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Equivalence of the Riesz Method and the λ-Limiting Process for the Classical Electromagnetic Field of a Point Source

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It is shown generally that the Riesz method of analytic continuation and the λ-limiting process give, in general, the same results for both the classical electromagnetic potential of a point source and all its derivatives with respect to the coordinates of the field point.

I. INTRODUCTION

It is well-known that the λ-limiting process developed by Wentzel, Dirac, and Pauli[1] leads to divergence-free results for classical wave fields of a point source. In the case of the electromagnetic field, the λ-limiting process gives, for a field point not on the world line of the particle, the Lienard-Wiechert retarded field just as in the ordinary theory. On the other hand, for a point on the world line, where the ordinary mathematical treatment gives divergent results, the λ-limiting process gives one-half of the radiation field, i.e., half the difference of the retarded and advanced fields, which is finite on the world line. Dirac[2] studied in detail the field strengths of the radiation field of a point charge. Generalized investigations for the fields of point sources possessing higher multipole moments were later made by Bhabha and Harish-Chandra for both the electromagnetic and meson fields.[3]

An alternative mathematical treatment of the problem was introduced by M. Riesz.[4] The application of Riesz' method has recently been developed by Gustafson[5] for the quantum theory and by Fremberg[6] for the classical theory. Fremberg has shown that, for the wave fields of a point charge, the Riesz method gives the same results as the λ-limiting process for the following field quantities: (i) the retarded potentials and field strengths for a point not on the world line; (ii) half the radiation field strengths for a point on the world line.

Unlike the retarded field quantities given by the formula of Lienard and Wiechert, the values on the world line of the derivatives of the radiation field potentials, with respect to the coordinates of the field point, cannot be derived from the corresponding values of the potentials by simple differentiation. The agreement of the two methods found by Fremberg therefore raises the question of the equivalence of the two methods in general. It is the main purpose of the present investigation to give a general proof of the equivalence of the two methods for the electromagnetic field generated by a point source. The case of the meson field can be studied in a similar manner.

It follows from the general equivalence of the two methods that the Riesz method provides us with an additional method of calculating the radiation fields. This is of practical importance for the sake of computation, as the Riesz method is usually simpler and more straightforward for evaluating the higher derivatives of the radiation fields, which appear in the equation of motion of a particle having a multipole moment.

Before giving the proof, we shall first derive in Section II a new expression for the Riesz field, which will help bring out the connection between the two methods. Explicit results for the potential and its first two derivatives on the world line will be given in Section III, which will be seen to agree with those of the λ-process. A general proof of the equivalence of the two methods will be given in Section IV.

II. THE RIESZ FIELD

We begin by considering the electromagnetic field generated by a source distribution **J** which

[1] G. Wentzel, Zeits. f. Physik **86**, 479 and 635 (1933); **87**, 726 (1934); P. A. M. Dirac, Ann. de l'Inst. Poincaré **9**, 13 (1939); W. Pauli, Phys. Rev. **64**, 332 (1943).
[2] P. A. M. Dirac, Proc. Roy. Soc. **167**, 148 (1938).
[3] H. J. Bhabha and Harish-Chandra, Proc. Roy. Soc. **A185**, 250 (1946); Harish-Chandra, Proc. Roy. Soc. **A185**, 269 (1946).
[4] M. Riesz, Conference de la Réunion internat. des math. tenue à Paris en Juillet 1937 (Paris, 1939).
[5] T. Gustafson, Kgl. Fys. Sallsk. i Lund Forhandl. **15**, No. 28 (1945); 16, No. 2 (1946).
[6] N. E. Fremberg, Meddelanden fran Lunds Univ. Mat. Seminarium **7** (1946); Proc. Roy. Soc. **188A**, 18 (1946).

may be a scalar, vector, or tensor of higher rank, and the field potential **A** determined by the wave equation

$$\Box \mathbf{A} = 4\pi \mathbf{J}. \tag{1}$$

The Riesz potential is defined as

$$\mathbf{A}^{(\alpha)}(P) = \frac{1}{H_4(\alpha)} \int_{D_S{}^P} \mathbf{A}(Q) R^{\alpha-4} dQ, \tag{2}$$

where $H_4(\alpha)$ denotes the factor

$$H_4(\alpha) = \pi 2^{\alpha-1} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{\alpha}{2}-1\right), \tag{3}$$

