, \quad (22a)$$

where β is a constant that can be arbitrarily increased, but not decreased beyond a certain limit. The equation satisfied by Φ is obtained by multiplying Eq. (2) by $e^{-\eta u}$, by making use of Eq. (5), and integrating respect to u. The term containing $a(\psi)$ must be modified as usual by means of the convolution theorem, that is by inverting the order of integrations respect to u and u'. Finally one finds easily:

$$\omega\cdot\mathrm{grad}\Phi+\Phi - \int g(\omega\cdot\omega',\eta)d\omega'\Phi(\mathbf{r},\eta,\omega')=s(\mathbf{r}). \quad (2')$$

We shall now examine two typical source distributions.

4.1 Wave Source

We consider a source of the sinusoidal plane wave type:

$$s(\mathbf{r})=e^{-kz}/4\pi, \quad (23)$$

where k is a pure imaginary, whenever the contrary is not stated explicitly. The choice of the z axis as the wave normal does not of course imply any essential limitation. We look for a solution of the type:

$$\Phi=e^{-kz}\phi(\mu), \quad (23a)$$

suggested by the symmetry of the problem. The equation for ϕ is then:

$$\phi(\mu)-k\mu\phi(\mu)-\int g(\omega\cdot\omega',\eta)d\omega'\phi(\mu')=1/4\pi, \quad (24)$$

which can be again transformed by means of Eqs. (11) and (11a) into the system:

$$(2n+1)[1-g_n(\eta)]\phi_n - k[n\phi_{n-1}+(n+1)\phi_{n+1}]=\delta_{n0}, \quad (24a)$$

differing from (15) only for the Kronecker symbol $\delta_{n0}(=1$ if $n=0$, $=0$ otherwise) on the right-hand side. Various methods can be devised to solve this system of equations. The solution may be given, for instance, in terms of a continued fraction:

$$\phi_0=\cfrac{1}{1-g_0-\cfrac{k^2/3}{1-g_1-\cdots-\cfrac{k^2/(4n^2-1)}{1-g_n-\cdots}}}. \quad (25)$$

This can be formally derived, if one uses the first of Eqs. (24a) to express ϕ_0 in terms of ϕ_1/ϕ_0, then the second to express ϕ_1/ϕ_0 in terms of ϕ_2/ϕ_1, and so on.[4] The fraction (25) defines ϕ_0 in the whole plane of the complex k-variable, except on the real strips:

$$1<k<+\infty; \quad -\infty<k<-1 \quad (26)$$

for reasons which are disclosed by the discussion of the homogeneous system (15) (see the text following Eq. (19)). If the plane is cut along the strips (26), the function ϕ_0 is analytic and one-valued and the only singularities it possesses are poles, at which of course the fraction (25) does not converge.

If our problem were merely to compute ϕ_0 for a few real values of η and ik, the continued fraction would be very suitable, since it converges fairly well for most values of ik. But if we are interested in an integral of the type (22a) we obviously need some expression which is easier to handle than (25). Without entering into too many details, we wish to mention various possible lines of attack.

One possible method consists in developing ϕ_n in powers of k, as in the homogeneous case (see Eq. (21)). The difference is that in the present

[4] For a rigorous justification see: Perron, *Die Lehre von den Kettenbrüchen* (B. G. Teubner, Leipzig), second edition, especially Section 57.

case η is given, and *must* not be developed in powers of k. One finds for instance:

$$\phi_0=(1-g_0)^{-1} + (k^2/3)(1-g_0)^{-2}(1-g_1)^{-1}+\cdots. \quad (27)$$

The convergence radius of this development will be determined by the position of the singular points, two of which (± 1) are fixed, whilst the remaining ones (the poles) are functions of η, see below. Thus the radius will in general be a function of η; when $\eta\to 0$, the convergence radius does the same, so that the series (27) becomes useless in a very important case. Another method, which may help to avoid this disadvantage, consists in using essentially the same type of development, but without taking into account at first the inhomogeneous equation:

$$(1-g_0)\phi_0-k\phi_1=1. \quad (24b)$$

The remaining system is homogeneous, and determines only the ratios between the ϕ's. One may then develop, in powers of k, the ratio ϕ_n/ϕ_N where N is some fixed value. Two values of N are especially interesting, namely $N=0$, and $N=\infty$. In the first case one finds for instance:

$$\phi_1/\phi_0=(1/3)k[1-g_1]^{-1} + (4/45)k^3[1-g_1]^{-2}[1-g_2]^{-1}+\cdots \quad (28)$$

