sional Schrödinger equation, $\Delta\psi+(\lambda-V)\psi=0$. For simplicity we assume that V is a finite function. Let u be any function with bounded second derivatives satisfying $u\geq\epsilon$ for a positive constant, ϵ. The continuous function $r=\psi/u$ vanishes on the boundary and may be assumed to have a positive maximum in R (otherwise take $-\psi$). At a maximum point the first derivatives of r vanish, and the second derivatives are negative or zero. Adding the second derivatives gives $\Delta r=(u\Delta\psi-\psi\Delta u)/u^2\leq 0$. Thus $\Delta u/u\geq\Delta\psi/\psi=V-\lambda$ so that $\lambda\geq V-\Delta u/u$. *A fortiori*, $\lambda\geq\min.(V-\Delta u/u)$ for all points of the region. This gives the theorem: *The eigenvalues of Schrödinger's equation satisfy the inequality* $\lambda\geq\min.(V-\Delta u/u)$ *where u is a positive function in the region.*

The rest of this note is mainly of mathematical interest and proves that on the boundary, S, it is only necessary to assume $u\geq 0$. In case the region is finite, we suppose that S is sufficiently regular so that there is a function, g, which satisfies $\Delta g+(h-V)g<0$ on S and $g\geq 1$ in R, where $h=\min.(V-\Delta u/u)$. If $w=u+\epsilon g$, then $\lambda\geq\min.(V-\Delta w/w)$ by what has just been shown. Clearly, $V-(\Delta u+\epsilon\Delta g)/(u+\epsilon g)\geq V+(hu-Vu-\epsilon\Delta g)/(u+\epsilon g)=h+\epsilon(Vg-hg-\Delta g)/(u+\epsilon g)$. In a neighborhood of the boundary the last term is positive, and in the remainder of the region it must vanish with ϵ, so that $\lambda\geq h$, Q. E. D.

In an infinite region we take $g=1$ in the above proof, that is, $w=u+\epsilon$. Assume, moreover, that outside a sufficiently large sphere $h-V<0$. Thus $V-\Delta w/w\geq h+\epsilon(V-h)/(u+\epsilon g)$. Outside the sphere the last term is positive, and inside the sphere it vanishes with ϵ, so that $\lambda\geq h$.

The problem discussed here could be generalized for an equation in which λ is multiplied by a positive function. On the other hand, the writer has been unable to treat by similar methods the eigenvalues of the acoustic equation, the equation of the clamped plate, or Maxwell's equations.

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On a Connection between the Fountain Effect, Second Sound, and Thermal Conductivity in Liquid Helium II

LOTHAR MEYER
Institute for the Study of Metals, University of Chicago, Chicago, Illinois
AND
WILLIAM BAND
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois
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WE have investigated the possible phenomenological relations between the observed properties of liquid helium II. We assume that in the absence of a constriction the internal forces responsible for the fountain effect are still present but produce an internal momentum density $\mathbf{M}$ such that

$$\mu\nabla T=\nabla p=-d\mathbf{M}/dt, \tag{1}$$

where T is the temperature and p is the fountain effect pressure difference. We then apply the first law of thermodynamics to a heat transported with the momentum

$$\nabla\cdot\mathbf{h}=\nabla\cdot(\epsilon\mathbf{M}/m)=-\rho C_v dT/dt. \tag{2}$$

We solve (1) and (2) simultaneously to give a wave equation for momentum and temperature similar to second sound. The phase velocity is

$$V_s=(\epsilon\mu/m\rho C_v)^{\frac{1}{2}}, \tag{3}$$

where μ is the fountain effect coefficient,[1,2] ρC_v is the specific heat[3] per cm^3, and ϵ is the heat associated with a mass m transported with the momentum. Empirical values of μ, V_s (second sound velocity),[4,5] and ρC_v below 1.8°K yield values of ϵ/m which fit a monotonically increasing curve from zero at 0°K to the heat content at the lambda point (T_λ).

From the equation

$$d(\epsilon\rho_x/m)/dT=\rho C_v \tag{4}$$

we calculated the concentration ρ_x/ρ of the liquid taking part in the momentum. The curve connecting these values is approximately equal to (T/T_λ),[6] going therefore through zero at 0°K and unity at T_λ, which it should do from general considerations.

The observed finite heat conductivity in liquid helium requires a wave equation having a relaxation (damping) term added to (1), namely,

$$\nabla\cdot\nabla T=\left\{\frac{d^2T}{dt^2}+\frac{1}{\tau}\frac{dT}{dt}\right\}\Big/V_s^2. \tag{5}$$

Combined with the data on thermal conductivity[3] this gives a time of relaxation τ of the order 10^{-3} to 10^{-4} sec., going to zero at T_λ. Equations (5) and (1) with the damping term lead to an alternative form of the heat current as a convection process:

$$\mathbf{h}=\rho Q\mathbf{u}_x, \tag{6}$$

where ρQ is the heat content per cm^3 of liquid helium, and $\mathbf{u}_x$ is the velocity associated with the steady momentum density as limited by the relaxation term. The data on thermal flow give $\mathbf{u}_x$ as a function of T and dT/dx, leading to values practically identical with those of the super critical velocities through slits.[2]

The relaxation causes an absorption and dispersion of second sound near the lambda point, formally analogous to the transmission of electromagnetic waves in a conducting medium. This may be responsible for the rapid approach to zero of the observed phase velocity as T approaches T_λ, and leads to the expectation that under ideal conditions (small amplitude and high frequency) the velocity of second sound should rise up to the immediate neighborhood of T_λ just as the maximum in the curves of the heat conductivity and the fountain effect shift towards the lambda point under appropriate conditions.[2]

We are studying the physical meaning of these relations.

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