$D_S{}^P$ is the domain of integration bounded by the retrograde cone with its vertex at the field point P and a space-like surface S, and R is the hyperbolic distance between the points P and Q. In the case of a point source moving along a world line $\mathbf{Z}(\tau)$, the source density is of the form

$$\mathbf{J}(\mathbf{X}) = \int \mathbf{S}(\tau) \delta(x_0 - z_0) \delta(x_1 - z_1) \times \delta(x_2 - z_2) \delta(x_3 - z_3) d\tau, \tag{4}$$

where τ is the proper time of the particle. In order to simplify our considerations we assume that there is a single source which begins to radiate at an instant τ_A and we take **S** to be a surface which cuts the world line at the point $\mathbf{Z}(\tau_A)$. It will also be assumed that there is no incoming field present. Under such conditions, the Riesz potential becomes

$$\mathbf{A}^{(\alpha)}(\mathbf{X}) = \frac{4\pi}{H_4(\alpha+2)} \int_{\tau_A}^{\tau_r} \mathbf{S}(\tau) R^{\alpha-2} d\tau, \tag{5}$$

where τ_r is the retarded proper time and R is the length of the vector $\mathbf{X}-\mathbf{Z}$. We denote the time and space components of **R** by r_0 and **r** and denote the length of **r** by r, so that

$$R = (r_0{}^2 - r^2)^{\frac{1}{2}}. \tag{6}$$

Equation (5) holds for the region $x_0 - z_0(\tau_A) > 0$, $(\mathbf{X} - \mathbf{Z}(\tau_A))^2 > 0$, i.e., the region where the Lienard-Wiechert potential does not vanish identically. The derivatives of $\mathbf{A}^{(\alpha)}(\mathbf{X})$ with respect to the coordinates of the field point **X** are of the form

$$\mathbf{F}^{(\alpha)\mu\nu\cdots}(\mathbf{X}) = \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\nu} \cdots \mathbf{A}^{(\alpha)}(\mathbf{X}). \tag{7}$$

The field quantities are to be calculated first for a value of α large enough for the integrals to be convergent, and the final results are obtained by analytic continuation to $\alpha = 0$.

It will be convenient to base our later investigation on a somewhat different expression for the Riesz potential, namely,

$$\mathbf{A}^{(\alpha)}(\mathbf{X}) = \alpha \int_0^{R_A} \mathbf{A}^{\text{ret}}(\mathbf{X}, R) R^{\alpha-1} dR. \tag{8}$$

Here R plays the role of a variable of integration, R_A is the length of the vector $\mathbf{X} - \mathbf{Z}(\tau_A)$, and $\mathbf{A}^{\text{ret}}(\mathbf{X}, \mathbf{R})$ denotes the value of the potential

$$\mathbf{A}^{\text{ret}}(\mathbf{X}, \mathbf{R}) = \mathbf{S}/(\mathbf{V}, \mathbf{R}), \tag{9}$$

at the retarded time when the vector $\mathbf{X} - \mathbf{Z}(\tau)$ has the magnitude **R**, and $\mathbf{V} = (v_0, \mathbf{v})$ denotes $d\mathbf{Z}/d\tau$. The retarded potential given by Eq. (9) differs from the ordinary Lienard-Wiechert potential in that the variable R is in general not zero.

To derive Eq. (8) from Eq. (6), we note that (6) may be written in the form

$$\mathbf{A}^{(\alpha)}(\mathbf{X}) = \frac{4\pi}{H_4(\alpha+2)} \int_{-\infty}^{\infty} \mathbf{S}(\tau) d\tau \times \int_0^{\rho_A} \frac{\delta[r_0 - (r^2+\rho^2)^{\frac{1}{2}}]}{(r^2+\rho^2)^{\frac{1}{2}}} \rho^{\alpha-1} d\rho, \tag{10}$$

where ρ_A is the length of $\mathbf{X} - \mathbf{Z}(\tau_A)$. This may be verified with the help of the general formula which states that

$$\int_0^{\rho_A} f(\rho) \delta[g(\rho)] d\rho = \int_0^{\rho_A} \frac{f(\rho)}{(dg(\rho)/d\rho)} \delta[g(\rho)] dg(\rho) = \frac{f(\rho_1)}{(dg/d\rho)_{\rho=\rho_1}} \text{ or } 0 \tag{11}$$

according as $g(\rho)$ has a zero ρ_1 between 0 and ρ_A or not, f and g being two arbitrary functions with g subject to the condition $dg/d\rho > 0$. We carry out the integration with respect to τ in (10) by applying (11) with ρ replaced by τ and using the relation

$$\frac{d}{d\tau}[(r^2+\rho^2)^{\frac{1}{2}}-r_0]=\frac{v_0(r^2+\rho^2)^{\frac{1}{2}}-(\mathbf{v},\mathbf{r})}{(r^2+\rho^2)^{\frac{1}{2}}}$$

$$=\frac{(\mathbf{V},\mathbf{R})_{R=\rho}}{(r^2+\rho^2)^{\frac{1}{2}}}. \tag{12}$$