and inserting this into Eq. (24b) one finds ϕ_0 as the reciprocal of a power series in k. The case $N=\infty$ is not so simple because actually $\phi_\infty=0$; it may be proved, however, that, provided the g's tend to zero sufficiently quickly when N tends to infinity, one has:

$$\phi_N\sim CN^{-\frac{1}{2}}\epsilon^N, \quad \text{for large } N\text{'s}; \quad (29)$$

what one can do then, is to develop the ratio ϕ_n/C in powers of k. Finally, taking into account (24b), one finds ϕ_0 as the quotient of two power series. Although this development is somewhat complicated, we think that it is interesting, because the convergence of the power series is only limited by the *fixed* singularities at $k=\pm 1$, as a more detailed consideration shows.

We wish to mention here also another possible method, which consists in developing the solution in powers of ϵ (instead of k). When k is small, one has from Eq. (19):

$$\epsilon=k/2+k^3/8+\cdots \quad (30)$$

so that the two variables k and ϵ are not essentially different; but for large values of k the situation changes considerably. In fact, in the plane of the complex variable ϵ, the interior of the circle:

$$|\epsilon|=1 \quad (31)$$

corresponds to the whole plane of the k-variable, cut along the strips (26); in particular the whole imaginary axis for k is represented on the segment from $\epsilon=-i$ to $\epsilon=+i$. Therefore, *as far as the convergence of the power series under examination is limited by the fixed singularities at:* $k=\pm 1$ (or: $\epsilon=\pm 1$), the power series in ϵ is *much more powerful* than that in k. In fact, the fixed singularities prevent us from reaching any imaginary values of k beyond $k=\pm i$, whilst there is no limitation whatever in the ϵ-development. These advantages will be fully exhibited in the next section.

Let us now turn our attention to the isolated poles of the solution. These arise in the following way. We have seen already that for each value of η there are in general k-values, for which the complete homogeneous system (15) (*including* the equation for $n=0$) admits a solution satisfying Eq. (16). From the preceding discussion it is clear that the inhomogeneous system (24a) does not then have any solution satisfying condition (16). In other words the eigenvalues k give the poles of the solution of the inhomogeneous case. It is possible to investigate the behavior of the function ϕ_0, Eq. (25), when k approaches one of these poles, say $\bar{k}$. In fact, call $\bar{\phi}_n$ the solution of the complete homogeneous system corresponding to $\bar{k}$, and write Eq. (15) for this $\bar{\phi}_n$. Multiplying this equation by ϕ_n and Eq. (24a) by $\bar{\phi}_n$ and subtracting one from the other, and summing with respect to n, one finds:

$$(k-\bar{k})\sum_{n=1}^{\infty} n(\phi_{n-1}\bar{\phi}_n+\phi_n\bar{\phi}_{n-1})=\bar{\phi}_0. \quad (32)$$

Now we must have:

$$\lim_{k=\bar{k}}(\bar{k}-k)\phi_n=N\bar{\phi}_n, \quad (33)$$

where N is a normalization factor. Equation (32) then yields:

$$2N\sum_{n=1}^{\infty} n\bar{\phi}_n\bar{\phi}_{n-1}=\bar{\phi}_0. \quad (34)$$

Equations (33) and (34) describe the behavior of the solution in the neighborhood of $\bar{k}$.

4.2 Point Source

A point source $s(\mathbf{r})=\delta(\mathbf{r})$ and the corresponding solution of Eq. (2′) can be built up from the wave source (23) and the corresponding solution, from the Fourier expansion

$$\delta(\mathbf{r})=(1/2\pi i)^3\int e^{-\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}, \tag{35}$$

where the integrations with respect to the three components of $\mathbf{k}$ run from $-i\infty$ to $+i\infty$. Since we are now considering wave-sources with an arbitrary direction of propagation, it is clear that Eqs. (23a) and (24) still hold, provided we understand by μ the cosine of the angle between the velocity vector (or $\boldsymbol{\omega}$) and the propagation vector $\mathbf{k}$. We then have the solution of Eq. (2′) in the form:

$$\Phi=(1/2\pi i)^3\int e^{-\mathbf{k}\cdot\mathbf{r}}\phi(\boldsymbol{\omega}\cdot\mathbf{k}/k)d\mathbf{k}\quad(k=|\mathbf{k}|). \tag{36}$$

