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The Double Focusing Beta-Ray Spectrometer

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The double focusing spectrometer recently proposed by Siegbahn and Svartholm has been analyzed and compared with the conventional semicircular spectrometer. A shaped magnetic field is required which possesses cylindrical symmetry as well as symmetry with respect to the median plane of the instrument. The magnetic field in the median plane is assumed to have the form

$$H=H_0-(r-a)\alpha H_0/a+(r-a)^2\beta H_0/a^2,$$

where α and β are experimentally disposable shaping constants. The analysis, which has been carried through the second order of approximation, shows that double focusing occurs when $\alpha=\frac{1}{2}$. In general there will exist, for all types of spectrometers, second order defocusing terms which tend to lower the resolving power. However, these may be almost completely eliminated in the double focusing spectrometer by choosing $\beta=\frac{1}{8}$. A figure is presented showing the form of the focused image for a semicircular spectrometer and for several cases of the double focusing spectrometer. It becomes evident that when $\alpha=\frac{1}{2}$ and $\beta=\frac{1}{8}$, the intensity of the image as well as the resolving power become many times greater than for the former type of instrument.

1. INTRODUCTION

IT is customary in beta-ray spectroscopy to focus electrons with either the magnetic semicircular spectrometer or the magnetic electron lens. The former requires a uniform field, which is created by an iron-core magnetic circuit with plane parallel pole faces. Large magnetic field strengths are feasible, and hence high energy electrons can easily be focused. The electron lens uses a magnetic circuit which is relatively free of iron. Much copper and power are necessary to produce large fields, and it is impractical to work with high energy electrons. In another respect, however, the electron lens seems superior. Assuming equal resolving power for both, the electron lens can focus a beam intensity as much as 15 to 20 times greater than the semicircular spectrometer.

Siegbahn and Svartholm[1] have proposed and constructed an electron focusing system which utilizes an inhomogeneous magnetic field produced by an iron-core magnetic circuit with shaped pole faces. The magnetic field is shaped to fall off radially as $1/r^{\frac{1}{2}}$. The electron beam is focused after sweeping out an angle $2^{\frac{1}{2}}\pi$ about the axis of symmetry. The focusing principle is basically a refinement of the semicircular system which permits a considerable increase of beam intensity without sacrificing resolving power. Such a spectrometer could compete more favorably with the electron lens in terms of focused beam intensity, while retaining the ability to focus high energy particles.

It is proposed in the paper which follows to analyze this focusing system, evaluate its performance, and compare it to the conventional semicircular type. The equations of motion of the electrons will be solved by means of the usual perturbation methods, and the calculations will be carried through the second order of approximation.

2. THEORY

It will be convenient to use the cylindrical coordinates r, θ, and z, and it will be assumed that the magnetic field is (a) independent of the azimuth angle θ and (b) symmetrical with respect to the median plane $z=0$. The vertical component of the field, H_z, as measured in the median plane, is taken to have the form

$$H_z=H_0-\left(\frac{r-a}{a}\right)\alpha H_0+\left(\frac{r-a}{a}\right)^2\beta H_0-\cdots, \quad (1)$$

where a is the radial coordinate of the center of the electron source (whose other coordinates are $\theta=0$, $z=0$), H_0 is a constant magnetic field, and α and β are arbitrary dimensionless constants. Clearly Eq. (1) represents a series development for H_z through second order, and assumes that the equilibrium or zeroth order orbit will be a circle with the radius $r=a$. If, for points in the

[1] Siegbahn and Svartholm, Nature **157**, 872 (1946).

median plane, H_z is assumed to fall off as $1/r^n$, the identifications $\alpha=n$ and $\beta=(n^2+n)/2$ can be made, but it should be emphasized that H_0, α, and β are experimentally disposable parameters in the present problem. Since $\operatorname{curl}H=0$, it follows that the radial component H_r of the magnetic field must be

$$H_r = zH_0\left[-\frac{\alpha}{a}+\frac{2\beta(r-a)}{a^2}\right],$$

and since $\operatorname{div}H=0$, it follows that a fourth term must be added to Eq. (1) in order to express H_z at points off the median plane. Thus,

$$H_z = H_0 - \left(\frac{r-a}{a}\right)\alpha H_0 + \left(\frac{r-a}{a}\right)^2 \beta H_0 - \frac{z^2}{a^2}\beta H_0.$$