The result is

$$\mathbf{A}^{(\alpha)}(\mathbf{X})=\alpha\int_0^{\rho_A}\left[\frac{\mathbf{S}}{(\mathbf{V},\mathbf{R})}\right]_{R=\rho}\rho^{\alpha-1}d\rho,$$

which is just Eq. (8) with $\mathbf{A}^{\text{ret}}(\mathbf{X}, R)$ given by Eq. (9). The factor

$$\frac{4\pi}{H_4(\alpha+2)}=\frac{4\pi\alpha}{\pi 2^{\alpha+2}\left[\Gamma\left(\frac{\alpha}{2}+1\right)\right]^2} \tag{13}$$

has been replaced for brevity by α, as the ratio of these two factors becomes unity when $\alpha=0$ and so its omission does not make any difference in the final result.

It follows from Eq. (8) that the derivatives of $\mathbf{A}^{(\alpha)}(\mathbf{X})$ in Eq. (7) are given by

$$\mathbf{F}^{(\alpha)}(\mathbf{X})=\alpha\int_0^{R_A}\mathbf{F}^{\text{ret}}(\mathbf{X},R)R^{\alpha-1}dR, \tag{14}$$

where $\mathbf{F}^{\text{ret}}(\mathbf{X}, R)$ denotes the field quantity that can be derived from the formula of Lienard and Wiechert by differentiation with respect to the x_μ with R kept fixed. We have neglected in Eq. (15) terms arising from the variation of R_A with the x_μ. These terms are independent of the variable R and therefore vanish when α is put equal to zero.

The analytic continuation to $\alpha=0$ is governed by the following general theorem: For any function $f(x)$ which can be expanded into a Laurent series of the form

$$f(x)=\sum_{-m}^{\infty}a_n x^n \quad (m=0, 1, 2, \cdots), \tag{15}$$

we have

$$\alpha\int_0^X f(x)x^{\alpha-1}dx \to a_0 \tag{16}$$

when analytic continuation is performed to $\alpha=0$, irrespective of the magnitude of the upper limit X. The coefficient a_0 may be regarded as the finite part ("partie finie") of the function $f(x)$. It follows from this general theorem, and Eqs. (8) and (14), that *the Riesz method consists in replacing the Lienard-Wiechert potential and its derivatives by their finite parts when R is taken as the independent variable.*

It was pointed out by M. Riesz* that the use of the square of R as the independent variable brings the retarded potential to a very simple form. Writing

$$R^2=\mathcal{R} \tag{17}$$

and

$$\mathbf{S}(\tau)=-\tfrac{1}{2}d\mathbf{T}/d\tau, \tag{18}$$

we have from Eq. (9)

$$\mathbf{A}^{\text{ret}}=d\mathbf{T}/d\mathcal{R}. \tag{19}$$

III. EXPLICIT EVALUATION OF THE FIELD QUANTITIES ON THE WORLD LINE

Equation (9) may be written for brevity

$$\mathbf{A}^{\text{ret}}=\mathbf{S}/\kappa, \tag{20}$$

where

$$\kappa=(\mathbf{V},\mathbf{R}). \tag{21}$$

The derivatives of $\mathbf{A}^{\text{ret}}$ can best be calculated by a method used by Dirac in his classical theory of radiating electrons. The first two derivatives of $\mathbf{A}^{\text{ret}}$ are

$$\frac{\partial}{\partial x_\mu}\mathbf{A}^{\text{ret}}=-\frac{1}{\kappa}\frac{d}{d\tau}\left(\frac{\mathbf{S}r^\mu}{\kappa}\right), \tag{22}$$

$$\frac{\partial^2}{\partial x_\mu\partial x_\nu}\mathbf{A}^{\text{ret}}=\frac{1}{\kappa}\left\{g^{\mu\nu}\frac{d}{d\tau}\left(\frac{\mathbf{S}}{\kappa}\right)+\frac{d}{d\tau}\left[\frac{1}{\kappa}\frac{d}{d\tau}\frac{\mathbf{S}r^\mu r^\nu}{\kappa}\right]\right\}. \tag{23}$$

* I am indebted to Professor Riesz for this remark.