We are mainly interested in the density of slow neutrons of a given velocity, irrespective of the direction of the velocity, which is:

$$\int\Phi d\omega=(1/2\pi i)^3\int e^{-\mathbf{k}\cdot\mathbf{r}}\phi_0(\eta,k)d\mathbf{k}, \tag{37}$$

where ϕ_0 is of course the function given by Eq. (25), but its dependence on both η and k is now explicitly indicated for the sake of clarity. Performing the integrations with respect to the direction of $\mathbf{k}$, one naturally finds that the density (37) only depends on the distance $r=|\mathbf{r}|$ from the point source, and not on the direction of $\mathbf{r}$:

$$\cdots=-(1/2\pi r)(1/2\pi i)\int_0^{+i\infty}(e^{kr}-e^{-kr})\phi_0(\eta,k)kdk,$$

which can be further simplified to:

$$\int\Phi d\omega=-(1/2\pi r)(\partial/\partial r)(1/2\pi i)\times\int_{-i\infty}^{+i\infty}e^{-kr}\phi_0(\eta,k)dk. \tag{38}$$

In order to obtain the density of neutrons of a given energy we must still invert the Laplace-Mellin transform by means of Eq. (22a) so that finally the quantity we are interested in is given by:

$$\int\psi d\omega=-(1/2\pi r)(\partial/\partial r)X(r,u)\text{ with:}$$

$$X(r,u)=(1/2\pi i)^2\int_{\beta-i\infty}^{\beta+i\infty}e^{\eta u}d\eta\int e^{-kr}\phi_0(\eta,k)dk. \tag{39}$$

It is not possible to evaluate these integrals in closed form. It is possible, however, to find approximate expressions for them in the limiting case when u is very large. Strictly speaking the expression obtainable is an asymptotic evaluation for $u\to\infty$ (and r kept fixed or increasing only like the square root of u).

Since we understand that this case has been studied very thoroughly, we shall be content with a few remarks. If u is large it is convenient in Eq. (39) to perform the integration with respect to η first. The integral:

$$\int e^{\eta u}\phi_0(\eta,k)d\eta \tag{40}$$

may be evaluated approximately by pulling the integration path to the left in the η-plane until one meets the first singularity. One may show that this is given by Eq. (21). The integral is thus given by the residue at this singularity, plus the contributions from the remaining singularities. One may show aso that the real part of the second singularity is smaller than that of the first by about $\frac{1}{2}$ (in the hydrogen case) for small k's, so that the contribution deriving from it is of the order of $e^{-u/2}$ with respect to the first residue. The computation of this first residue is quite easy if k is small, and it may be shown later that only small values of k contribute appreciably to the final result in Eq. (39). Namely, if k is small Eq. (21) converges rapidly, so that the solution of the homogeneous system (indicated later with $\bar{\phi}$) is known, and it is possible to express the normalization factor N in Eq. (34) by means of a rapidly convergent series in k. The residue is easily expressed in terms of N and of the derivative $d\eta/dk$, η being the first pole for a given k (comp. Eqs. (33) and (21)). If the residue is indicated by

$R(k)$, being a power series in k, Eq. (39) takes the form:

$$X(r, u) = (1/2\pi i)\int_{-i\infty}^{+i\infty} e^{-kr+u\eta(k)}R(k)dk. \quad (41)$$

Now since k is imaginary, η is negative, and the factor $e^{u\eta(k)}$ decreases very rapidly (large u!) as k increases. One may therefore retain only the first term of the second Eq. (21), or if a better approximation is wanted use an expansion:

$$e^{u\eta(k)} = \exp(u\eta_1 k^2)(1+u\eta_2 k^4+\cdots)$$

so that finally (41) reduces to a Gauss integral, plus terms of higher order, also derivable from the Gauss integral.

5. NON-HOMOGENEOUS EQUATION. VARIABLE MEAN FREE PATH

Quite often, for instance, if the slowing-down medium contains hydrogen, the approximation of constant mean free path is very far from being satisfactory. We want to show now that with suitable assumptions on the velocity dependence of the mean free path, one can still develop an accurate method of solution. The experimental results on this dependence can often be represented quite well by a law:

$$l = a+bv^c \quad (42)$$

with suitable constants a, b, c. A law of this kind may be studied with the methods now to be described. But the problem certainly becomes very involved. If, however, the constant a is neglected, one has a problem that, although more complicated than the case of constant mean free path under some respects, nevertheless presents some features which make it easily amenable to numerical treatment.