In any purely magnetic field, the motion of a charged particle must be such that its velocity, and consequently its mass, will remain constant. The equations of motion for the particle then reduce to the following expressions:

$$\frac{d\dot{r}}{dt} - r\dot{\theta}^2 = \frac{e}{mc} r\dot{\theta}H_z, \tag{2}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = \frac{er}{mc}(\dot{z}H_r - \dot{r}H_z), \tag{3}$$

$$\frac{d\dot{z}}{dt} = -\frac{e}{mc} r\dot{\theta}H_r. \tag{4}$$

As is well known, an integral of the motion may be obtained by multiplying (2) by $\dot{r}$, (3) by $\dot{\theta}$, and (4) by $\dot{z}$, summing, and integrating with respect to the time. This integral is

$$\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 = v^2 = \text{constant} \tag{5}$$

where v is the initial velocity of the electron. Any three of the Eqs. (2), (3), (4), and (5) would be sufficient to describe the motion, and it will prove convenient in the present case to eliminate (3) and use only (2), (4), and (5).

Assume that an electron originates in the source at the point $r=a$, $z=0$, $\theta=0$ and has the initial velocity components $\dot{r}=0$, $\dot{z}=0$, $r\dot{\theta}=v=-eH_0a/mc$. Under these conditions, the solution of the equations of motion becomes

$$r=a, \quad z=0, \quad \theta=\omega t,$$

where $\omega=v/a$. This circular orbit is hereafter designated as the equilibrium or zeroth order orbit. In general, however, the initial conditions describing an electron orbit will be

$$\left.\begin{aligned} r&=a+\delta r\\ z&=\delta z\\ \theta&=0\\ \dot{r}/v&=\varphi_r\\ \dot{z}/v&=\varphi_z \end{aligned}\right\} t=0,$$

where φ_r and φ_z are evidently angles which specify the direction of the initial velocity, and δr and δz are coordinates of the initial position relative to the center of the source.

It is proposed to solve Eqs. (2), (4), and (5) by standard perturbation methods, and to obtain solutions for r, θ, and z which are complete through second order. Let

$$\begin{aligned} r&=a+\lambda\rho_1+\lambda^2\rho_2,\\ \dot{\theta}&=\omega+\lambda\theta_1+\lambda^2\theta_2,\\ z&=\quad\lambda\zeta_1+\lambda^2\zeta_2, \end{aligned} \tag{6}$$

where ρ_1, θ_1, ζ_1 and ρ_2, θ_2, ζ_2 are functions of the time, and λ is a parameter of smallness. It will be noticed that the equations of motion involve $\dot{\theta}$ rather than θ, and consequently $\dot{\theta}$ has been specified above in place of θ. The variables (6) are substituted directly into Eqs. (2), (4), and (5), and the three resulting equations are classified according to powers of the parameter λ. By equating the coefficients of λ^0, λ^1, and λ^2 to zero, equations are obtained which describe the equilibrium orbit and the first- and second-order corrections to the equilibrium orbit respectively.

The zeroth-order equations yield the following relations, which determine the magnetic field H_0 as a function of the momentum of the electron.

$$v=\omega a, \quad \omega=-eH_0/mc. \tag{7}$$

The first-order equations are

$$\begin{aligned} &\lambda\frac{d\dot{\rho}_1}{dt}+\lambda\omega^2(1-\alpha)\rho_1=0,\\ &\lambda\frac{d\dot{\zeta}_1}{dt}+\lambda\omega^2\alpha\zeta_1=0,\\ &\theta_1=-\omega\rho_1/a. \end{aligned} \tag{8}$$

It is evident that the first-order solutions $\lambda\rho_1$, $\lambda\theta_1$, and $\lambda\zeta_1$ will be sinusoidal, with circular frequencies $\omega(1-\alpha)^{\frac{1}{2}}$, $\omega(1-\alpha)^{\frac{1}{2}}$, and $\omega\alpha^{\frac{1}{2}}$, respectively.