For the explicit evaluation of the radiation field quantities at a point **X** on the world line, it is convenient to take as the independent variable $\sigma=\tau_0-\tau$, τ_0 being the time when $\mathbf{Z}(\tau)=\mathbf{X}$. Since

$$RdR=\kappa d\sigma, \tag{24}$$

we have

$$\mathbf{F}^{(\alpha)}(\mathbf{X})=\alpha\int_0^{\sigma_A}\kappa\mathbf{F}R^{\alpha-2}d\sigma, \tag{25}$$

with **F** given by Eqs. (20), (22), and (23) for the potential and its first two derivatives, σ_A being the value of σ corresponding to τ_A. Using the expansions

$$\mathbf{S}=\mathbf{S}_0-\left(\frac{d\mathbf{S}}{d\tau}\right)_0\sigma+\frac{1}{2}\left(\frac{d^2\mathbf{S}}{d\tau^2}\right)_0\sigma^2-\frac{1}{6}\left(\frac{d^3\mathbf{S}}{d\tau^3}\right)_0\sigma^3+\cdots, \tag{26}$$

$$\mathbf{R}=\mathbf{X}-\mathbf{Z}=\mathbf{V}_0\sigma-\frac{1}{2}\left(\frac{d\mathbf{V}}{d\tau}\right)_0\sigma^2+\frac{1}{6}\left(\frac{d^2\mathbf{V}}{d\tau^2}\right)_0\sigma^3-\frac{1}{24}\left(\frac{d^3\mathbf{V}}{d\tau^3}\right)_0\sigma^4+\cdots, \tag{27}$$

$$R=\sigma\left\{1-\frac{1}{24}\left(\frac{d\mathbf{V}}{d\tau}\right)_0^2\sigma^2+\frac{1}{24}\left(\frac{d\mathbf{V}}{d\tau},\frac{d^2\mathbf{V}}{d\tau^2}\right)_0\sigma^3+\cdots\right\}, \tag{28}$$

$$R^{\alpha-2}=\sigma^{\alpha-2}\left\{1-\frac{\alpha-2}{24}\left(\frac{d\mathbf{V}}{d\tau}\right)_0^2\sigma^2+\frac{\alpha-2}{24}\left(\frac{d\mathbf{V}}{d\tau},\frac{d^2\mathbf{V}}{d\tau^2}\right)_0\sigma^3+\cdots\right\}, \tag{29}$$

where the suffix 0 denotes the value at time τ_0, we obtain from Eq. (25) and the general relation of Eq. (16)

$$\mathbf{A}(\mathbf{X})=-(d\mathbf{S}/d\tau)_0, \tag{30}$$

$$\frac{\partial\mathbf{A}(\mathbf{X})}{\partial x^\mu}=-\left\{\tfrac{1}{3}\mathbf{S}v_\mu\left(\frac{d\mathbf{V}}{d\tau}\right)^2+\tfrac{1}{3}\mathbf{S}\left(\frac{d^2v_\mu}{d\tau^2}\right)+\frac{dv_\mu}{d\tau}\frac{d\mathbf{S}}{d\tau}+v_\mu\frac{d^2\mathbf{S}}{d\tau^2}\right\}_0, \tag{31}$$

and a rather lengthy expression for $\partial^2\mathbf{A}/\partial x_\mu\partial x_\nu$, which will not be written down here. These results are just half the corresponding quantities of the radiation field given by the λ-limiting process, namely,

$$\mathbf{A}(\mathbf{X})=\tfrac{1}{2}\{(\mathbf{S}/\kappa)_{\text{ret}}+(\mathbf{S}/\kappa)_{\text{adv}}\}, \tag{32}$$

$$\mathbf{F}^{\mu\nu\cdots}(\mathbf{X})=\frac{1}{2}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\cdots\left\{\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{ret}}+\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{adv}}\right\}. \tag{33}$$

The calculation here is simpler than the previous calculation of the radiation field because the Riesz field has finite values on the world line from the beginning, so that we do not have to consider its values in the vicinity of the world line.