Instead of developing the method in general we prefer to examine this case thoroughly, that is, also numerically, for one special example, namely a mixture of hydrogen and of infinitely heavy nuclei, which may be regarded as a sufficiently good model for water or paraffin. We neglect capture, and assume for simplicity that the ratio of the scattering mean free paths l_1 for hydrogen, and l_2 for the heavy nuclei is independent of velocity, so that Eq. (5) is valid. More specifically:

$$p(\omega\cdot\omega', u-u') = (l/2\pi l_1)e^{u'-u}\delta(\omega\cdot\omega' - e^{\frac{1}{2}(u'-u)}) + (l/4\pi l_2)\delta(u-u'),$$

so that, setting $\omega\cdot\omega' = \mu$, one easily finds:

$$g(\mu, \eta) = (l/\pi l_1)\mu^{\eta+\frac{1}{2}} + (l/4\pi l_2) \quad \text{if} \quad \mu>0, \quad (43)$$
$$= l/4\pi l_2 \quad \text{if} \quad \mu<0.$$

Multiplying both sides of Eq. (11a) by $P_n(\mu)$ and integrating with respect to μ from -1 to $+1$, one then finds:

$$g_n = (2l/l_1)\int_0^1 \mu^{2\eta+1}P_n(\mu)d\mu + (l/l_2)\delta_{n0}, \quad (43a)$$

e.g.,

$$g_0 = (l/l_1)(\eta+1)^{-1} + l/l_2$$
$$g_1 = (l/l_1)(\eta+\tfrac{3}{2})^{-1} \quad (43b)$$
$$g_2 = (l/l_1)[\tfrac{3}{2}(\eta+2)^{-1} - \tfrac{1}{2}(\eta+1)^{-1}]; \text{ etc.}$$

Making use of the equation:

$$(n+1)P_{n+1} - \mu P'_{n+1} = -nP_{n-1} - \mu P'_{n-1}$$

one can also easily prove the recurrence relation:

$$g_{n+1} = [(2\eta+2-n)/(2\eta+3+n)]g_{n-1}, \quad \text{for} \quad n>1, \quad (43c)$$

which allows one to compute quickly all g's.

We have found that the law:

$$l = (v/v_0)l(v_0) \quad (42a)$$

represents the data for water or paraffin with very good accuracy down to energies of the order of 100 kev. In other words in Eq. (42) we shall set $a=0$, and $c=1$. But the development would be almost exactly the same if c were $\neq 1$. Of course the assumption that the mean free path tends to zero at zero velocity means that we will grossly underestimate the diffusion of the neutrons in the last stages of slowing down. On the other hand the region between say 6 Mev and 100 kv is already large enough, and the corresponding variation of the mean free path is so strong that it is interesting to investigate what happens in this phase of the slowing down.

We take now $l(v_0)$ as a unit of length. Insert-

ing (42a) into Eq. (2) that is writing:

$$l(u)=e^{-u/2},$$

assuming again that $S(\mathbf{r}, u)=S(\mathbf{r})\delta(u)$, and using the transformation (40) one finds instead of Eq. (2′):

$$\boldsymbol{\omega}\cdot\mathrm{grad}\Phi(\eta+\tfrac{1}{2})+\Phi(\eta)$$
$$-\int g(\boldsymbol{\omega}\cdot\boldsymbol{\omega}', \eta)d\boldsymbol{\omega}'\Phi(\mathbf{r}, \eta, \boldsymbol{\omega}')=s(\mathbf{r}). \quad (44)$$

Again we consider a wave source, Eq. (23); going again through the same steps, Eqs. (23a), (24), and (24a), we find instead of Eq. (24a) the following

$$(2n+1)[1-g_n(\eta)]\phi_n(\eta)-k\{n\phi_{n-1}(\eta+\tfrac{1}{2})$$
$$+(n+1)\phi_{n+1}(\eta+\tfrac{1}{2})\}=\delta_{n0}. \quad (45)$$

Mathematically this is a much more difficult system of equations than (24a); η, instead of appearing simply as a parameter, is a variable within the equation itself. In fact, (45) is a difference equation in *two* independent variables, n and η; very little is known on the mathematical properties of these equations. One may guess, however, what they are, on the simplifying assumption that one may neglect all ϕ_n's with an n larger than some large value N.

This is an approximate way of taking into account condition (16). One has then to do with a finite system of difference equations, and one can apply known theorems.[2] It appears then that in order to define the solution completely, it is necessary to take into account the behavior of $\phi(\eta)$ when η tends to infinity in the complex plane in a direction forming an acute angle with the positive real axis. This behavior can be easily inferred from Eq. (22), because when $\eta\to\infty$ as stated the contribution to the integral arises entirely from the neutrons of highest energy, i.e., the primary neutrons. One then finds the condition:

$$\lim_{\eta\to\infty}\phi_n(\eta)=Q_n(1/k)$$
$$=\tfrac{1}{2}\int_{-1}^{+1}P_n(z)dz/(1-kz). \quad (46)$$

This condition selects among the infinity of possible solutions the so-called "fundamental solution" which is, however, generally defined by means of formulae so involved that they are of little use for computation.