The second-order equations become after some simplification

$$\lambda^2\left[\frac{d\dot{\rho}_2}{dt}+\omega^2(1-\alpha)\rho_2\right]=\lambda^2\left[\frac{\omega^2}{a}\rho_1{}^2-\frac{\beta\omega^2}{a}(\rho_1{}^2-\zeta_1{}^2)-\frac{1}{2a}(\dot{\rho}_1{}^2+\dot{\zeta}_1{}^2)\right], \tag{9}$$

$$\lambda^2\left[\frac{d\dot{\zeta}_2}{dt}+\omega^2\alpha\zeta_2\right]=2\lambda^2\frac{\beta\omega^2}{a}\rho_1\zeta_1, \tag{10}$$

$$\lambda^2\theta_2=\lambda^2\left[\frac{\omega}{a^2}\rho_1{}^2-\frac{\omega}{a}\rho_2-\frac{(\dot{\rho}_1{}^2+\dot{\zeta}_1{}^2)}{2a^2\omega}\right]. \tag{11}$$

Both Eqs. (9) and (10) consist of a homogeneous part, to the left of the equality sign, and an inhomogeneous part. The solution of the homogeneous part of each may be neglected, since it is identical to the solution of the corresponding first order Eq. (8), and thus contributes nothing new. The inhomogeneous parts which remain are now known functions of the time, and may be computed from the first order solutions. The equations can be integrated by elementary processes.

The final expressions for r, z, and θ, complete through second order, are

$$\begin{aligned}
r=a&+\left[\delta r+\frac{\delta r^2}{a}\frac{(\beta-\alpha)}{3(1-\alpha)}+a\varphi_r{}^2\frac{(4\beta-\alpha-3)}{6(1-\alpha)^2}+\frac{\delta z^2}{a}\frac{(3\alpha\beta-\beta-\alpha^2)}{(1-\alpha)(1-5\alpha)}+a\varphi_z{}^2\frac{(4\beta-3\alpha+1)}{2(1-\alpha)(1-5\alpha)}\right]\cos\omega(1-\alpha)^{\frac{1}{2}}t\\
&+\frac{1}{(1-\alpha)^{\frac{1}{2}}}\left[a\varphi_r-\frac{(\alpha+2\beta-3)}{3(1-\alpha)}\varphi_r\delta r-\frac{(\alpha+2\beta)}{(1-5\alpha)}\varphi_z\delta z\right]\sin\omega(1-\alpha)^{\frac{1}{2}}t\\
&+\frac{1}{4a(1-\alpha)}\left[(\alpha-2\beta+1)\left(\delta r^2+\frac{a^2\varphi_r{}^2}{1-\alpha}\right)+(2\beta-\alpha)\left(\delta z^2+\frac{a^2\varphi_z{}^2}{\alpha}\right)\right]\\
&+\frac{(\alpha+2\beta-3)}{12a(1-\alpha)}\left(\delta r^2-\frac{a^2\varphi_r{}^2}{1-\alpha}\right)\cos 2\omega(1-\alpha)^{\frac{1}{2}}t+\frac{(\alpha+2\beta)}{4a(1-5\alpha)}\left(\delta z^2-\frac{a^2\varphi_z{}^2}{\alpha}\right)\cos 2\omega\alpha^{\frac{1}{2}}t\\
&+\frac{(\alpha+2\beta-3)}{6(1-\alpha)^{\frac{3}{2}}}\varphi_r\delta r\sin 2\omega(1-\alpha)^{\frac{1}{2}}t+\frac{(\alpha+2\beta)}{2(1-5\alpha)\alpha^{\frac{1}{2}}}\varphi_z\delta_z\sin 2\omega\alpha^{\frac{1}{2}}t,
\end{aligned} \tag{12}$$

