

(i.e., a point of Hilbert's space). Direct inspections show that all those quantum-mechanical matrices Q, P which have been calculated explicitly (linear oscillator, hydrogen atom, etc.) fail to be bounded in Hilbert's sense.

It is natural to expect that this failure is independent of the explicit choice of the particular Hamiltonians H in question. In fact, the unboundedness of Q, P is indicated by heuristic considerations concerning the location of possible energy levels. However, the literature consulted does not contain a general proof. The purpose of this note is the construction of such a proof.

If Planck's constant is expressed in appropriate units, Heisenberg's relation can be written in the form

$$QP - PQ = \frac{1}{2}iE, \quad (1)$$

where E denotes the unit matrix and Q, P are supposed to be Hermitian. Without any reference to a Hamiltonian H , it will be shown that these assumptions prevent the boundedness of Q and P .

Let A^* denote the transposed matrix of the complex-conjugate of an arbitrary matrix A . Thus A is Hermitian if and only if $A^* = A$. Since Q and P are Hermitian, it follows that A^* is $Q - iP$, if A denotes $Q + iP$. Hence, Eq. (1) can be written in the form

$$A^*A - AA^* = E. \quad (2)$$

Furthermore, if P and Q are bounded, then A is bounded. It will be shown that these assumptions lead to a contradiction.

By the spectrum of a bounded matrix A is meant the set of those real or complex values λ for which the bounded matrix $\lambda E - A$ fails to have a (unique) bounded reciprocal $(\lambda E - A)^{-1}$. It is known that the spectrum of every bounded matrix is contained in a sufficiently large circle about the origin of the λ -plane, and that every bounded matrix has a spectrum¹ (i.e., that the spectrum contains at least one λ -value). Because of the first of these two properties, $A - \lambda E$ must have a bounded reciprocal matrix if the absolute value of λ is large enough. On the other hand, it is clear that the expression to the left side of Eq. (2) remains unaltered if A is replaced by $A - \lambda E$, where λ is any real value. Hence, if $A - \lambda E$, where λ is real and large enough, is denoted by A , there results a bounded matrix A satisfying Eq. (2) and possessing a bounded reciprocal A^{-1} . But Eq. (2) can then be written in the form

$$B - ABA^{-1} = E, \quad (3)$$

where $B = A^*A$. In fact, the product of bounded matrices is bounded, and their multiplication is associative. In particular, B is bounded.

It is clear from the above definition of a spectrum that if A, A^{-1} and B are bounded, then B and its transform ABA^{-1} have the same spectrum. Since Eq. (3) means that ABA^{-1} is identical with $B - E$, it follows that B and $B - E$ have the same spectrum. Finally, the definition of a spectrum also shows that the spectrum of $B - E$ results from that of B by a rigid motion, representing a translation (of unit span) of the λ -plan into itself.

Accordingly, the spectrum of B remains invariant under a certain translation. Consequently, either the spectrum of

B contains every point of a line in the λ -plane or the spectrum of B has no point at all. However, as mentioned above,¹ both of these possibilities are ruled out by the boundedness of B . This contradiction proves the unboundedness of $B = A^*A$, and therefore that of $A = Q + iP$.

¹ A. Wintner, *Spektraltheorie der Unendlichen Matrizen* (Hirzel, Leipzig, 1929) pp. 143-44.

On the Dependence of Nuclear Radius on the Mass Number

E. AMALDI AND B. N. CACCIAPUOTI
Istituto di Fisica della Università, Centro di Fisica Nucleare
del C.N.R., Rome, Italy
April 17, 1947

IN an earlier paper¹ we reported some measurements on nuclear radius for 19 elements with Z between 4 and 83, and sought a linear representation of our results by an expression of the form

$$R = b + r_0 A^{1/3}. \quad (1)$$

By graphical interpolation, in which we intentionally enhanced the importance of results corresponding to elements with high atomic number, we obtained for b and r_0 the following values:

$$b = 1.5 \text{ to } 2.0 \times 10^{-13} \text{ cm}, \quad r_0 = 1.3 \text{ to } 1.0 \times 10^{-13} \text{ cm}. \quad (2)$$

The quantity b represents the surface effect of the nucleus, that is, the contribution of nuclear force range of superficial particles to the value of R ; in fact our measurements refer to a nuclear radius R defined as the distance from the center of a nucleus at which a fast neutron interacts with the nucleus. Other definitions may be given; for instance,² when the binding energies of contiguous and specular isobar pairs are compared, the nuclear radius can be defined as the radius of the proton-distribution sphere. In all cases in which a comparison was possible, the values obtained using this second definition were found to be smaller than ours by about 1×10^{-13} cm.

In our former paper¹ it was mentioned that, notwithstanding the accuracy of the values which we found for R , it is not possible to determine the values of b and r_0 with great precision, owing to the appreciable deviations of the individual points from the linear representation of Eq. (1).

Therefore we thought it worth while to treat the question again by statistical methods in order to test the correctness of fit of a theoretical curve to a set of experimental values, although such procedures are generally open to some criticism.

The second column of Table I gives the values of $R_i + \delta R_i$ for 17 elements. The results obtained for C and Au have been omitted; the geometrical conditions for the C measurements were not quite the same as for the other elements, and the results obtained with Au were much less accurate than for other elements.

The third column gives the ratio of the statistical error to the standard deviation; the values δR_i of the second column are in all cases the greater of the two errors. A detailed account of the calculation of these two errors was given in reference 1; we shall only stress the fact that in

TABLE I.