There is, however, the fortunate circumstance that if we try to develop ϕ in powers of k, we find that the development is perfectly definite, and the coefficients of the development satisfy the condition (46) for the fundamental solution. There is little doubt, therefore, although no rigorous proof is available, that for small values of k one gets in this way the correct solution. Actually, instead of the k-development, we shall use, as indicated in Section 4, the more powerful development in powers of ϵ, which allows us to extend the solution analytically to a much wider field, and in fact, can be used for numerical computation up to sufficiently large imaginary values of k, as will be presently shown.

Let us, however, first discuss an obvious, but important, property of the system (45). We know that for large values of $\Re(\eta)$ the solution is a regular analytic function of η, this being a general property of Laplace transforms. Suppose for a moment the solution is known in a strip: $\eta_0\leq\Re(\eta)\leq\eta_0+\frac{1}{2}$, where η_0 is some large positive number; then by means of Eq. (45) we can move in steps $\Delta\eta=-\frac{1}{2}$ towards the left in the complex η-plane, extending the solution to smaller and smaller values of $\Re(\eta)$. No singularity can occur in this process, until some of the factors $(1-g)$ become zero, and in that event we shall find a pole.

In the special case, Eq. (43), the first pole one meets is the root $\eta=0$ of the equation $1-g_0=0$, and this must be true also in many other cases. The next pole is the root $\eta=-\frac{3}{2}+l/l_1$ of the equation $1-g_1=0$, and so on. For higher n-values one finds also complex roots. The essential point in all this is that the poles are entirely determined by the coefficients g, and *do not at all depend on* k. One can easily convince oneself that this behavior which is so different from what we found in the case of constant mean free path, is entirely caused by the fact that here the mean free path is assumed to tend to zero for zero velocity. The general law (42) would not lead to such a simple behavior, and therein lies the great simplification implied in the law (42a). One can say indeed that for purposes of numerical computation the present case is simpler than the case

TABLE I. Computation of $f(k)$.

$ik=0$	0.12	0.18	0.289	0.471	0.615	0.703	0.912	1.12	1.33_5	1.56_5	1.96	2.10
$f(k)=1$	0.98	0.96	0.894	0.756	0.64	0.57	0.43	0.32	0.24	0.18	0.1	0.06

TABLE II.

$r=0$	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.5	3.26	3.89	4.76	5.18
$I(r)=2.30$	2.07	1.55	1.05	0.68	0.44	0.29	0.18	0.12	0.05	0.025	0.0084	0.0024	0.0013

of constant mean free path, notwithstanding the higher complexity of Eq. (45).

If one is interested in the distribution of neutrons having velocities much smaller than the initial one, it will be enough to know ϕ_0 in the neighborhood of the pole $\eta=0$; this will enable one to compute the residue of the integral (22a) at this pole, while the contribution from poles lying more to the left will be negligible. Now Eq. (45) for $n=0$, shows that, when $\eta\approx 0$:

$$\phi_0(\eta)\approx[1-g_0(\eta)]^{-1}\{1+k\phi_1(\tfrac{1}{2})\} \tag{47}$$

so that it is sufficient to have the development of ϕ_1 for one definite value $\eta=\frac{1}{2}$. This is the simplification which makes a numerical treatment easily possible. In fact, once the variable η is eliminated, it becomes a possible program to compute ϕ_0 numerically for closely spaced values of k, so that one can compute, again numerically, the Fourier integral of Eq. (38), for a sufficient number of values of the distance r. The method becomes inaccurate only at very large values of r, where, however, a different procedure can be adopted (see below).