$$\begin{aligned}
z=&\left[\delta z-\frac{2\beta}{a}\left(\frac{\delta r\delta z}{5\alpha-1}-\frac{2a^2\varphi_r\varphi_z}{5\alpha^2-6\alpha+1}\right)\right]\cos\omega\alpha^{\frac{1}{2}}t\\
&+\frac{1}{\alpha^{\frac{1}{2}}}\left[a\varphi_z-2\beta\left(\frac{\varphi_z\delta r}{1-5\alpha}+\frac{(3\alpha-1)}{5\alpha^2-6\alpha+1}\varphi_r\delta z\right)\right]\sin\omega\alpha^{\frac{1}{2}}t\\
&+\frac{\beta\sin\omega[(1-\alpha)^{\frac{1}{2}}+\alpha^{\frac{1}{2}}]t}{\alpha-1-2(\alpha-\alpha^2)^{\frac{1}{2}}}\left[\frac{\varphi_r\delta z}{(1-\alpha)^{\frac{1}{2}}}+\frac{\varphi_z\delta r}{\alpha^{\frac{1}{2}}}\right]+\frac{\beta\sin\omega[(1-\alpha)^{\frac{1}{2}}-\alpha^{\frac{1}{2}}]t}{\alpha-1+2(\alpha-\alpha^2)^{\frac{1}{2}}}\left[\frac{\varphi_r\delta z}{(1-\alpha)^{\frac{1}{2}}}-\frac{\varphi_z\delta r}{\alpha^{\frac{1}{2}}}\right]\\
&+\frac{\beta\cos\omega[(1-\alpha)^{\frac{1}{2}}+\alpha^{\frac{1}{2}}]t}{\alpha-1-2(\alpha-\alpha^2)^{\frac{1}{2}}}\left[\frac{\delta r\delta z}{a}-\frac{a\varphi_r\varphi_z}{(\alpha-\alpha^2)^{\frac{1}{2}}}\right]+\frac{\beta\cos\omega[(1-\alpha)^{\frac{1}{2}}-\alpha^{\frac{1}{2}}]t}{\alpha-1+2(\alpha-\alpha^2)^{\frac{1}{2}}}\left[\frac{\delta r\delta z}{a}+\frac{a\varphi_r\varphi_z}{(\alpha-\alpha^2)^{\frac{1}{2}}}\right],
\end{aligned} \tag{13}$$

$$\theta=\left[1+\frac{(2\beta-\alpha-\alpha^2)}{4(1-\alpha)}\left(\frac{\delta r^2}{a^2}+\frac{\varphi_r^2}{1-\alpha}\right)+\frac{(\alpha^2-2\beta)}{4(1-\alpha)}\left(\frac{\delta z^2}{a^2}+\frac{\varphi_z^2}{\alpha}\right)\right]\omega t$$

$$-\frac{1}{(1-\alpha)^{\frac{1}{2}}}\left[\frac{\delta r}{a}+\frac{\delta r^2}{a^2}\frac{(\beta-\alpha)}{3(1-\alpha)}+\varphi_r^2\frac{(4\beta-\alpha-3)}{6(1-\alpha)^2}+\frac{\delta z^2}{a^2}\frac{(3\alpha\beta-\beta-\alpha^2)}{(1-\alpha)(1-5\alpha)}\right.$$

$$\left.+\varphi_z^2\frac{(4\beta-3\alpha+1)}{2(1-\alpha)(1-5\alpha)}\right]\sin\omega(1-\alpha)^{\frac{1}{2}}t+\frac{1}{1-\alpha}\left[\varphi_r-\frac{\varphi_r\delta r}{3a}\frac{(\alpha+2\beta-3)}{(1-\alpha)}-\frac{\varphi_z\delta z}{a}\frac{(\alpha+2\beta)}{(1-5\alpha)}\right]\cos\omega(1-\alpha)^{\frac{1}{2}}t$$