Element	$(R_i \pm \delta R_i) \cdot 10^{-13}$ cm	Statistical error Standard deviation
4 Be	3.22 ± 0.074	1.5
5 B	4.30 ± 0.24	—
12 Mg	5.40 ± 0.13	0.55
13 Al	5.54 ± 0.11	0.63
16 S	5.02 ± 0.14	2
26 Fe	6.63 ± 0.06	1.2
28 Ni	6.47 ± 0.074	1.5
29 Cu	6.75 ± 0.15	0.3
30 Zn	6.96 ± 0.17	3
34 Se	7.31 ± 0.20	0.55
47 Ag	7.81 ± 1.09	1.3
48 Cd	8.23 ± 0.07	2.3
50 Sn	8.50 ± 0.085	4
51 Sb	8.33 ± 0.096	10
80 Hg	9.50 ± 0.16	0.58
82 Pb	9.00 ± 0.071	0.87
83 Bi	9.08 ± 0.097	3.7

about half of the cases we found that this ratio was smaller, and in the other half was greater than unity, so we believe that the main error is purely statistical. This means that the probability for the real value to deviate by an amount ΔR_i from the observed value may be represented by means of a normal distribution (Gaussian law).

Under this reasonable assumption, we may calculate the probability that a given distribution will deviate from the interpolated straight line as much, or more, than our observed points.

Such a probability is obviously given by

$$P = \int_{(\chi^2)_\frac{1}{2}}^{\infty} e^{-\frac{1}{2}\chi^2} \chi^{n-1} d\chi / \int_0^{\infty} e^{-\frac{1}{2}\chi^2} \chi^{n-1} d\chi, \quad (3)$$

where

$$\chi^2 = \sum_1^n (\Delta R_i / \delta R_i)^2 \quad (4)$$

and n is the number of measured points used for comparison with the interpolated straight line of Eq. (1); in our case $n=17$. The deviations of the observed points from the interpolated line are indicated with ΔR_i and the corresponding experimental errors are δR_i (see the second column of Table I).

The function defined by (3) is formally identical to the function P introduced by Pearson for his " χ^2 test," although in our case the definition of χ^2 is slightly different from his. In order to calculate P , the most convenient linear representation (1) is the one that makes χ^2 a minimum. According to the definition (4) of χ^2 , this obviously means that the constants b and r_0 of (1) must be determined by the least-squares method, considering the different precision δR_i of the single points.

In this way we get

$$\begin{aligned} b &= (0.696 \pm 0.082) \times 10^{-13} \text{ cm}, \\ r_0 &= (1.52 \pm 0.02) \times 10^{-13} \text{ cm}, \end{aligned} \quad (5)$$

that is, two values quite different from those obtained with the graphical interpolation (2), in which we intentionally over-rated the points corresponding to heavy elements.

Applying definition (4), with the experimental values of Table I, and through the representations (1) and (5), we obtain for χ^2 the value $\chi^2 \approx 270$ which corresponds, for $n=17$, to a very small value of P defined by (3). From Pearson's tables,³ it is seen that, for $n=17$, P is smaller than 10^{-6} , even for $\chi^2=70$.

Thus we must argue that representation (1) is not a good one if we assume purely accidental errors; in fact, the probability of obtaining a distribution such as the one we found for the deviations of the experimental values from a linear law (1) is exceedingly small. However, it is well known that a small value of P does not necessarily indicate that the experimental deviations from law (1) are systematic, because the "goodness of fit" test has meaning only in the case of purely accidental errors.

We believe that the very small value obtained by us for P gives good evidence for a dependence of the nuclear radius on the number of particles in a nucleus, which is more complex than the simple law (1).

¹ E. Amaldi, D. Bocciarelli, B. N. Cacciapuoti, and G. C. Trabacchi, *Nuovo. Cimento*, in press. Presented at the conference on elementary particles and low temperature at Cambridge, England, July, 1946.

² H. A. Bethe, *Phys. Rev.* **54**, 436 (1938); E. P. Wigner, *Phys. Rev.* **56**, 519 (1939).

³ K. Pearson, *Tables for Statisticians and Biometricians*, Part I.

Search for Element 87

HORIA HULUBEI
Bucarest, Rumania
March 31, 1947

THE article by F. R. Hirsh, Jr.¹ "The Search for Element 87" came to my notice only recently because of the interruption of postal relations between America and Rumania during the war. I wish to comment on the numerical data employed in his research and on the research itself. He adopts 0.812A and 1.045A for the presumptive wave-lengths of the absorption line L_{III} and the emission line $L\alpha$, respectively. The former departs by 0.011A from the value deduced, by Moseley's law, from the most reliable experimental values obtained for neighboring elements. The latter departs by 0.017A from the value to be expected for $L\alpha_1$, and by 0.011A from the average of the to-be-expected values of $L\alpha_1$ and $L\alpha_2$. Convincing evidence for the element 87 could not be found, even assuming the element to be present in the sample, with such expectations as to the wave-lengths. I suggest that his categorical objections to the inferences of other authors are based on very inexact estimates of the emission-lines which are to be expected.

I find the experiments of Hirsh far from convincing. He used a tungsten filament and two molybdenum anticathodes, and had to clean the anticathodes frequently to remove the pulverized tungsten from them. A photograph taken with an exposure of 36 hours, and having given no indication whatever of the tungsten L lines under the conditions of his observations, should not be expected to reveal the presence of an element present very probably in much smaller quantities than the tungsten remaining on the anticathode even after hourly cleanings. In addition, the $L\gamma$ lines of tungsten would mask the presumptive line $L\alpha_1$ of element 87 if they were both emitted. Similarly, the lines of molybdenum should have made their appearance: whether or not they were sufficiently excited by the narrow residual band of continuous x-rays springing from the primary anticathode under a 25-kv bombardment and falling