Here is, very briefly, how the computation can be slightly simplified. First we put:

$$\begin{aligned} U_n&=\{\Gamma(\tfrac{1}{2})\Gamma(1+\tfrac{1}{2}n)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n)\}\phi_n \\ E_n(\eta)&=4\{\Gamma(1+\tfrac{1}{2}n)/\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n)\}^2 \\ &\quad\times(2n+1)^{-1}[1-g_n(\eta)]^{-1}, \end{aligned} \tag{48}$$

whereby Eq. (45) becomes:

$$\begin{aligned} &4U_n(\eta)-2kE_n(\eta)\{U_{n-1}(\eta+\tfrac{1}{2}) \\ &\qquad+U_{n+1}(\eta+\tfrac{1}{2})\}=\pi E_0(\eta)\delta_{n0}. \end{aligned} \tag{49}$$

On account of Eq. (19) we then set $k=2\epsilon/(1+\epsilon^2)$ and multiply the whole equation by $(1+\epsilon^2)$. Developing U_n in a series of powers of ϵ which (comp. Eq. (31)) starts with ϵ^n:

$$U_n=\epsilon^n\{U_{n0}+U_{n1}\epsilon^2+U_{n2}\epsilon^4+\cdots\} \tag{50}$$

we find the recurrence relation:

$$\begin{aligned} &U_{nm}(\eta)+U_{n,m-1}(\eta)-E_n(\eta) \\ &\qquad\times\{U_{n-1,m}(\eta+\tfrac{1}{2})+U_{n+1,m-1}(\eta+\tfrac{1}{2})\} \\ &\qquad=(\pi/4)E_0(\eta)\delta_{n0}(\delta_{m0}+\delta_{m1}), \end{aligned} \tag{51}$$

which yields the various coefficients in the following order: U_{00}, U_{10}, U_{01} and U_{20}, U_{11} and U_{30}, $\cdots$. As indicated in Eq. (47), we need only the coefficients of the development for one value of η, or $\eta=\frac{1}{2}$, but in the course of the computations owing to the peculiar structure of Eq. (49) values $\eta=1$, $\eta=\frac{3}{2}$, $\eta=2$, $\cdots$ occur. In order to compute the residue of ϕ_0 at $\eta=0$ up to the tenth power of ϵ^2, we found it necessary for instance to make use of the values of $E_n(\eta)$ for the following values of the variables:

$$n=0,\ 1,\ 2,\ \cdots,\ 10 \quad\text{and}\quad \eta=m+\tfrac{1}{2}n,$$
$$\text{with}\quad m=0,\ 1,\ \cdots,\ 10-n.$$

In order to determine the value of the ratio l_1/l_2 to be used in the computation it was necessary to make some compromise between the experimental values at high and at low energies. The value $l_1/l_2=0.27$ was finally adopted. With this value the first ten coefficients in the development of ϕ_0 were computed, finding:

$$\begin{aligned} \phi_0(\eta)=U_0(\eta)\approx(l_1/l)(1/\eta)\{&1+5.586\epsilon^2 \\ &+15.609\epsilon^4+31.143\epsilon^6+52.240\epsilon^8 \\ &+79.420\epsilon^{10}+113.259\epsilon^{12}+154.40\epsilon^{14} \\ &+203.52\epsilon^{16}+261.37\epsilon^{18} \\ &+330.35\epsilon^{20}+\cdots\} \end{aligned} \tag{52}$$

for $\eta\approx 0$. The bracketed series is a function of ϵ

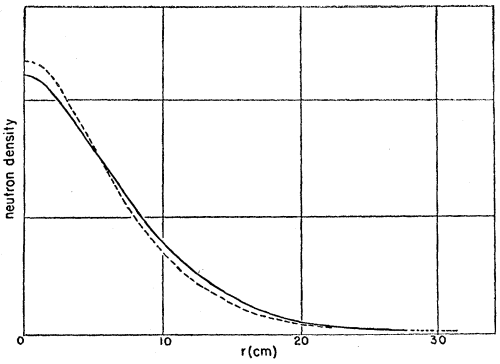

FIG. 2. Dependence of neutron density on the distance for a D+D source. Dotted line theoretical, full line experimental.

and hence of k, which we shall indicate by $f(k)$. The values in Table I were computed by means of Eq. (52). The table indicates that it is quite feasible to compute the function $f(k)$ by means of Eq. (52) up to a point where it is already fairly small. A diagram of the function resembles roughly a gaussian curve, but the tail falls off more slowly than a gaussian, and rather more like an exponential. This also permits one to extrapolate the values of $f(k)$ to values of $ik>2.1$.

Inserting (52) into the integrals (22a), (39), one finds that for large u, or for low velocities, ψ becomes velocity independent. This can be easily understood if one reflects that with a law $l\sim v$, the diffusion of neutrons comes practically to a stop for sufficiently low velocities so that the space distribution tends to a limit for $v\rightarrow 0$. The local energy spectrum must then finally coincide with the over-all spectrum which is given by the well-known Fermi law: $dN=l(v)dv/v^2$ which means $\psi=$const.