$$+\frac{(3\alpha^2-13\alpha-2\beta+12)}{24(1-\alpha)^{\frac{3}{2}}}\left[\frac{\delta r^2}{a^2}-\frac{\varphi_r^2}{1-\alpha}\right]\sin 2\omega(1-\alpha)^{\frac{1}{2}}t-\frac{(5\alpha^2+2\beta)}{8\alpha^{\frac{1}{2}}(1-5\alpha)}\left[\frac{\delta z^2}{a^2}-\frac{\varphi_z^2}{\alpha}\right]\sin 2\omega\alpha^{\frac{1}{2}}t$$

$$-\frac{(3\alpha^2-13\alpha-2\beta+12)}{12(1-\alpha)^2}\frac{\varphi_r\delta r}{a}\cos 2\omega(1-\alpha)^{\frac{1}{2}}t+\frac{(5\alpha^2+2\beta)}{4\alpha(1-5\alpha)}\frac{\varphi_z\delta z}{a}\cos 2\omega\alpha^{\frac{1}{2}}t$$

$$-\frac{\varphi_r}{(1-\alpha)}+\frac{(\alpha^2-3\alpha+2\beta)}{4(1-\alpha)^2}\frac{\varphi_r\delta r}{a}-\frac{(\alpha^2+2\beta)}{4\alpha(1-\alpha)}\frac{\varphi_z\delta z}{a}. \tag{14}$$

The integration constants have been chosen so that θ, r, and z satisfy the initial conditions given earlier, namely, at $t=0$

$$r=a+\delta r,\quad \theta=0,\quad z=\delta z,\quad \dot{r}=a\omega\varphi_r=v\varphi_r,\quad \dot{z}=a\omega\varphi_z=v\varphi_z,\quad v^2=\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2.$$

It should be noted that in the orbital Eqs. (12), (13), and (14) those terms which do not contain δr, δz, φ_r or φ_z are the zeroth-order terms, while those which are linear and quadratic in these quantities represent the first and second order terms respectively.

3. THE SEMICIRCULAR SPECTROMETER

When $\alpha=\beta=0$, Eqs. (12), (13), and (14) describe the electron orbits in a semicircular spectrometer, and it will be fruitful to consider this case in some detail before proceeding further. Equations (12) and (13) become

$$r=a+\left[\delta r-\frac{a}{2}(\varphi_r^2-\varphi_z^2)\right]\cos\omega t+\varphi_r(a+\delta r)\sin\omega t-\frac{1}{4a}(\delta r^2-a^2\varphi_r^2)\cos 2\omega t$$

$$-\tfrac{1}{2}\varphi_r\delta r\sin 2\omega t+\frac{1}{4a}(\delta r^2+a^2\varphi_r^2)-\frac{a\varphi_z^2}{2}, \tag{15}$$

$$z=\delta z+a\varphi_z\omega t. \tag{16}$$

When the magnetic field is uniform, as it is here, there exists no restoring force in the z direction, and consequently there can be no z focusing. However, the first order sinusoidal terms in r show that r focusing will occur when $\omega t=\pi$, and that the image will be reversed in r with respect to the source. Let θ^*, r^*, z^* be the coordinates of a focused electron, where $\theta^*=\pi$. Let t^* be the time required for an electron to travel from the source to the focused position. In zeroth order

$$\theta=\omega t,\quad \theta^*=\omega t^*=\pi, \tag{17}$$

while in first order

$$\begin{gathered}\theta=\omega t-\frac{\delta r}{a}\sin\omega t+\varphi_r\cos\omega t-\varphi_r,\\ \theta^*=\pi=\omega t^*-2\varphi_r,\quad \omega t^*=\pi+2\varphi_r.\end{gathered} \tag{18}$$

It will not be necessary for the present purpose to obtain the second-order correction to t^*. Substituting t^* from (17) and (18) into the second- and first-order terms, respectively, of (15) and (16), r^* and z^* are found to be

$$r^* = a - \delta r - a(\varphi_r^2 + \varphi_z^2), \tag{19}$$

$$z^* = \delta z + \pi a \varphi_z + 2a\varphi_r\varphi_z. \tag{20}$$

It is evident that the second-order terms in Eq. (19) represent a departure from perfect focusing which is strongly dependent on φ_r and φ_z, and this will be discussed later. Since the orbits are all circular in form, it is clear that rigorous expressions for r^* and z^* may be derived from purely geometric considerations. When these exact expressions are developed through second order, they yield Eqs. (19) and (20), as they must.