IV. GENERAL PROOF OF THE EQUIVALENCE OF THE TWO METHODS

We give now a general proof that the two methods give the same results for the classical electromagnetic field. It is obvious that the two methods are equivalent for a point not on the world line, since at such a point the retarded field quantities given by the formula of Lienard and Wiechert are finite, and therefore the finite parts given by the Riesz method are just the retarded field quantities, in agreement with the λ-limiting process.

This is no longer the case for a point on the world line where the Lienard-Wiechert potential becomes infinite. To show the equivalence of the two methods in such a case we use the expansion method introduced by Dirac and developed in detail by Bhabha and Harish-Chandra.

Let us consider a field point **X** near the world line at a distance ϵ from its "contemporary point" **Z**c, i.e., the point at which $\kappa=0$. Using the suffix c to denote the value of a quantity at the contemporary point and writing $\tau-\tau_c=\Delta\tau$, we have, as shown by the above authors,

$$\mathbf{S}=\mathbf{S}_c+\left(\frac{d\mathbf{S}}{d\tau}\right)_c\Delta\tau+\frac{1}{2}\left(\frac{d^2\mathbf{S}}{d\tau^2}\right)_c(\Delta\tau)^2+\frac{1}{6}\left(\frac{d^3\mathbf{S}}{d\tau^3}\right)_c(\Delta\tau)^3+\cdots,$$
$$\kappa=-(1-\kappa_c')\Delta\tau+\tfrac{1}{2}\kappa_c''(\Delta\tau)^2+\frac{1}{6}\left(\left(\frac{d\mathbf{V}}{d\tau}\right)_c^2+\kappa_c'''\right)(\Delta\tau)^3+\cdots, \tag{34}$$

$$R^2=-\epsilon^2+(1-\kappa_c')(\Delta\tau)^2-\tfrac{1}{3}\kappa_c''(\Delta\tau)^3-\frac{1}{12}\left(\left(\frac{d\mathbf{V}}{d\tau}\right)_c^2+\kappa_c'''\right)(\Delta\tau)^4+\cdots \tag{35}$$

where κ_c', κ_c'', etc. denote $(\mathbf{R},\, d\mathbf{V}/d\tau)_c$, $(\mathbf{R},\, d^2\mathbf{V}/d\tau^2)_c$, etc. From Eq. (35) with R set equal to zero,

$$\Delta\tau=\pm\epsilon(1+\tfrac{1}{2}\kappa_c'+\tfrac{3}{8}\kappa_c'^2)+\tfrac{1}{6}\kappa_c''\epsilon^2\pm(1/24)(d\mathbf{V}/d\tau)_c^2\epsilon^3+\cdots, \tag{36}$$

where the upper and lower signs are to be taken for the advanced and retarded times, respectively. From Eqs. (34) and (36),

$$\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{ret}}=\frac{\mathbf{S}_c}{\epsilon}+\left(-\frac{d\mathbf{S}}{d\tau}+\frac{\mathbf{S}\kappa'}{2\epsilon}\right)_c+\left[\mathbf{S}\left(\frac{3\kappa'^2}{8\epsilon}-\frac{\kappa''}{3}-\left(\frac{d\mathbf{V}}{d\tau}\right)^2\frac{\epsilon}{24}\right)-\kappa'\frac{d\mathbf{S}}{d\tau}+\frac{\epsilon}{2}\frac{d^2\mathbf{S}}{d\tau^2}\right]_c+\cdots. \tag{37}$$

The value of the same quantity at the advanced time is obtained by writing $-\epsilon$ for ϵ in the right-hand side. It follows therefore,

$$\frac{1}{2}\left[\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{ret}}+\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{adv}}\right]=-\left(\frac{d\mathbf{S}}{d\tau}\right)_c-\left(\frac{1}{3}\mathbf{S}\kappa''+\frac{d\mathbf{S}}{d\tau}\kappa'\right)_c+\cdots. \tag{38}$$

With the help of the formulas

$$\frac{\partial\tau_c}{\partial x_\mu}=\left(\frac{v^\mu}{1-\kappa'}\right)_c, \tag{39}$$

$$\frac{\partial r_c^\mu}{\partial x_\nu}=g^{\mu\nu}-\left(\frac{v^\mu v^\nu}{1-\kappa'}\right)_c, \tag{40}$$

$$\frac{\partial\epsilon^2}{\partial x_\mu}=-2r_c^\mu, \tag{41}$$

Bhabha and Harish-Chandra reached the conclusion that the expression in Eq. (38) and all its derivatives are of the form

$$\sum_0^\infty \mathbf{a}_n\epsilon^{2n},$$

where the $\mathbf{a}_n$ are functions of τ_c and $\mathbf{R}_c$ which are finite on the world line. It follows therefore that the radiation potential and all its derivatives are finite when $\epsilon=0$, i.e., when the point **X** lies on the world line.