The space distribution is given by the formula:

$$\int \psi d\omega \sim r^{-1}\int_0^{i\infty} kf(k)\sin(ikr)dk. \qquad (53)$$

By means of Table I, this function has been evaluated numerically for the first ten r-values in Table II. When r becomes very large the evaluation by this method becomes too uncertain; accordingly, the last four values listed were obtained by means of the saddle-point method to be described later. It has to be remembered that r in Table II is really the distance from the source measured in terms of the mean free path of the primary neutrons. Thus on assumption (42a), the limiting distribution in space is given by a universal function; if the initial energy is varied, one has merely to change the unit in which distances are measured. This is of course only approximately true, mainly owing to the fact that the mean free path does not really become zero at zero velocities. Actually formula (53) should hold for neutrons whose final energy is sufficiently small (so that the contribution from the neglected poles is really negligible) and at the same time sufficiently large for formula (42a) to be still valid. It would, however, be perfectly possible to compute the contribution from the next pole to the left of $\eta=0$.

5.1 Comparison With Experiment

If the function on the right-hand side of Eq. (53), which we may indicate by $I(r)$, were correct down to 1 ev it could be directly compared with measurements on Rh or Ag resonance neutrons. The most accurate measurements, however, are made on thermal neutrons, and in order to get the distribution for these neutrons it becomes necessary to apply diffusion theory. One then finds the distribution:

$$I_{th}(r)=(1-A'\Delta)^{-1}I(r)$$
$$=(1/r)\int_0^{\infty} yf(iy)\sin(yr)\times(1+A'y^2)^{-1}dy, \qquad (54)$$

where $A'=D\tau/l^2(v_0)$, D being the diffusion constant and τ the mean lifetime of thermal neutrons.

The integral (54) can be computed just as easily as (53). It is moreover possible in this computation to include also a correction for the defects of formula (42a) below 100 kev. Indeed if we start with neutrons of say 6 Mev, it is clear that by the time their energy is reduced by a factor 60, their distribution in space has flattened and their angular distribution has become isotropic to such an extent that age theory is henceforward quite a good approximation. The distribution of resonance neutrons should then be given by:

$$I_{res}(r)=e^{A\Delta}I(r)=(1/r)\int_0^{\infty} yf(iy)\sin(yr)\times\exp(-Ay^2)dy, \qquad (55)$$

where A is the "age" of 1-ev neutrons starting from 100 kev, or preferably the difference between the ages as computed from Eq. (42a) and the real experimental dependence of the mean free path on velocity. According to Bethe[5] the age in water is given by:

$$\text{Age}=0.8\int^{u} l^2(u)du, \tag{56}$$

and in this way an age difference was computed: $A=7$ cm^2 when formula (42a) was adjusted to fit the experimental value at an energy of 3.5 Mev. Applying corrections (54) and (55) contemporarily one then should take:

$$I_{th}=(1-A'\Delta)^{-1}e^{A\Delta}I(r) \tag{57}$$

expressing of course A in units $l^2(v_0)$.

A comparison with experiment was made in two cases. First we took the distribution of thermal neutrons from the reaction D+D, as measured by Amaldi, Hafstad, and Tuve[6] with incident deuterons of 0.74 Mev. This source is not quite isotropic, but we assumed for the energy of the primary neutrons the value which may be computed for the neutrons emitted in the direction of observation (at 40° from the direction of the deuteron beam), namely 3.5 Mev. In this case we assumed $l(v_0)=6.8/1.27=5.35$ cm. In Fig. 2 the theoretical curve (dotted) was normalized to coincide with the experimental (full) curve at 5 cm. It should be realized that no adjustment in the unit of length was effected in order to obtain the fairly good agreement of the results. The values for large r were computed with the same saddle-point method, that was used for the computation of $I(r)$.

Another comparison was made with measurements on thermal neutrons from the D+B reaction,[7] with 0.85-Mev deuterons. In this case we did not expect a good agreement, because we assumed the spectrum of the primary neutrons to consist of four lines of energy 14, 9.77, 6.85, and 4.69 Mev respectively, while there are indications that also considerably slower neutrons are emitted. We did not take these into account because we had no reliable data. The relative intensities of the lines were assumed to be in the ratios: 1:2:1:3. The distributions corresponding to the four lines were computed separately with the following data:

$$\begin{aligned} l(v_0)&=17.7 \quad 12.1 \quad 8.7 \quad 6.45 \text{ cm} \\ 1/(A'+A)&=15.2 \quad 11.2 \quad 7.1 \quad 4.3 \end{aligned}$$

(the value of A had to be revised each time so as to take into account the deviation of the experimental law from formula (42a). The computation was greatly simplified by assuming that one could replace

$$(1-A'\Delta)^{-1}e^{A\Delta} \quad \text{with} \quad (1-[A'+A]\Delta)^{-1},$$

and moreover that one could interpolate linearly with respect to $(A'+A)$. This should not be very inaccurate since A' and A are small. The result in Fig. 3 shows a deviation in the direction to be expected if the spectrum contains slower neutrons in addition to the four lines taken into account.