4. THE DOUBLE FOCUSING SPECTROMETER

When the first-order sinusoidal frequencies in Eqs. (12) and (13) are made precisely equal, a simultaneous focusing occurs in both the r and z directions. The necessary condition for this two-dimensional focusing is

$$\omega(1-\alpha)^{\frac{1}{2}} = \omega\alpha^{\frac{1}{2}}, \quad \alpha = \tfrac{1}{2}.$$

In this case, (12) and (13) become

$$\begin{aligned} r = a &+ \left[\delta r + \frac{(2\beta-1)}{3a}(\delta r^2 - \delta z^2) + \frac{(8\beta-7)}{3}a\varphi_r^2 + \frac{(1-8\beta)}{3}a\varphi_z^2\right]\cos\frac{\omega t}{2^{\frac{1}{2}}} \\ &+ 2^{\frac{1}{2}}\left[a\varphi_r + \frac{(5-4\beta)}{3}\varphi_r\delta_r + \frac{(1+4\beta)}{3}\varphi_z\delta z\right]\sin\frac{\omega t}{2^{\frac{1}{2}}} + \frac{(3-4\beta)}{4a}(\delta r^2 + 2a^2\varphi_r^2) + \frac{(4\beta-1)}{4a}(\delta z^2 + 2a^2\varphi_z^2) \\ &+ \frac{1}{12a}[(4\beta-5)(\delta r^2 - 2a^2\varphi_r^2) - (1+4\beta)(\delta z^2 - 2a^2\varphi_z^2)]\cos 2^{\frac{1}{2}}\omega t \\ &+ \frac{2^{\frac{1}{2}}}{6}[(4\beta-5)\varphi_r\delta r - (1+4\beta)\varphi_z\delta z]\sin 2^{\frac{1}{2}}\omega t, \end{aligned} \tag{21}$$

$$\begin{aligned} z = &\left[\delta z - \frac{4\beta}{3a}\delta r\delta z - \frac{16\beta}{3}a\varphi_r\varphi_z\right]\cos\frac{\omega t}{2^{\frac{1}{2}}} + 2^{\frac{1}{2}}\left[a\varphi_z + \frac{4\beta}{3}(\varphi_z\delta r + \varphi_r\delta z)\right]\sin\frac{\omega t}{2^{\frac{1}{2}}} \\ &+ 2\beta\left[\frac{\delta r\delta z}{a} + 2a\varphi_r\varphi_z\right] - \frac{2\beta}{3}\left[\frac{\delta r\delta z}{a} - 2a\varphi_r\varphi_z\right]\cos 2^{\frac{1}{2}}\omega t - \frac{2^{\frac{1}{2}}\beta}{3}[\varphi_z\delta r + \varphi_r\delta z]\sin 2^{\frac{1}{2}}\omega t. \end{aligned} \tag{22}$$

The first order sinusoidal terms in (21) and (22), namely those terms which are linear in δr, δz, φ_r or φ_z, show that focusing will take place when $\omega t/2^{\frac{1}{2}} = \pi$, and that the image will be reversed in r and inverted in z. Let θ^*, r^*, z^* and t^* be defined as before. Then, in zeroth approximation,

$$\theta = \omega t, \quad \theta^* = \omega t^* = 2^{\frac{1}{2}}\pi \sim 254°\ 33', \tag{23}$$

while in first order

$$\begin{aligned} \theta &= \omega t - 2^{\frac{1}{2}}\frac{\delta r}{a}\sin\frac{\omega t}{2^{\frac{1}{2}}} + 2\varphi_r\cos\frac{\omega t}{2^{\frac{1}{2}}} - 2\varphi_r, \\ \theta^* &= 2^{\frac{1}{2}}\pi = \omega t^* - 4\varphi_r, \quad \omega t^* = 2^{\frac{1}{2}}\pi + 4\varphi_r. \end{aligned} \tag{24}$$