For the Riesz field R is not necessarily zero. For the purpose of evaluating the finite parts, however, it is sufficient to consider values of R in the neighborhood of zero. We can then take R to be a small number and solve again Eq. (35) for $\Delta\tau$, the result being

$$\Delta\tau=\pm(1+\tfrac{1}{2}\kappa'+\tfrac{3}{8}\kappa'^2)_c(\epsilon^2+R^2)^{\frac{1}{2}}+\tfrac{1}{6}\kappa_c''(\epsilon^2+R^2)\pm(1/24)(d\mathbf{V}/d\tau)_c^2(\epsilon^2+R^2)^{\frac{3}{2}}+\cdots \tag{42}$$

for the advanced and retarded times respectively. Equation (42) differs from Eq. (36) only in having $(\epsilon^2+R^2)^{\frac{1}{2}}$ instead of ϵ. Similarly,

$$\left(\frac{\mathbf{S}}{\kappa}\right)_{\text{ret}}=\frac{\mathbf{S}_c}{(\epsilon^2+R^2)^{\frac{1}{2}}}+\left[-\frac{d\mathbf{S}}{d\tau}+\frac{\mathbf{S}\kappa'}{2(\epsilon^2+R^2)^{\frac{1}{2}}}\right]_c$$

$$+\left[\mathbf{S}\left(\frac{3\kappa'^2}{8(\epsilon^2+R^2)^{\frac{1}{2}}}-\frac{\kappa''}{3}-\frac{(\epsilon^2+R^2)^{\frac{1}{2}}}{24}\left(\frac{d\mathbf{V}}{d\tau}\right)^2\right)-\kappa'\frac{d\mathbf{S}}{d\tau}+\frac{(\epsilon^2+R^2)^{\frac{1}{2}}}{2}\frac{d^2\mathbf{S}}{d\tau^2}\right]_c+\sum_0^\infty \mathbf{a}_n(\epsilon^2+R^2)^n+\sum_0^\infty \mathbf{b}_n(\epsilon^2+R^2)^{n+\frac{1}{2}}, \tag{43}$$

where the $\mathbf{b}_n$ denote functions similar to the $\mathbf{a}_n$. Using the formulas (39), (40), (41) and

$$\frac{\partial(\epsilon^2+R^2)}{\partial x_\mu}=\frac{\partial\epsilon^2}{\partial x_\mu}=-2r_c^{\mu}, \tag{44}$$

we see that both the expression in Eq. (43) and all its derivatives are of the form

$$\sum_0^\infty \mathbf{a}_n(\epsilon^2+R^2)^n+\sum_{-m}^\infty \mathbf{b}_n(\epsilon^2+R^2)^{n+\frac{1}{2}}, \tag{45}$$

when m is either zero or a positive integer.

The expansion in Eq. (45) contains the results of both methods for the field quantities on the world line. The λ-limiting process consists in putting $R=0$, taking the mean value

$$\frac{1}{2}\left\{\left(\sum_0^\infty \mathbf{a}_n\epsilon^{2n}+\sum_{-m}^\infty \mathbf{b}_n\epsilon^{2n+1}\right)+\left(\sum_0^\infty \mathbf{a}_n\epsilon^{2n}-\sum_{-m}^\infty b_n\epsilon^{2n+1}\right)\right\}=\sum_0^\infty \mathbf{a}_n\epsilon^{2n} \tag{46}$$

and finally putting $\epsilon=0$. On the other hand, the Riesz method consists in putting $\epsilon=0$ and replacing the expansion

$$\sum_0^\infty \mathbf{a}_nR^{2n}+\sum_{-m}^\infty \mathbf{b}_nR^{2n+1} \tag{47}$$

by its finite part. We see therefore quite generally that the two methods are equivalent.

I am indebted to Professor W. Pauli for a discussion which suggested the present investigation, and for his advice.