5.2 Evaluation of the Integrals (53), (54) for Large r

Let us write for instance:

$$I(r)=(1/2ir)\int_{-i\infty}^{i\infty} kf(k)e^{kr}dk, \tag{58}$$

and deform the integration path in the complex

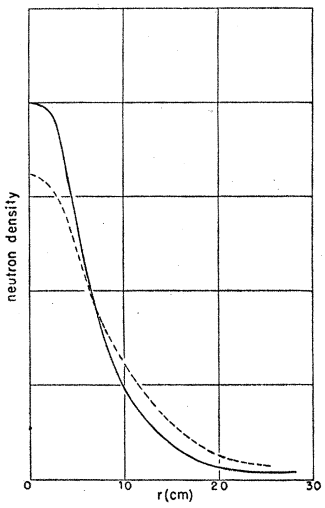

FIG. 3. Neutron density for a D+B source. Full line experimental. Dotted theoretical, not including slow part of primary spectrum.

[5] H. A. Bethe, Rev. Mod. Phys. 9, 69 (1937), esp. p. 129.
[6] E. Amaldi, L. R. Hafstad, and M. A. Tuve, Phys. Rev. 51, 896 (1937).
[7] Amaldi, unpublished.

k-plane to the left, so that it passes through a real point k_0 such that $-1<k_0<0$. If k_0 moves towards the point $k=-1$, $f(k_0)$ tends rapidly to infinity while the exponential factor decreases rapidly (if r is large). As a consequence there is a saddle-point, and we let k_0 coincide with this saddle-point. Moreover we put:

$$f(k)=e^{g(k)} \tag{59}$$

and develop $g(k)$ in powers of $k-k_0$ in the usual way. The known coefficients in Eq. (52) are in sufficient number to allow the computation of $g(k)$ from $k=0$ to about $k=-0.85$ or $\epsilon^2=0.3$. Now it is quite certain, on the basis of formula (61) below, that the conditions for a rigorous application of the saddle-point method are not satisfied even for a very large r. The situation is somewhat similar to that encountered, if one were trying to compute the integral:

$$(1/2\pi i)\int_{\beta-i\infty}^{\beta+i\infty} e^{kr}k^{-\rho}dk=r^{\rho-1}/\Gamma(\epsilon) \tag{60}$$

by a saddle-point method. One would find the value:

$$r^{\rho-1}/\{\rho^{\rho-1}e^{-\rho}(2\pi\rho)^{\frac{1}{2}}\},$$

which according to Stirling's formula coincides with (60) only if ρ is large. In our case we may assume $\rho=1.77$, and therefore the condition is not satisfied. However, the error in Stirling's formula is only about ten percent: we have applied, therefore, the saddle-point formula as an empirical method to extrapolate the results to larger distances. In this way the four last values were computed in Table II, and the same was done in the computations underlying the diagrams 2 and 3.

We may also venture a hypothesis about the asymptotic behavior of such integrals for large r. We could of course derive this behavior rigorously if we knew what type of singularity $f(k)$ possesses at the point $k=-1$, but we have no method for knowing this. We may observe, however, that the coefficients in the development (52) are remarkably close to those of a function:

$$f=C(1-\epsilon^2)^{-2\rho}$$

with $C=2.15$ and $2\rho=3.55$. The first coefficients of course do not coincide, but the agreement becomes quite good after the fourth coefficient. This might indicate that the singularity is well represented by a formula of the above type, which when translated in terms of the variable k would give:

$$f(k)\sim 0.054(1+k)^{-\rho}. \tag{61}$$

Inserting this into formula (53) and remembering Eq. (60) one finds easily (one should really use a Tauberian theorem):

$$I(r)\sim 0.18r^{-0.22}e^{-r},$$

which should hold, however, only for values considerably larger than any in the table. It is unnecessary to stress that this is merely a guess.

Our thanks are due the Nuclear Physics Center of the C.N.R. for support given to this work.