Substituting t^* from Eqs. (23) and (24) into the second- and first-order terms, respectively, of

(21) and (22), r^* and z^* are found to be

$$r^*=a-\delta r+\frac{(2-4\beta)}{3}\frac{(\delta r^2-\delta z^2)}{a}+\frac{(2-16\beta)}{3}a(\varphi_r{}^2-\varphi_z{}^2), \tag{25}$$

$$z^*=-\delta z+\frac{8\beta}{3}\frac{\delta r\delta z}{a}+\frac{(32\beta-12)}{3}a\varphi_r\varphi_z. \tag{26}$$

As before, the second-order terms in (25) and (26) represent departures from perfect focusing in r and z, respectively. The arbitrary constant β, which determines the second-order correction to the magnetic field, still remains as a disposable parameter and can be chosen so as to minimize the second order defocusing.

For the spectrometer of Siegbahn and Svartholm, it is stated that the magnetic field was shaped to fall off as $1/r^{\frac{1}{2}}$. This is identical with Eq. (1) if $\alpha=\frac{1}{2}$ and $\beta=\frac{3}{8}$. When $\beta=\frac{3}{8}$, Eqs. (25) and (26) become

$$r^*=a-\delta r+\frac{(\delta r^2-\delta z^2)}{6a}-\frac{4}{3}a(\varphi_r{}^2-\varphi_z{}^2), \tag{27}$$

$$z^*=-\delta z+\frac{\delta r\delta z}{a}. \tag{28}$$

As in the semicircular spectrometer, the second order defocusing in r is strongly dependent upon φ_r and φ_z, and this point will be discussed in greater detail later.

With a different choice of the field parameter β, it is possible to eliminate φ_r and φ_z entirely from the second order defocusing in r. Thus, when $\beta=\frac{1}{8}$, Eqs. (25) and (26) become

$$r^*=a-\delta r+\frac{(\delta r^2-\delta z^2)}{2a}, \tag{29}$$

$$z^*=-\delta z+\frac{\delta r\delta z}{3a}-\frac{8}{3}a\varphi_r\varphi_z. \tag{30}$$

Comparing Eqs. (19) and (20), (27) and (28), and (29) and (30), the following conclusions may be drawn. It will be assumed throughout that the source is rectilinear in shape, with its long dimension parallel to the z axis.

(a) For the semicircular spectrometer, the negative term $a(\varphi_r{}^2+\varphi_z{}^2)$ in Eq. (19) predicts that the center of gravity of the image will be shifted to smaller values of r, and that the image will be correspondingly broadened.

(b) For the double focusing spectrometer where $\beta=\frac{3}{8}$, the term $4a(\varphi_r{}^2-\varphi_z{}^2)/3$ in Eq. (27) may be either positive or negative. The image will be broadened toward both larger and smaller values of r. Its center of gravity may or may not shift; the magnitude and direction of the shift depend upon the relative maximum values of φ_r and φ_z. The term $(\delta r^2-\delta z^2)/6a$ in Eq. (27) predicts that the image will be slightly concave toward the origin of coordinates.

(c) For the double focusing spectrometer where $\beta=\frac{1}{8}$, Eq. (29) predicts a curvature of the image similar to that discussed under (b) above. However, the absence of φ_r and φ_z means that the image will be sharp, and its center of gravity is at $r=a$.

(d) Terms of similar order of magnitude involving $a\varphi_r\varphi_z$ appear in each of the z^* Eqs. (20) and (30), but not in (28). In addition, Eq. (20), for the semicircular spectrometer, contains the much stronger term $\pi a\varphi_z$. Such terms mean that the image will be lengthened vertically, and this effect is especially prominent in the semicircular type.

The conclusions listed under (a) through (d) above are illustrated graphically in Fig. 1. The rectilinear source has been assumed to have the dimensions 20 mm$\times$2 mm, while the radius a has been arbitrarily chosen as $a=10$ cm. It is further assumed that the angles φ_r and φ_z are limited by a system of baffles so that $(\varphi_r{}^2+\varphi_z{}^2)^{\frac{1}{2}}\leqq 0.1$ radian. Figure 1 illustrates the image formed in each of the three spectrometer types which have been discussed; only the upper half of each image is shown, since the image is symmetrical with respect to the median plane. The shaded rectangle represents a perfect image of the source, centered at $r=a$; it could also be considered as representing an exit slit, through which the focused beam passes into a Geiger counter. The dotted contours represent the extreme boundaries of the focused image; it should be emphasized that the intensity distribution within the boundary is not uniform. Nevertheless, it is evident that a large proportion of the focused particles in a semicircular spectrometer miss the exit slit entirely, and consequently the counting rate may be expected to be relatively low. On the other hand, the double focusing spectrometer, especially when the field parameter is $\beta=\frac{1}{8}$, makes very efficient use of the focused beam, and the counting rate will be relatively high. Moreover, when $\beta=\frac{1}{8}$, the sharpness of the image leads to a considerably higher resolving power. If the exit slit is properly curved and elongated to fit the image, maximum counting rate and maximum resolving power will be attained simultaneously.

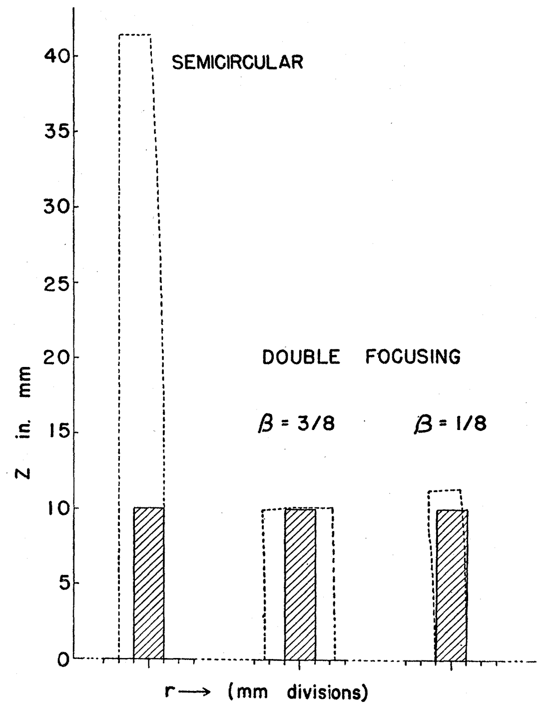

FIG. 1. The focused images for a semicircular spectrometer and for double focusing spectrometers with $\beta=\frac{3}{8}$ and $\frac{1}{8}$, respectively.

5. CONCLUSION

The analysis has shown that, through the use of a shaped magnetic field, the beta-ray spectrometer proposed by Siegbahn and Svartholm may be made into a very effective instrument. The resolving power as well as the intensity of the focused beam are much higher than those of a semicircular spectrometer. Some technical difficulties may be encountered in the construction of the instrument, however, since it is necessary that the shape of the magnetic field shall remain independent of the magnitude of the field over the useful range of operation.

With the shaped field where $\beta=\frac{1}{8}$ and where a curved exit slit is used, the resolving power is very high; in fact, the equations show that the image will almost precisely coincide with the slit. This nearly perfect predicted performance may be slightly illusory in the sense that third and higher order terms in the development of the solution will again lead to some defocusing. It is clear, however, that these will be small and of the order of magnitude of, say, $\delta z^3/a^2$ or of $a\varphi_r^3$. For the spectrometer dimensions chosen in the present example, these terms would have extreme values of about 0.1 mm. In principle, through the use of a further shaping constant γ for the magnetic field, it might be possible to cancel out much of the third order defocusing. This would appear to be too refined a process to be practical however.